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Additive and multiplicative duals for American option pricing

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Abstract We investigate and compare two dual formulations of the American option pricing problem based on two decompositions of supermartingales: the additive dual of Haugh and Kogan (Oper. Res. 52:258–270, 2004) and Rogers (Math. Finance 12:271–286, 2002) and the multiplicative dual of Jamshidian (Minimax optimality of Bermudan and American claims and their Monte-Carlo upper bound approximation. NIB Capital, The Hague, 2003). Both provide upper bounds on American option prices; we show how to improve these bounds iteratively and use this to show that any multiplicative dual can be improved by an additive dual and vice versa. This iterative improvement converges to the optimal value function. We also compare bias and variance under the two dual formulations as the time horizon grows; either method may have smaller bias, but the variance of the multiplicative method typically grows much faster than that of the additive method. We show that in the case of a discrete state space, the additive dual coincides with the dual of the optimal stopping problem in the sense of linear programming duality and the multiplicative method arises through a nonlinear duality.

Keywords Optimal stopping · Monte Carlo methods · Variance reduction

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1 Introduction

In the pricing of American options, Monte Carlo methods become potentially attractive when the number of underlying assets or state variables is large, as is often the case in interest rate models, for example. Because pricing an American option entails solving an optimal stopping problem, it is generally infeasible to develop an unbiased Monte Carlo estimator, so Broadie and Glasserman [4,5] introduced methods that pair two estimators, one biased high and one biased low, to produce conservative confidence intervals for the true price. Low-biased estimates are generated by methods based on suboptimal stopping rules; these include, for example, Andersen [1], Kolodko and Schoenmakers [12], and Longstaff and Schwartz [13]. High-biased estimates result from combining backward induction with simulation or, more systematically, from the duality results of Haugh and Kogan [8] and Rogers [15]. In these dual formulations of optimal stopping problems, a maximization over stopping times is replaced by a minimization over martingales; a suboptimal martingale thus produces an upper bound. Andersen and Broadie [2] show how to estimate the dual value associated with a suboptimal policy, again to bound the option price from both above and below.

The dual formulations of Haugh and Kogan [8] and Rogers [15] are rooted in Doob's decomposition of a supermartingale as the sum of a martingale and a decreasing process. Jamshidian [9] shows that a multiplicative decomposition of positive supermartingales—as products of martingales and decreasing processes—leads to an alternative dual formulation of the American option pricing problem. This multiplicative dual is appealing in a financial context because of its similarity to the mechanics of discounting and measure transformations commonly used in pricing.

The objective of this paper is to compare and further investigate these additive and multiplicative duals to the American option pricing problem. We consider only the "Bermudan" case of a finite number of exercise opportunities. There is a trivial sense in which the two duals are equally effective: neither has a duality gap, meaning that in both cases the optimal value of the dual problem coincides with the optimal value of the original optimal stopping problem. We go further and establish the following:

- We develop transformations of martingales that reduce the dual value associated with the martingales and thus improve the quality of the associated upper bounds. Using these transformations, we show that any additive dual can be improved by a multiplicative dual and vice versa. Moreover, this iterative improvement converges to the optimal value.
- We compare the growth in the bias of additive and multiplicative duals as the time horizon T grows. The multiplicative bias is $O(\sqrt{T})$ under a

bound on the relative error in approximating the value function, whereas the additive bias is $O(\sqrt{T})$ under a similar bound on the absolute error.

- We compare the growth in the variance of estimates based on the two duals as the time horizon grows. We give conditions under which the variance using the multiplicative dual grows *at least* exponentially and the variance using the additive method grows *at most* quadratically.
- We examine the two methods in the case of a finite state space and show that the additive dual coincides with ordinary linear programming duality and the multiplicative dual has an interpretation in terms of nonlinear programming duality.

Our comparisons involve conditions and qualifications; a universal comparison does not seem feasible or even meaningful. However, we consider the variance comparison the most compelling distinction between the two methods—one that shows up quickly in numerical tests. Given that there is no clear ordering of the quality of the upper bounds produced by the two methods, the variance comparison favors the additive formulation.

Section 2 reviews the two dual problems. Section 3 presents our iterative improvement method and Sect. 4 relates it to dynamic programming. Section 5 presents our bias comparisons and Sect. 6 our variance comparisons. Section 7 examines the connection with linear programming duality.

2 Doob's decomposition and duality

We formulate a discrete-time optimal stopping problem, starting from a stochastic process $X = \{X_t, t = 0, 1, \ldots, T\}$. Let $\mathcal{F} = \{\mathcal{F}_t : 0 \leq t \leq T\}$ be the filtration generated by X. Exercising (or stopping) at time t yields a payoff of $h(t, \mathbf{X}_t) \geq 0$, where $\mathbf{X}_t = \{X_0, \ldots, X_t\}$. Let $\mathcal{T}$ be the set of all stopping times with respect to $\mathcal{F}$. The value of the optimal stopping problem is then

$$V_0 = \sup_{\tau \in \mathcal{T}} E[h(\tau, \mathbf{X}_\tau)|\mathcal{F}_0]. \tag{2.1}$$

(We have not included explicit discounting in Eq. (2.1), but discounting may be incorporated by appropriately defining h and $\mathbf{X}$.)

More generally, let $\mathcal{F}^j = \{\mathcal{F}_t : j \leq t \leq T\}$ and let $\mathcal{T}^j$ be the set of all stopping times with respect to $\mathcal{F}^j$, restricted to take values in $\{j, j+1, \ldots, T\}$. Then

$$V_j = \sup_{\tau \in \mathcal{T}^j} E[h(\tau, \mathbf{X}_\tau)|\mathcal{F}_j], \quad 0 \leq j \leq T \tag{2.2}$$

represents the value of the optimal stopping problem if exercise prior to time j is precluded. This may also be thought of as the value of a new American option issued at time j, given the history $\mathcal{F}_j$.

It is standard (and easily verified) that $V = \{V_j, j = 0, 1, \ldots, T\}$ is a supermartingale with respect to $\mathcal{F}$. As such, it admits an (additive) Doob decomposition,

$$V_t = M_t + D_t, \quad t = 0, 1, \ldots, T, \tag{2.3}$$

in which M is a martingale with $M_0 = 0$ and D is a nonincreasing predictable process (*predictable* here meaning that $D_t \in \mathcal{F}_{t-1}, t = 1, \ldots, T$). If V is positive (which holds if, e.g., $h > 0$) then it also admits a multiplicative decomposition

$$V_t = B_t A_t, \quad t = 0, 1, \ldots, T, \tag{2.4}$$

in which B is a positive martingale with $B_0 = 1$ and A is a nonincreasing predictable process.

Haugh and Kogan [8] and Rogers [15] give dual formulations of optimal stopping problems in discrete and continuous time, respectively. We record their results in our setting as follows:

Proposition 2.1 *Let $\mathcal{M}^0$ denote the set of all martingales M with respect to $\mathcal{F}$ having $M_0 = 0$. Then*

$$V_0 = \inf_{M \in \mathcal{M}^0} E\left[\max_{0 \le t \le T}(h(t, \mathbf{X}_t) - M_t)\middle|\mathcal{F}_0\right].$$

Moreover, the infimum is attained by the martingale in (2.3).

This formulation converts the problem of maximizing over stopping times into one of minimizing over martingales. Finding the optimal martingale is as difficult as finding the optimal value function, so Haugh and Kogan [8] and Rogers [15] suggest approximating V_0 using suboptimal martingales and the inequality

$$V_0 \le E\left[\max_{0 \le t \le T}(h(t, \mathbf{X}_t) - M_t)\middle|\mathcal{F}_0\right] \tag{2.5}$$

for any $M \in \mathcal{M}^0$. Andersen and Broadie [2] combine this upper bound with lower bounds on the optimal value and construct martingales from stopping rules.

Jamshidian [9] proposes a dual formulation of the optimal stopping problem based on the multiplicative decomposition (2.4). He proves the following:

Proposition 2.2 *Let $\mathcal{B}^0$ denote the set of all positive martingales B with respect to $\mathcal{F}$ having $B_0 = 1$. Then*

$$V_0 = \inf_{B \in \mathcal{B}^0} E^B\left[\max_{0 \le t \le T}\frac{h(t, \mathbf{X}_t)}{B_t}\middle|\mathcal{F}_0\right] := \inf_{B \in \mathcal{B}^0} E\left[\max_{0 \le t \le T}\frac{h(t, \mathbf{X}_t)}{B_t}B_T\middle|\mathcal{F}_0\right].$$

If V is positive (*e.g., if $h > 0$*)*, then the infimum is attained by the martingale in* (2.4).

For the rest of this paper, we assume for simplicity that h is strictly positive. The case of merely nonnegative h can be handled by adding $\epsilon > 0$ to h and then letting ϵ decrease to 0, as in [9].

As is implicit in the statement of this result, E^B denotes expectation under the change of measure defined by B_T; i.e., $E^B[Y] = E[YB_T]$ for all nonnegative $Y \in \mathcal{F}_T$. Jamshidian [9] suggests methods of constructing positive martingales to approximate the optimal martingale and thus to compute approximations to V_0. Also, any $B \in \mathcal{B}^0$ generates an upper bound

$$V_0 \leq E\left[\max_{0\leq t\leq T} \frac{h(t,\mathbf{X}_t)}{B_t} B_T \middle| \mathcal{F}_0\right]. \tag{2.6}$$

3 Iterative improvement of upper bounds

In this section, we present a method for improving the upper bounds (2.5) and (2.6) on V_0 obtained from dual formulations of the optimal stopping problem. We give an iterative construction that stops only when the optimal value is reached. A consequence of this construction is a strong equivalence result between the quality of the additive and multiplicative duals: any multiplicative upper bound can be improved by an additive upper bound and, if $h > 0$, any additive upper bound can be improved by a multiplicative upper bound. As a step in this analysis, we introduce the following definition:

Definition 3.1 *A* supersolution *is a supermartingale W that satisfies $W_t \geq h(t,\mathbf{X}_t)$ for all $0 \leq t \leq T$.*

A supersolution W thus satisfies

$$W_t \geq \max\{h(t,\mathbf{X}_t), E[W_{t+1}|\mathcal{F}_t]\}, \tag{3.1}$$

whereas the value function itself solves the dynamic programming equation

$$V_t = \max\{h(t,\mathbf{X}_t), E[V_{t+1}|\mathcal{F}_t]\}.$$

Because $V_T = h(T,\mathbf{X}_T)$, it follows that $W_j \geq V_j, j = 0, 1, \ldots, T$.

Our iterative construction is based on defining martingales from supersolutions and supersolutions from martingales. The first step is standard—it is an explicit construction of the Doob and multiplicative decompositions—but we include it for completeness:

Lemma 3.2 *If W is a supersolution, then the martingale in* (2.3) *is given by*

$$M_j^W = \sum_{t=1}^{j}(W_t - E[W_t|\mathcal{F}_{t-1}]). \tag{3.2}$$

If W is a positive supersolution (*e.g., if $h > 0$*), *then the martingale in* (2.4) *is given by*

$$B_j^W = \prod_{t=1}^{j} \frac{W_t}{E[W_t|\mathcal{F}_{t-1}]}. \tag{3.3}$$

Our next result shows how to go in the opposite direction—from martingales to supersolutions.

Lemma 3.3 *If $M \in \mathcal{M}^0$ and*

$$W_j^M = E\left[\max_{j \le t \le T}(h(t, \mathbf{X}_t) - M_t)\middle|\mathcal{F}_j\right] + M_j, \quad j = 0, 1, \ldots, T, \tag{3.4}$$

then W^M is a supersolution. If $B \in \mathcal{B}^0$ and

$$W_j^B = E^B\left[\max_{j \le t \le T}\frac{h(t, \mathbf{X}_t)}{B_t}\middle|\mathcal{F}_j\right]B_j, \quad j = 0, 1, \ldots, T, \tag{3.5}$$

then W^B is a positive supersolution.

Proof In the case of Eq. (3.4), we have, for all $j = 0, 1, \ldots, T$,

$$W_j^M = E\left[\max_{j \le t \le T}(h(t, \mathbf{X}_t) - M_t)\middle|\mathcal{F}_j\right] + M_j \ge h(j, \mathbf{X}_j) - M_j + M_j = h(j, \mathbf{X}_j).$$

And W_j^M is a supermartingale because

$$\begin{aligned} E\left[W_j^M\middle|\mathcal{F}_{j-1}\right] &= E\left[\max_{j \le t \le T}(h(t, \mathbf{X}_t) - M_t)\middle|\mathcal{F}_{j-1}\right] + M_{j-1} \\ &\le E\left[\max_{j-1 \le t \le T}(h(t, \mathbf{X}_t) - M_t)\middle|\mathcal{F}_{j-1}\right] + M_{j-1} = W_{j-1}^M. \end{aligned}$$

For Eq. (3.5), we have

$$W_j^B = E^B\left[\max_{j \le t \le T}\frac{h(t, \mathbf{X}_t)}{B_t}\middle|\mathcal{F}_j\right]B_j \ge (h(j, \mathbf{X}_j)/B_j)B_j = h(j, \mathbf{X}_j)$$

and

$$E\left[W_j^B\middle|\mathcal{F}_{j-1}\right] = E\left[E^B\left[\max_{j \le t \le T}\frac{h(t, \mathbf{X}_t)}{B_t}\middle|\mathcal{F}_j\right]B_j\middle|\mathcal{F}_{j-1}\right].$$

Using the Bayes rule for conditional expectations, $E^B[Y|\mathcal{F}_k] = E[YB_T|\mathcal{F}_k]/B_k$, for all $Y \ge 0$ (as in [10], Lemma 5.3),

$$\begin{aligned} E\left[W_j^B\middle|\mathcal{F}_{j-1}\right] &= E\left[E\left[\max_{j \le t \le T}\frac{h(t, \mathbf{X}_t)}{B_t}\frac{B_T}{B_j}\middle|\mathcal{F}_j\right]B_j\middle|\mathcal{F}_{j-1}\right] \\ &= E\left[\max_{j \le t \le T}\frac{h(t, \mathbf{X}_t)}{B_t}B_T\middle|\mathcal{F}_{j-1}\right] \\ &\le E\left[\max_{j-1 \le t \le T}\frac{h(t, \mathbf{X}_t)}{B_t}B_T\middle|\mathcal{F}_{j-1}\right]. \end{aligned}$$

Another application of Bayes rule gives

$$E\left[W_j^B\middle|\mathcal{F}_{j-1}\right] \le E^B\left[\max_{j-1 \le t \le T}\frac{h(t, \mathbf{X}_t)}{B_t}B_{j-1}\middle|\mathcal{F}_{j-1}\right] = W_{j-1}^B.$$

□

The main result of this section is that combining these two lemmas provides an iterative improvement of upper bounds: If we start with a supersolution, extract a martingale (as in Lemma 3.2), and then construct a new supersolution from the martingale (as in Lemma 3.3), we get an improvement over the original supersolution.

Theorem 3.4 *Suppose that a supersolution W has a decomposition*

$$W = M + D \quad \text{or} \quad W = BA.$$

Let W^M and W^B be as in (3.4) *and* (3.5)*, respectively. Then, for $j = 0, 1, \ldots, T$,*

$$V_j \leq W_j^M \leq W_j \tag{3.6}$$

and

$$V_j \leq W_j^B \leq W_j. \tag{3.7}$$

Proof The first inequalities in both Eqs. (3.6) and (3.7) follow from the supersolution property; see the comments following Eq. (3.1). For the second inequalities,

$$\begin{aligned} W_j^M &= E\left[\max_{j \leq t \leq T} (h(t, \mathbf{X}_t) - M_t) \middle| \mathcal{F}_j\right] + M_j \\ &\leq E\left[\max_{j \leq t \leq T} (W_t - M_t) \middle| \mathcal{F}_j\right] + M_j \\ &= E\left[\max_{j \leq t \leq T} D_t \middle| \mathcal{F}_j\right] = D_j + M_j = W_j, \end{aligned}$$

because D is nonincreasing; and

$$\begin{aligned} W_j^B &= E^B\left[\max_{j \leq t \leq T} \frac{h(t, \mathbf{X}_t)}{B_t} \middle| \mathcal{F}_j\right] B_j \\ &\leq E^B\left[\max_{j \leq t \leq T} \frac{W_t}{B_t} \middle| \mathcal{F}_j\right] B_j \\ &= E^B\left[\max_{j \leq t \leq T} A_t \middle| \mathcal{F}_j\right] B_j = A_j B_j = W_j \end{aligned}$$

because A is nonincreasing. □

As a consequence of Theorem 3.4, we establish an equivalence between the quality of the bounds achievable with the additive and multiplicative duals, using essentially the same information. Given a martingale and a dual value under one method, we can explicitly construct a new martingale that gives a dual value using the opposite method that is better (or at least no worse).

Corollary 3.5 *Let M be a martingale, $M_0 = 0$, with $W \equiv W^M$ the corresponding supersolution in* (3.4)*. If $W > 0$ (e.g., if $h > 0$), let B^W be the martingale constructed from W as in* (3.3)*. Then*

$$E^{B^W}\left[\max_{0\le t\le T}\frac{h(t,\mathbf{X}_t)}{B_t^W}\bigg|\mathcal{F}_0\right]\le W_0.$$

Let B be a positive martingale, $B_0 = 1$, with $W \equiv W^B$ the corresponding supersolution in (3.5). Let M^W be the martingale constructed from M as in (3.2). Then

$$E\left[\max_{0\le t\le T}\left(h(t,\mathbf{X}_t)-M_t^W\right)\bigg|\mathcal{F}_0\right]\le W_0.$$

Theorem 3.4 and Corollary 3.5 provide the basis for an iterative method of improving upper bounds. However, because these results use weak inequalities, they do not preclude the possibility that an iterative scheme could get stuck at a suboptimal fixed point. The next result rules out this possibility: equality holds only if the supersolution is the optimal value function.

Theorem 3.6 *Let W be a supersolution and M^W and B^W the martingales in (3.2) and (3.3). If*

$$E\left[\max_{j\le t\le T}\left(h(t,\mathbf{X}_t)-M_t^W\right)\bigg|\mathcal{F}_j\right]+M_j^W=W_j\quad\text{for all } 0\le j\le T,\tag{3.8}$$

or

$$E^{B^W}\left[\max_{j\le t\le T}\frac{h(t,\mathbf{X}_t)}{B_t^W}\bigg|\mathcal{F}_0\right]B_j^W=W_j\quad\text{for all } 0\le j\le T,\tag{3.9}$$

then $W_0 = V_0$, the value of the optimal stopping problem.

Proof Consider the additive method first. We claim that if Eq. (3.8) holds, $\max_{j\le t\le T}(h(t,\mathbf{X}_t)-M_t^W)$ is $\mathcal{F}_j$-measurable. That is,

$$\max_{j\le t\le T}\left(h(t,\mathbf{X}_t)-M_t^W\right)+M_j^W=W_j.\tag{3.10}$$

Indeed,

$$E\left[\max_{j\le t\le T}(h(t,\mathbf{X}_t)-M_t^W)\bigg|\mathcal{F}_j\right]=W_j-M_j^W=D_j^W,$$

where D^W is the nonincreasing part of the Doob decomposition of W. On the other hand,

$$\max_{j\le t\le T}\left(h(t,\mathbf{X}_t)-M_t^W\right)\le\max_{j\le t\le T}\left(W_t-M_t^W\right)=\max_{j\le t\le T}D_t^W=D_j^W.$$

Thus,

$$\max_{j\le t\le T}\left(h(t,\mathbf{X}_t)-M_t^W\right)=D_j^W=W_j-M_j^W.$$

For $j=0$,

$$\begin{aligned}W_0&=\max_{0\le t\le T}\left(h(t,\mathbf{X}_t)-M_t^W\right)=\max_{0\le t\le T}\left(h(t,\mathbf{X}_t)-W_t+D_t^W\right)\\&\le\max_{0\le t\le T}(h(t,\mathbf{X}_t)-W_t)+\max_{0\le t\le T}D_t^W\le\max_{0\le t\le T}D_t^W=D_0^W=W_0.\end{aligned}$$

Thus,

$$\max_{0 \le t \le T} (h(t, \mathbf{X}_t) - W_t) = 0.$$

From Eq. (3.10), we know that

$$\begin{aligned}
W_j &= \max_{j \le t \le T} \left(h(t, \mathbf{X}_t) - M_t^W \right) + M_j^W \\
&= \max \left(h(j, \mathbf{X}_j), \max_{j+1 \le t \le T} h(t, \mathbf{X}_t) - \left(M_t^W - M_j^W \right) \right) \\
&= \max \left(h(j, \mathbf{X}_j), \max_{j+1 \le t \le T} h\left(t, \mathbf{X}_t - M_t^W \right) + M_{j+1}^W - \left(M_{j+1}^W - M_j^W \right) \right) \\
&= \max \left(h(j, \mathbf{X}_j), W_{j+1} - \left(M_{j+1}^W - M_j^W \right) \right). \qquad (3.11)
\end{aligned}$$

Define a stopping time $\tau = \inf\{t \ge 0 : h(t, \mathbf{X}_t) = W_t\}$. According to the definition of τ and (3.11), for any j,

$$\begin{aligned}
\{\tau = j\} &= \{W_0 > h(0, \mathbf{X}_0)\} \cap \{W_1 > h(1, \mathbf{X}_1)\} \\
&\quad \cap \cdots \cap \{W_{j-1} > h(j-1, \mathbf{X}_{j-1})\} \cap \{W_j = h(j, \mathbf{X}_j)\} \\
&= \left\{ W_0 = W_1 - \left(M_1^W - M_0^W \right) \right\} \\
&\quad \cap \cdots \cap \left\{ W_{j-1} = W_j - \left(M_j^W - M_{j-1}^W \right) \right\} \cap \{W_j = h(j, \mathbf{X}_j)\}.
\end{aligned}$$

In words, W would be a martingale before the stopping time τ. Indeed,

$$W_{t \wedge \tau} = W_0 + \sum_{j=1}^{t} \left(W_{j \wedge \tau} - W_{(j-1) \wedge \tau} \right) = W_0 + \sum_{j=1}^{t} \mathbf{1}_{\{j \le \tau\}} \left(M_{j \wedge \tau}^W - M_{(j-1) \wedge \tau}^W \right).$$

Thus, $\{W_{t \wedge \tau} : 0 \le t \le T\}$ is a martingale because of Durrett [6], Theorem 2.7 and the fact that $\mathbf{1}_{\{j \le \tau\}}$ is $\mathcal{F}_{j-1}$-measurable and $\left(M_{j \wedge \tau}^W \right)$ is a martingale. By the optional sampling theorem, for any stopping time γ, noting that $W_{\cdot}$ is a supermartingale and $W_{\cdot \wedge \tau}$ is a martingale,

$$W_0 = E[W_{0 \wedge \tau} | \mathcal{F}_0] = E[W_{\gamma \wedge \tau} | \mathcal{F}_0] \ge E[W_\gamma | \mathcal{F}_0] \ge E[h(\gamma, \mathbf{X}_\gamma) | \mathcal{F}_0].$$

On the other hand,

$$W_0 = E[W_{0 \wedge \tau} | \mathcal{F}_0] = E[W_{\gamma \wedge \tau} | \mathcal{F}_0] = E[W_{T \wedge \tau} | \mathcal{F}_0] = E[W_\tau | \mathcal{F}_0].$$

Thus, W_0 is the optimal value. The argument for the multiplicative case is very similar. □

Theorems 3.4 and 3.6 may be contrasted with an iterative method of Kolodko and Schoenmakers [12]. Their method iteratively constructs stopping times. The value associated with a suboptimal stopping rule provides a lower bound on the optimal value function, and Kolodko and Schoenmakers [12] show that the lower bounds they construct increase monotonically. They also construct dual values for stopping rules, thus pairing each lower bound with an upper bound. However, it is not in general the case that their upper bounds improve monotonically together with their lower bounds.

4 Dynamic programming perspective

In this section, we examine the iterative construction of the previous section from a dynamic programming perspective, starting with the following lemma:

Lemma 4.1 *Let $\tilde{W}$ be either of the supersolutions W^M and W^B in* (3.2) *and* (3.3). *Then*

$$\tilde{W}_j \le \max\{h(j, \mathbf{X}_j), E[W_{j+1}|\mathcal{F}_j]\}.$$

Proof We prove the case $\tilde{W} = W^M$, the case of W^B being very similar. Indeed,

$$W_t^M = E\left[\max_{t\le i\le T}(h(i,\mathbf{X}_i) - M_i)\middle|\mathcal{F}_t\right] + M_t = E\left[\max_{t\le i\le T}(h(i,\mathbf{X}_i) - (M_i - M_t)\middle|\mathcal{F}_t\right]$$

and

$$M_i - M_t = \sum_{\ell=t+1}^{i}\{W_\ell - E[W_\ell|\mathcal{F}_{\ell-1}]\}.$$

Thus,

$$\begin{aligned}
W_t^M &= E\left[\max_{t\le i\le T}\left(h(i,\mathbf{X}_i) - \sum_{\ell=t+1}^{i}\{W_\ell - E[W_\ell|\mathcal{F}_{\ell-1}]\}\right)\middle|\mathcal{F}_t\right] \\
&= E\left[\max\left(h(t,\mathbf{X}_t), \max_{t+1\le i\le T}\left(h(i,\mathbf{X}_i) - \sum_{\ell=t+1}^{i}\{W_\ell - E[W_\ell|\mathcal{F}_{\ell-1}]\}\right)\right)\middle|\mathcal{F}_t\right] \\
&= E\Bigg[\max\Bigg(h(t,\mathbf{X}_t), \max_{t+1\le i\le T}\Bigg((h(i,\mathbf{X}_i) - W_i) \\
&\qquad + \sum_{\ell=t+2}^{i}\{E[W_\ell|\mathcal{F}_{\ell-1}] - W_{\ell-1}\} + E[W_{t+1}|\mathcal{F}_t]\Bigg)\Bigg)\Bigg|\mathcal{F}_t\Bigg].
\end{aligned}$$

Note that $h(i,\mathbf{X}_i) \le W_i$ and $E[W_\ell|\mathcal{F}_{\ell-1}] \le W_{\ell-1}$. Then,

$$W_t^M \le E\Big[\max\big(h(t,\mathbf{X}_t), E[W_{t+1}|\mathcal{F}_t]\big)\Big|\mathcal{F}_t\Big] = \max\big(h(t,\mathbf{X}_t), E[W_{t+1}|\mathcal{F}_t]\big).$$

□

Now consider a sequence of supersolutions $W^0, W^1, \ldots$, each obtained from the previous one using either Eqs. (3.4) or (3.5). Using Lemma 4.1, we can show that if $W_T^0 = h_T(T, \mathbf{X}_T)$, then the sequence reaches the optimal value function V in T steps. Moreover, the agreement between the supersolution and V moves backward from time T to time 0.

Theorem 4.2 *Consider a sequence of supersolutions* $W^0, W^1, \ldots, W^T$ *with decompositions* $W^k = M^k + D^k$ *and* $W^k = B^k A^k$ *and*

$$W_j^k = E\left[\max_{j \le t \le T}(h(t, \mathbf{X}_t) - M_t^{k-1})\middle|\mathcal{F}_j\right] + M_j^{k-1}$$

or

$$W_j^k = E^{B^{k-1}}\left[\max_{j \le t \le T}\frac{h(t, \mathbf{X}_t)}{B_t^{k-1}}\middle|\mathcal{F}_j\right] B_j^{k-1}.$$

Suppose $W_T^0 = h_T(T, \mathbf{X}_T)$*. Then, for all* $0 \le j \le T$*,*

$$W_t^{T-j} = V_t \quad \text{for all } t \ge j,$$

where V_t *is the optimal value in* (2.2).

Proof The proof is by induction. We have $W_T^0 = V_T$. Suppose therefore that $W_t^{T-j-1} = V_t$, for all $t = j+1, \ldots, T$. Then, for $t = j, j+1, \ldots, T$, Lemma 4.1 implies

$$W_t^{T-j} \le \max\{h(t, \mathbf{X}_t), E[V_{t+1}|\mathcal{F}_t]\} = V_t.$$

But we also have $W^{T-j} \ge V$ because it is a supersolution, so we must have $W_t^{T-j} = V_t, t = j, \ldots, T$. □

5 Bias comparisons

In Sect. 3, we argued that the additive and multiplicative duals produce upper bounds of equal quality, in the sense that either could be used to improve the other in a systematic way. In this section, we turn to other ways of comparing these bounds—i.e., of comparing the bias in the dual estimates compared with the optimal value. In the next section, we compare variances.

We know from Sect. 2 that both the additive and multiplicative duals can achieve the optimal value once we find the corresponding optimal martingale. But in practice, finding the optimal martingale is as difficult as solving the original problem (2.1). Thus, a general strategy is to approximate the optimal martingale and use this approximation in a Monte Carlo simulation to estimate upper bounds for the optimal value. Such a strategy introduces two kinds of errors: bias, from using an approximating martingale, and sampling variability, from the Monte Carlo simulation. Both are captured by an estimator's mean square error, which is the sum of its variance and the square of its bias. We investigate the growth in bias and variance as the time horizon in the optimal stopping problem grows.

5.1 Bias of the multiplicative method

First we point out that the worst case of bias of the multiplicative dual grows as a linear function of T. We use $\tilde{B}$ to denote an arbitrary positive martingale with $\tilde{B}_0 = 1$. Think of this as an approximation to the optimal B.

Proposition 5.1 *The bias of the multiplicative method satisfies*

$$0 \leq E^{\widetilde{B}}\left[\max_{0\leq t\leq T} \frac{h(t,\mathbf{X}_t)}{\tilde{B}_t}\middle|\mathcal{F}_0\right] - V_0 \leq V_0 \cdot (T-1),$$

where V_0 is the value of the optimal stopping problem.

Proof The first inequality is easy because $\widetilde{B}$ is a martingale and $E^{\widetilde{B}}\left[\max_{0\leq t\leq T} \frac{h(t,\mathbf{X}_t)}{\tilde{B}_t}\middle|\mathcal{F}_0\right]$ gives us an upper bound on V_0. For the second inequality,

$$\begin{aligned} E^{\widetilde{B}}\left[\max_{0\leq t\leq T} \frac{h(t,\mathbf{X}_t)}{\tilde{B}_t}\middle|\mathcal{F}_0\right] - V_0 &\leq E^{\widetilde{B}}\left[\max_{0\leq t\leq T} \frac{V_t}{\tilde{B}_t}\middle|\mathcal{F}_0\right] - V_0 \\ &= E^{\widetilde{B}}\left[\max_{0\leq t\leq T} \frac{B_t A_t}{\tilde{B}_t}\middle|\mathcal{F}_0\right] - V_0 \\ &\leq A_0 E^{\widetilde{B}}\left[\max_{0\leq t\leq T} \frac{B_t}{\tilde{B}_t}\middle|\mathcal{F}_0\right] - V_0, \end{aligned} \tag{5.1}$$

where the first line uses $h(t,\mathbf{X}_t) \leq V_t$, the second applies the multiplicative decomposition of the supermartingale V, and the third uses the fact that A is a nonincreasing process. Note that $V_0 = A_0 B_0$ and $B_0 = 1$. By Eq. (5.1),

$$\begin{aligned} E^{\widetilde{B}}\left[\max_{0\leq t\leq T} \frac{h(t,\mathbf{X}_t)}{\tilde{B}_t}\middle|\mathcal{F}_0\right] - V_0 &\leq V_0 \cdot \left(E^{\widetilde{B}}\left[\max_{0\leq t\leq T} \frac{B_t}{\tilde{B}_t}\middle|\mathcal{F}_0\right] - 1\right) \\ &\leq V_0 \cdot \left(E^{\widetilde{B}}\left[\sum_{t=0}^{T} \frac{B_t}{\tilde{B}_t}\middle|\mathcal{F}_0\right] - 1\right) \\ &= V_0 \cdot (T-1). \end{aligned}$$

□

This upper bound can be improved dramatically with a more accurate approximation of the optimal martingale. As discussed in Glasserman [7], Sect. 8.7, there are two broad approaches to approximating the optimal martingale—by approximating the optimal value function or by approximating the optimal stopping rule. Given a (positive) approximation $\tilde{V}$ to the (positive) value function V, one may construct a positive martingale

$$\tilde{B}_t = \prod_{i=1}^{t} \frac{\tilde{V}_i}{E[\tilde{V}_i|\mathcal{F}_{i-1}]}.$$

Given a sequence of stopping times $\tilde{\tau}^j$, $j = 0, 1, \ldots, T$, taking values in $\{j, j+1, \ldots, T\}$ and measurable with respect to $\mathcal{F}^j$, one may construct a martingale

$$\tilde{B}_t = \prod_{i=1}^{t} \frac{E[h(\tilde{\tau}^i, \mathbf{X}_{\tilde{\tau}^i})|\mathcal{F}_i]}{E[h(\tilde{\tau}^i, \mathbf{X}_{\tilde{\tau}^i})|\mathcal{F}_{i-1}]}.$$

The next result shows the improvement achieved if either of these approximations has uniformly small relative error.

Proposition 5.2 *Suppose that for all* $i = 0, 1, \ldots, T$,

$$1 - \epsilon \le \frac{\tilde{V}_i}{V_i} \le 1 + \epsilon \quad \text{or} \quad 1 - \epsilon \le \frac{E[h(\tilde{\tau}^i, \mathbf{X}_{\tilde{\tau}^i})|\mathcal{F}_i]}{V_i} \le 1 + \epsilon.$$

Then,

$$0 \le E^{\tilde{B}}\left[\max_{0 \le t \le T} \frac{h(t, \mathbf{X}_t)}{\tilde{B}_t}\middle|\mathcal{F}_0\right] - V_0 \le V_0\sqrt{T}O(\epsilon).$$

Proof The first inequality is immediate, so we prove only the second one. First,

$$\begin{aligned}
E^{\tilde{B}}\left[\max_{0 \le t \le T} \frac{h(t, \mathbf{X}_t)}{\tilde{B}_t}\middle|\mathcal{F}_0\right] - V_0 &\le E^{\tilde{B}}\left[\max_{0 \le t \le T} \frac{V_t}{\tilde{B}_t}\middle|\mathcal{F}_0\right] - V_0 \\
&= E^{\tilde{B}}\left[\max_{0 \le t \le T} \frac{B_t D_t}{\tilde{B}_t}\middle|\mathcal{F}_0\right] - V_0 \\
&\le D_0 E^{\tilde{B}}\left[\max_{0 \le t \le T} \frac{B_t}{\tilde{B}_t}\middle|\mathcal{F}_0\right] - V_0 \\
&= V_0\left(E^{\tilde{B}}\left[\max_{0 \le t \le T} \frac{B_t}{\tilde{B}_t}\middle|\mathcal{F}_0\right] - 1\right).
\end{aligned}$$

By Doob's inequality and the fact that $B_t/\tilde{B}_t - 1$ is a martingale under the probability measure $P^{\tilde{B}}$,

$$\begin{aligned}
E^{\tilde{B}}\left[\max_{0 \le t \le T} \frac{h(t, \mathbf{X}_t)}{\tilde{B}_t}\middle|\mathcal{F}_0\right] - V_0 &\le V_0 E^{\tilde{B}}\left[\max_{0 \le t \le T}\left(\frac{B_t}{\tilde{B}_t} - 1\right)\middle|\mathcal{F}_0\right] \\
&\le V_0 \cdot 2\sqrt{E^{\tilde{B}}\left[\left(\frac{B_T}{\tilde{B}_T} - 1\right)^2\middle|\mathcal{F}_0\right]}. \qquad (5.2)
\end{aligned}$$

We know that, by the orthogonality of martingale differences (see, e.g., [11], p. 331),

$$E^{\tilde{B}}\left[\left(\frac{B_T}{\tilde{B}_T} - 1\right)^2\middle|\mathcal{F}_0\right] = \sum_{t=0}^{T} E^{\tilde{B}}\left[\left(\frac{B_t}{\tilde{B}_t} - \frac{B_{t-1}}{\tilde{B}_{t-1}}\right)^2\middle|\mathcal{F}_0\right]. \qquad (5.3)$$

From Eqs. (5.2) and (5.3),

$$E^{\widetilde{B}}\left[\max_{0\le t\le T}\frac{h(t,\mathbf{X}_t)}{\tilde{B}_t}\middle|\mathcal{F}_0\right]-V_0\le 2V_0\sqrt{\sum_{t=0}^{T}E^{\widetilde{B}}\left[\left(\frac{B_t}{\tilde{B}_t}-\frac{B_{t-1}}{\tilde{B}_{t-1}}\right)^2\middle|\mathcal{F}_0\right]}$$
$$=2V_0\sqrt{\sum_{t=0}^{T}E^{\widetilde{B}}\left[\left(\frac{B_{t-1}}{\tilde{B}_{t-1}}\right)^2\left(\frac{B_t/B_{t-1}}{\tilde{B}_t/\tilde{B}_{t-1}}-1\right)^2\middle|\mathcal{F}_0\right]}.$$

For the ratio $\frac{B_t/B_{t-1}}{\tilde{B}_t/\tilde{B}_{t-1}}$, we know that

$$\frac{B_t/B_{t-1}}{\tilde{B}_t/\tilde{B}_{t-1}}=\frac{\Delta_t/\tilde{\Delta}_t}{E[\Delta_t|\mathcal{F}_{t-1}]/E[\tilde{\Delta}_t|\mathcal{F}_{t-1}]},$$

where

$$\frac{1}{1+\epsilon}\le\frac{\Delta_t}{\tilde{\Delta}_t}=\frac{\tilde{V}_t}{V_t}\quad\left(\text{or }\frac{E[h(\tilde{\tau}^t,\mathbf{X}_{\tilde{\tau}^t})|\mathcal{F}_t]}{V_t}\le\frac{1}{1-\epsilon}\right)$$

which implies

$$\left(\frac{B_t/B_{t-1}}{\tilde{B}_t/\tilde{B}_{t-1}}-1\right)^2\le\left(\frac{2\epsilon}{1-\epsilon}\right)^2.$$

Thus,

$$E^{\widetilde{B}}\left[\max_{0\le t\le T}\frac{h(t,\mathbf{X}_t)}{\tilde{B}_t}\middle|\mathcal{F}_0\right]-V_0\le 2V_0\sqrt{\sum_{t=0}^{T}E^{\widetilde{B}}\left[\left(\frac{B_{t-1}}{\tilde{B}_{t-1}}\right)^2\left(\frac{B_t/B_{t-1}}{\tilde{B}_t/\tilde{B}_{t-1}}-1\right)^2\middle|\mathcal{F}_0\right]}$$
$$\le V_0\frac{4\epsilon}{1-\epsilon}\sqrt{\sum_{t=0}^{T}E^{\widetilde{B}}\left[E^{\widetilde{B}}\left[\left(\frac{B_{t-1}}{\tilde{B}_{t-1}}\right)^2\middle|\mathcal{F}_{t-2}\right]\middle|\mathcal{F}_0\right]}$$
$$=V_0\frac{4\epsilon}{1-\epsilon}\sqrt{\sum_{t=0}^{T}E^{\widetilde{B}}\left[E^{B}\left[\frac{B_{t-1}}{\tilde{B}_{t-1}}\middle|\mathcal{F}_{t-2}\right]\middle|\mathcal{F}_0\right]}$$

where we use the Bayes formula for conditional expectations to get the last equality. Note that $B_{t-1}/\tilde{B}_{t-1}\le(1+\epsilon)^{t-1}/(1-\epsilon)^{t-1}$. Thus,

$$E^{\widetilde{B}}\left[\max_{0\le t\le T}\frac{h(t,\mathbf{X}_t)}{\tilde{B}_t}\middle|\mathcal{F}_0\right]-V_0\le V_0\frac{4\epsilon}{1-\epsilon}\sqrt{\sum_{t=1}^{T}\left(\frac{1+\epsilon}{1-\epsilon}\right)^{t-1}}=V_0\sqrt{T}O(\epsilon).$$

□

5.2 Bias of the additive method

A similar bound applies in the additive case when the absolute (rather than relative) error in the approximating martingale is uniformly small. For the following, define a martingale $\tilde{M}$ by setting $\tilde{M}_0 = 0$ and

$$\tilde{M}_i - \tilde{M}_{i-1} = \tilde{V}_i - E[\tilde{V}_i|\mathcal{F}_{i-1}]$$

or

$$\tilde{M}_i - \tilde{M}_{i-1} = E\left[h\left(\tilde{\tau}^i, \mathbf{X}_{\tilde{\tau}^i}\right)|\mathcal{F}_i\right] - E\left[h\left(\tilde{\tau}^i, \mathbf{X}_{\tilde{\tau}^i}\right)|\mathcal{F}_{i-1}\right].$$

Proposition 5.3 *Suppose that for all* $i = 0, 1, \ldots, T$,

$$-\epsilon \le \tilde{V}_i - V_i \le \epsilon \quad \text{or} \quad -\epsilon \le E[h(\tilde{\tau}^i, \mathbf{X}_{\tilde{\tau}^i})|\mathcal{F}_i] - V_i \le \epsilon. \tag{5.4}$$

Then,

$$0 \le E\left[\max_{0\le t\le T}(h(t, \mathbf{X}_t) - \tilde{M}_t)\middle|\mathcal{F}_0\right] - V_0 \le 4\sqrt{T}\epsilon.$$

Proof We only need to prove the second inequality. Note that

$$\begin{aligned} E\left[\max_{0\le t\le T}(h(t, \mathbf{X}_t) - \tilde{M}_t)\middle|\mathcal{F}_0\right] - V_0 &\le E\left[\max_{0\le t\le T}(V_t - \tilde{M}_t)\middle|\mathcal{F}_0\right] - V_0 \\ &= E\left[\max_{0\le t\le T}(D_t + M_t - \tilde{M}_t)\middle|\mathcal{F}_0\right] - V_0 \\ &\le D_0 + E\left[\max_{0\le t\le T}(M_t - \tilde{M}_t)\middle|\mathcal{F}_0\right] - V_0 \\ &= E\left[\max_{0\le t\le T}(M_t - \tilde{M}_t)\middle|\mathcal{F}_0\right], \end{aligned}$$

where $D_0 = V_0$ in Doob's decomposition. Again, we use Doob's martingale inequality and the orthogonality property of martingale differences to get that

$$\begin{aligned} &E\left[\max_{0\le t\le T}(h(t, \mathbf{X}_t) - \tilde{M}_t)\middle|\mathcal{F}_0\right] - V_0 \\ &\le E\left[\max_{0\le t\le T}(M_t - \tilde{M}_t)\middle|\mathcal{F}_0\right] \le 2\sqrt{E[(M_T - \tilde{M}_T)^2|\mathcal{F}_0]} \\ &= 2\sqrt{\sum_{t=1}^{T} E[((M_t - \tilde{M}_t) - (M_{t-1} - \tilde{M}_{t-1}))^2|\mathcal{F}_0]} \\ &= 2\sqrt{\sum_{t=1}^{T} E[((\Delta_t - \tilde{\Delta}_t) - E[(\Delta_t - \tilde{\Delta}_t)|\mathcal{F}_{t-1}])^2|\mathcal{F}_0]} \\ &\le 2\sqrt{\sum_{t=1}^{T} E[(\Delta_t - \tilde{\Delta}_t)^2|\mathcal{F}_0]}. \end{aligned}$$

Under the assumption (5.4), $(\Delta_t - \tilde{\Delta}_t)^2 \leq 4\epsilon^2$. Thus,

$$E\left[\max_{0\leq t\leq T}(h(t,\mathbf{X}_t) - \tilde{M}_t)\middle|\mathcal{F}_0\right] - V_0 \leq 4\sqrt{T}\epsilon.$$

□

A comparison of Propositions 5.2 and 5.3 indicates that in both cases the bias is $O(\sqrt{T})$ when the corresponding optimal martingale can be approximated accurately.

5.3 Bounds from approximate value functions

We have thus far investigated upper bounds on the optimal value function constructed from martingales that approximate the optimal martingales in dual formulations of the optimal stopping problem. We now turn to an alternative approach in which one starts from approximations to the optimal value function and constructs processes from those approximations that would be optimal martingales if one started from the optimal value function.

To simplify the discussion, in this section we take X to be Markov and we assume that the payoff function h is dependent only on time and the current state of process X. The optimal value function V_j satisfies the dynamic programming recursion

$$\begin{aligned} V_T(X_T) &= h(T,X_T);\\ V_j(X_j) &= \max\left(h(j,X_j), E[V_{j+1}(X_{j+1})|X_j]\right). \end{aligned}$$

Let $\tilde{C}_j(\cdot)$, $j = 0,1,\ldots,T-1$, denote a sequence of approximations to the conditional expectations $E[V_{j+1}(X_{j+1})|X_j = \cdot]$. From these, define approximate value functions

$$\begin{aligned} \tilde{V}_T(X_T) &= h(T,X_T);\\ \tilde{V}_j(X_j) &= \max\left(h(j,X_j), \tilde{C}_j(X_j)\right). \end{aligned}$$

In addition, define two residual processes by

$$\tilde{\epsilon}_i = \tilde{V}_i(X_i) - \tilde{C}_{i-1}(X_{i-1}) \quad \text{or} \quad \tilde{\delta}_i = \frac{\tilde{V}_i(X_i)}{\tilde{C}_{i-1}(X_{i-1})}, \tag{5.5}$$

in the second case assuming $\tilde{C}_{i-1} > 0$. From these residuals we define two further processes

$$\tilde{M}_i = \sum_{j=1}^{i}\tilde{\epsilon}_j, \quad \hat{M}_0 = 0 \qquad \text{and} \qquad \tilde{B}_i = \prod_{j=1}^{i}\tilde{\delta}_j, \quad \tilde{B}_0 = 1. \tag{5.6}$$

The following lemma elaborates on an observation in [7, pp. 477–478] on the connection between upper bounds computed through duality and dynamic programming.

Lemma 5.4

$$\tilde{V}_0(X_0) = \max_{0\le t\le T}(h(t,X_t) - \tilde{M}_t) = \max_{0\le t\le T}\frac{h(t,X_t)}{\tilde{B}_t}, \quad \text{a.s.}$$

Proof We prove the additive case only; the multiplicative case is similar. So

$$\begin{aligned}
&\max\left(h(T-1,X_{T-1}) - \tilde{M}_{T-1}, h(T,X_T) - \tilde{M}_T\right)\\
&\quad = \max\left(h(T-1,X_{T-1}), h(T,X_T) - \tilde{\epsilon}_T\right) - \tilde{M}_{T-1}\\
&\quad = \max\left(h(T-1,X_{T-1}), h(T,X_T) - \tilde{V}_T(X_T) + \tilde{C}_{T-1}(X_{T-1})\right) - \tilde{M}_{T-1}\\
&\quad = \max\left(h(T-1,X_{T-1}), \tilde{C}_{T-1}(X_{T-1})\right) - \tilde{M}_{T-1} = \tilde{V}_{T-1}(X_{T-1}) - \tilde{M}_{T-1}.
\end{aligned}$$

Suppose that for time $i+1$, we have

$$\max_{i+1\le t\le T}(h(t,X_t) - \tilde{M}_t) = \tilde{V}_{i+1}(X_{i+1}) - \tilde{M}_{i+1}.$$

Then we get for time i

$$\begin{aligned}
\max_{i\le t\le T}(h(t,X_t) - \tilde{M}_t) &= \max\left(h(i,X_i) - \tilde{M}_i, \max_{i+1\le t\le T}(h(t,X_t) - \tilde{M}_t)\right)\\
&= \max\left(h(i,X_i) - \tilde{M}_i, \tilde{V}_{i+1}(X_{i+1}) - \tilde{M}_{i+1}\right)\\
&= \max\left(h(i,X_i), \tilde{V}_{i+1}(X_{i+1}) - \tilde{\epsilon}_{i+1}\right) - \tilde{M}_i\\
&= \max\left(h(i,X_i), \tilde{C}_{i+1}(X_{i+1})\right) - \tilde{M}_i = \tilde{V}_i(X_i) - \tilde{M}_i.
\end{aligned}$$

By induction, we know that

$$\max_{i\le t\le T}(h(t,X_t) - \tilde{M}_t) = \tilde{V}_0(X_0) - \tilde{M}_0 = \tilde{V}_0(X_0).$$

$\square$

If $\tilde{V}$ and $\tilde{C}$ were the optimal value and continuation functions, $\tilde{M}$ and $\tilde{B}$ would be the optimal martingales. Because $\tilde{C}_j$ need not be the conditional expectation of $\tilde{V}_{j+1}$, $\tilde{M}$ and $\tilde{B}$ need not even be martingales. Observe that $\tilde{M}$ is a supermartingale if and only if $\tilde{B}$ is a supermartingale and $\tilde{M}$ is a submartingale if and only if $\tilde{B}$ is a submartingale. The following result gives a sense in which neither the additive nor the multiplicative bound constructed in this way universally dominates the other.

Proposition 5.5 *If $\tilde{M}$ is a supermartingale, then*

$$E\left[\max_{0\le t\le T}\frac{h(t,X_t)}{\tilde{B}_t}\tilde{B}_T\middle|\mathcal{F}_0\right] \le E\left[\max_{0\le t\le T}(h(t,X_t) - \tilde{M}_t)\middle|\mathcal{F}_0\right]. \tag{5.7}$$

If $\tilde{B}$ is a submartingale, then

$$E\left[\max_{0\le t\le T}\frac{h(t,X_t)}{\tilde{B}_t}\tilde{B}_T\middle|\mathcal{F}_0\right] \ge E\left[\max_{0\le t\le T}(h(t,X_t) - \tilde{M}_t)\middle|\mathcal{F}_0\right]. \tag{5.8}$$

Remark 5.6 When $\tilde{M}$ is a supermartingale, $E[\max_{0\le t\le T}(h(t,X_t)-\tilde{M}_t)|\mathcal{F}_0]$ gives an upper bound of the optimal value. But Eq. (5.7) does not imply that the multiplicative method is better than the additive method because $E\left[\max_{0\le t\le T}\frac{h(t,X_t)}{\tilde{B}_t}\tilde{B}_T\middle|\mathcal{F}_0\right]$ is not necessarily an upper bound on V_0. Similar comments apply to Eq. (5.8).

Proof of Proposition 5.5 Suppose that $\tilde{M}$ is a supermartingale. Then, for any stopping time τ, by the optional sampling theorem of supermartingales,

$$\begin{aligned} E\left[\max_{0\le t\le T}(h(t,X_t)-\tilde{M}_t)\middle|\mathcal{F}_0\right] &\ge E[(h(\tau,X_\tau)-\tilde{M}_\tau)|\mathcal{F}_0] \ge E[h(\tau,X_\tau)|\mathcal{F}_0]-\tilde{M}_0 \\ &= E[h(\tau,X_\tau)|\mathcal{F}_0]. \end{aligned}$$

i.e., $E[\max_{0\le t\le T}(h(t,X_t)-\tilde{M}_t)|\mathcal{F}_0]$ gives us an upper bound of the optimal value. Recall that $\tilde{M}$ is a supermartingale if and only if $\tilde{B}$ is a supermartingale. Using Lemma 5.4,

$$\begin{aligned} E\left[\max_{0\le t\le T}\frac{h(t,X_t)}{\tilde{B}_t}\tilde{B}_T\middle|\mathcal{F}_0\right] &= E[\tilde{V}_0(X_0)\,\tilde{B}_T|\mathcal{F}_0] = \tilde{V}_0(X_0)E[\tilde{B}_T|\mathcal{F}_0] \\ &\le \tilde{V}_0(X_0) = E\left[\max_{0\le t\le T}(h(t,X_t)-\tilde{M}_t)\middle|\mathcal{F}_0\right]. \end{aligned}$$

A similar argument applies when $\tilde{B}$ and $\tilde{M}$ are submartingales. □

6 Variance comparison

In previous sections, we have argued from several perspectives that the quality of the bounds obtained using the additive and multiplicative duals are equivalent. In this section, compare the variance of the two methods and argue that this represents a significant advantage for the additive method.

6.1 Variance of the multiplicative method

Definition 6.1 (Relative entropy) *Let μ_1 and μ_2 be two mutually absolutely continuous measures. The* relative entropy *of μ_2 with respect to μ_1 is given by*

$$R(\mu_2\|\mu_1) = \int \log\left[\frac{d\mu_2}{d\mu_1}\right] d\mu_2.$$

Remark 6.2 Jensen's inequality implies that $R(\mu_2\|\mu_1)\ge 0$ and that equality holds only if $\mu_2=\mu_1$. Thus the relative entropy can be viewed as a kind of "distance" between two measures, though it is not symmetric.

Using the Bayes formula for conditional expectations ([10], Lemma 5.3), we get

Lemma 6.3

$$E^{\tilde{B}}[\tilde{B}_{i+1}|\mathcal{F}_i] \geq \tilde{B}_i \exp\Big(R\big(P^{\tilde{B}}(\cdot|\mathcal{F}_i)\big\|P(\cdot|\mathcal{F}_i)\big)\Big)$$

where $R(P^{\tilde{B}}(\cdot|\mathcal{F}_i)\|P(\cdot|\mathcal{F}_i))$ is the relative entropy of $P^{\tilde{B}}(\cdot|\mathcal{F}_i)$ with respect to $P(\cdot|\mathcal{F}_i)$.

Proof It is easy to see that

$$\begin{aligned} E^{\tilde{B}}[\tilde{B}_{i+1}|\mathcal{F}_i] &= \tilde{B}_i E^{\tilde{B}}\left[\frac{\tilde{B}_{i+1}}{\tilde{B}_i}\middle|\mathcal{F}_i\right] = \tilde{B}_i E^{\tilde{B}}\left[\exp\left(\log\frac{\tilde{B}_{i+1}}{\tilde{B}_i}\right)\middle|\mathcal{F}_i\right] \\ &\geq \tilde{B}_i \exp\left(E^{\tilde{B}}\left[\log\frac{\tilde{B}_{i+1}}{\tilde{B}_i}\middle|\mathcal{F}_i\right]\right) \end{aligned}$$

where the inequality follows the concavity of log and Jensen's inequality. On the other hand, by the Bayes formula, for all $A \in \mathcal{F}_{i+1}$,

$$P^{\tilde{B}}(A|\mathcal{F}_i) = E^{\tilde{B}}[\mathbf{1}_A|\mathcal{F}_i] = \frac{1}{\tilde{B}_i}E[\mathbf{1}_A\tilde{B}_{i+1}|\mathcal{F}_i].$$

In other words, $\tilde{B}_{i+1}/\tilde{B}_i$ is the Radon–Nikodým derivative of $P^{\tilde{B}}(\cdot|\mathcal{F}_i)$ over $P(\cdot|\mathcal{F}_i)$. The result now follows from the definition of relative entropy. □

Proposition 6.4 *If $\tilde{B}_t \in \mathcal{B}_0$ satisfies*

$$R\big(P^{\tilde{B}}(\cdot|\mathcal{F}_i)\big\|P(\cdot|\mathcal{F}_i)\big) \geq \epsilon, \quad 0 \leq i \leq T, \tag{6.1}$$

then the variance of the resulting multiplicative estimate has the lower bound

$$\mathrm{Var}\left[\max_{0\leq t\leq T}\frac{h(t,\mathbf{X}_t)}{\widetilde{B}_t}\widetilde{B}_T\middle|\mathcal{F}_0\right] \geq \exp(T\epsilon)E[h^2(1,\mathbf{X}_1)|\mathcal{F}_0] - V_0^2T^2.$$

Remark 6.5 Though perhaps difficult to verify, condition (6.1) is, we believe, broadly applicable. The optimal probability measure P^B for pricing American option is invariably different from the physical probability measure, so $R(P^{\tilde{B}}(\cdot|\mathcal{F}_i)\|P(\cdot|\mathcal{F}_i)) > 0$ for all i. Condition (6.1) requires that the relative entropy be bounded away from 0. For fixed T, this is no more restrictive than requiring that the relative entropy be positive for all i, so the real content of the condition is that it should continue to hold as T grows.

Proof of Proposition 6.4 We know that

$$\begin{aligned} \mathrm{Var}\left[\max_{0\leq t\leq T}\frac{h(t,\mathbf{X}_t)}{\widetilde{B}_t}\widetilde{B}_T\middle|\mathcal{F}_0\right] &= E\left[\max_{0\leq t\leq T}\left(\frac{h(t,\mathbf{X}_t)}{\widetilde{B}_t}\widetilde{B}_T\right)^2\middle|\mathcal{F}_0\right] \\ &\quad - \left(E\left[\max_{0\leq t\leq T}\frac{h(t,\mathbf{X}_t)}{\widetilde{B}_t}\widetilde{B}_T\middle|\mathcal{F}_0\right]\right)^2. \end{aligned} \tag{6.2}$$

For the first term on the right, we have

$$
\begin{aligned}
E\left[\max_{0\le t\le T}\left(\frac{h(t,\mathbf{X}_t)}{\widetilde{B}_t}\widetilde{B}_T\right)^2\middle|\mathcal{F}_0\right] &\ge E\left[\left(\frac{h(1,\mathbf{X}_1)}{\widetilde{B}_1}\widetilde{B}_T\right)^2\middle|\mathcal{F}_0\right]\\
&= E\left[\frac{h^2(1,\mathbf{X}_1)}{\widetilde{B}_1^2}E[\widetilde{B}_T^2|\mathcal{F}_1]\middle|\mathcal{F}_0\right]\\
&= E\left[\frac{h^2(1,\mathbf{X}_1)}{\widetilde{B}_1}E^{\widetilde{B}}[\widetilde{B}_T|\mathcal{F}_1]\middle|\mathcal{F}_0\right],
\end{aligned}
$$

using the Bayes rule for conditional expectations in the last equality. Using Lemma 6.3 and induction, we know that

$$
\begin{aligned}
&E\left[\frac{h^2(1,\mathbf{X}_1)}{\widetilde{B}_1}E^{\widetilde{B}}[\widetilde{B}_T|\mathcal{F}_1]\middle|\mathcal{F}_0\right]\\
&= E\left[\frac{h^2(1,\mathbf{X}_1)}{\widetilde{B}_1}E^{\widetilde{B}}\left[E^{\widetilde{B}}[\widetilde{B}_T|\mathcal{F}_{T-1}]\middle|\mathcal{F}_1\right]\middle|\mathcal{F}_0\right]\\
&\ge E\left[\frac{h^2(1,\mathbf{X}_1)}{\widetilde{B}_1}E^{\tilde{B}}\left[\tilde{B}_{T-1}\exp\left(R\big(P^{\tilde{B}}(\cdot|\mathcal{F}_{T-1})\|P(\cdot|\mathcal{F}_{T-1})\big)\right)\middle|\mathcal{F}_1\right]\middle|\mathcal{F}_0\right]\\
&\ge E\left[\frac{h^2(1,\mathbf{X}_1)}{\widetilde{B}_1}E^{\widetilde{B}}[\tilde{B}_{T-1}\exp(\epsilon)|\mathcal{F}_1]\middle|\mathcal{F}_0\right] \ge \cdots\\
&\ge \exp(T\epsilon)E[h^2(1,\mathbf{X}_1)|\mathcal{F}_0].
\end{aligned}
$$

On the other hand, by Proposition 5.1, the second term on the right side of Eq. (6.2) satisfies

$$
\left(E\left[\max_{0\le t\le T}\frac{h(t,\mathbf{X}_t)}{\widetilde{B}_t}\widetilde{B}_T\middle|\mathcal{F}_0\right]\right)^2 \le (V_0T)^2.
$$

Thus,

$$
\mathrm{Var}\left[\max_{0\le t\le T}\frac{h(t,\mathbf{X}_t)}{\widetilde{B}_t}\widetilde{B}_T\middle|\mathcal{F}_0\right] \ge \exp(T\epsilon)E[h^2(1,\mathbf{X}_1)|\mathcal{F}_0] - V_0^2T^2.
$$

□

6.2 Variance of the additive method

Proposition 6.6 *Suppose that the martingale $\tilde{M}\in\mathcal{M}_0$ and the optimal martingale M satisfy, for all* $0\le i\le T$,

$$
E[(\tilde{M}_i-\tilde{M}_{i-1})-(M_i-M_{i-1})]^2 \le K.
$$

Then the variance of the additive method satisfies

$$
\mathrm{Var}\left[\max_{0\le t\le T}(h(t,\mathbf{X}_t)-\tilde{M}_t)\middle|\mathcal{F}_0\right] \le 4V_0\sqrt{T}K+2T^2K^2.
$$

Remark 6.7 This proposition suggests that good estimations of optimal martingales (i.e., small K) will lead to small variance.

Proof of Proposition 6.6 By the definition of variance,

$$\begin{aligned}\mathrm{Var}\left[\max_{0\le t\le T}(h(t,\mathbf{X}_t)-\tilde{M}_t)\middle|\mathcal{F}_0\right] &\le E\left[\left(\max_{0\le t\le T}(h(t,\mathbf{X}_t)-\tilde{M}_t)\right)^2\middle|\mathcal{F}_0\right]\\ &\quad-\left(E\left[\max_{0\le t\le T}(h(t,\mathbf{X}_t)-\tilde{M}_t)\middle|\mathcal{F}_0\right]\right)^2\\ &\le E\left[\left(\max_{0\le t\le T}(h(t,\mathbf{X}_t)-\tilde{M}_t)\right)^2\middle|\mathcal{F}_0\right]-V_0^2.\end{aligned}$$

Note that $h(t,\mathbf{X}_t)\le V_t$. Then

$$E\left[\left(\max_{0\le t\le T}(h(t,\mathbf{X}_t)-\tilde{M}_t)\right)^2\middle|\mathcal{F}_0\right]\le E\left[\left(\max_{0\le t\le T}(V(t,\mathbf{X}_t)-\tilde{M}_t)\right)^2\middle|\mathcal{F}_0\right].$$

Doing Doob's decomposition for the supermartingale V and noting that D is a decreasing process yields

$$\begin{aligned}E\left[\left(\max_{0\le t\le T}(V(t,\mathbf{X}_t)-\tilde{M}_t)\right)^2\middle|\mathcal{F}_0\right] &= E\left[\left(\max_{0\le t\le T}(M_t+D_t-\tilde{M}_t)\right)^2\middle|\mathcal{F}_0\right]\\ &\le E\left[\left(D_0+\max_{0\le t\le T}(M_t-\tilde{M}_t)\right)^2\middle|\mathcal{F}_0\right].\end{aligned}$$

A little algebra and Doob's inequality lead to

$$\begin{aligned}&E\left[\left(D_0+\max_{0\le t\le T}(M_t-\tilde{M}_t)\right)^2\middle|\mathcal{F}_0\right]\\ &= D_0^2+2D_0E\left[\max_{0\le t\le T}(M_t-\tilde{M}_t)\middle|\mathcal{F}_0\right]+E\left[\max_{0\le t\le T}(M_t-\tilde{M}_t)^2\middle|\mathcal{F}_0\right]\\ &\le V_0^2+4V_0\sqrt{E[(M_T-\tilde{M}_T)^2|\mathcal{F}_0]}+\sum_{0\le t\le T}E[(M_t-\tilde{M}_t)^2|\mathcal{F}_0].\end{aligned}$$

Using the orthogonality property of martingale differences, we get

$$\begin{aligned}
\mathrm{Var}\left[\max_{0\le t\le T}(h(t,\mathbf{X}_t)-\tilde{M}_t)\middle|\mathcal{F}_0\right] &\le 4V_0\sqrt{E[(M_T-\tilde{M}_T)^2|\mathcal{F}_0]} \\
&\quad + \sum_{0\le t\le T} E[(M_t-\tilde{M}_t)^2|\mathcal{F}_0] \\
&= 4V_0\sqrt{\sum_{i=1}^{T} E[(\tilde{M}_i-\tilde{M}_{i-1})-(M_i-M_{i-1})]^2} \\
&\quad +2\sum_{t=1}^{T}\sum_{j=1}^{t} E[(\tilde{M}_i-\tilde{M}_{i-1})-(M_i-M_{i-1})]^2 \\
&\le 4V_0\sqrt{T}K+2T^2K^2.
\end{aligned}$$

□

Comparison of Propositions 6.4 and 6.6 indicates a strong advantage to computing upper bounds through the additive dual rather than the multiplicative. We illustrate this result numerically in Sect. 6.3. Before doing so, we point out a property of the optimal martingales under the two methods which is consistent with the variance advantage of the additive method. The following result follows from Lemma 5.4.

Proposition 6.8 *Suppose the optimal value process admits the decompositions $V = M + D$ and $V = BA$, with B positive. Then*

$$V_0 = \max_{0\le t\le T}(h(t,\mathbf{X}_t)-M_t) = \max_{0\le t\le T}\frac{h(t,\mathbf{X}_t)}{B_t}. \tag{6.3}$$

The expression in the middle of Eq. (6.3) is the additive dual estimate; thus, with the optimal martingale, this method has zero variance. But the expression on the far right of Eq. (6.3) is not the multiplicative dual estimate because it is missing a factor of B_T. Multiplying by B_T gives this expression positive variance. Thus, even when the optimal martingales are known, the additive method produces strictly smaller variance. The expression on the far right in Eq. (6.3) becomes an unbiased estimate of V_0 under P^{B_T}; in other words, achieving zero variance with the multiplicative method requires applying a change of measure in the Monte Carlo simulation. Bolia et al. [3] use this observation to try and develop near-optimal measure changes from near-optimal martingales, but the variance of the multiplicative method is problematic there as well.

6.3 Numerical illustration

Suppose the interest rate $r = 4\%$ and the underlying stock price follows the Black–Scholes model with volatility $\sigma = 30\%$. Assume that the current position of the stock price is $S_0 = 100$. We consider a family of American put options with strike price $K = 100$. In this example, the interval between two exercise dates is 0.01 and we use the values of European options with the same maturities and the same strike prices as approximations from which to define martingales. The number of sample paths is 100,000. For the purpose of comparison, we use the same sample paths in both methods. The following table shows estimated upper bounds and their variances in parentheses:

Exercise chances	True value	Additive	Multiplicative
$N = 10$	3.5601	3.6028 (0.0020)	3.6212 (26.5825)
$N = 50$	7.5495	7.6135 (0.0887)	7.5638 (110.0690)
$N = 100$	10.1993	10.3291 (0.4313)	10.3101 (196.4814)
$N = 150$	12.0428	12.2623 (1.0718)	12.1688 (267.4694)

This example shows that both methods have very close biases while the multiplicative method incurs much higher variance than the additive method.

Our next example is extreme. We still consider American puts, but we set the interest rate to be a very high 20% and we make the options deep in-the-money by setting $S_0 = 50$. With these parameters, the optimal strategy is to exercise the options immediately. But we still use the European options as our approximations. The table below shows that, again, the biases are similar but the variance of the multiplicative method is much greater.

Exercise chances	True value	Additive	Multiplicative
$N = 10$	50	50 (0)	50.0127 (24.6831)
$N = 50$	50	50 (0)	50.1964 (171.6656)
$N = 100$	50	50.0001 (2.7452×10^{-6})	50.7807 (490.4612)
$N = 150$	50	50.0482 (0.0093)	51.1385 (907.2639)

7 Duality and linear programming

In this section, we show that in the case of a finite state space, the additive dual coincides with duality in the sense of linear programming. We establish a related property for the multiplicative dual.

Suppose that the stochastic process X has a discrete state space $\mathcal{S} = \{S_1, \ldots, S_N\}$. Define a *path* as a sequence of nodes $(0, S_{j_0}) \to (1, S_{j_1}) \to \cdots \to (t, S_{j_t})$, where $S_{j_1}, \ldots, S_{j_t} \in \mathcal{S}$ and $0 \le t \le T$. Let $\mathcal{P}_t$ denote the set of all paths of length t. For any path q in $\mathcal{P}_T$, let $q|_{(0,t)}$ denote the truncation of q to its initial $t + 1$ nodes. This is an element of $\mathcal{P}_t$.

The process X induces a probability measure on the set $\{\mathcal{P}_t : 0 \le t \le T\}$. Let ω_q denote the probability of path q. For any two paths $q, q' \in \mathcal{P}_T$, let $P_{t,q,q'} = P\left(\mathbf{X}_{t+1} = q'|_{(0,t+1)} \middle| \mathbf{X}_t = q|_{(0,t)}\right)$; this is the probability that the path evolves from $q|_{(0,t)}$ to $q'|_{(0,t+1)}$. For any $q \in \mathcal{P}_T$, define $h(t, q)$ to be the payoff when the process is stopped at time t on path q. Because we model a problem in which the payoff is adapted, we require that the function h have the property that $h(t,q') = h(t,q)$ if $q|_{(0,t)} = q'|_{(0,t)}$, for all $q, q' \in \mathcal{P}_T$. Introduce variables $\{M(t,q), t = 0, 1, \ldots, T; q \in \mathcal{P}_T\}$ and consider the following problem of minimizing over these variables:

$$\min \sum_{q \in \mathcal{P}_T} \omega_q \max_t \left(h(t,q) - M(t,q) \right)$$

$$\text{s.t. } M(0,q) = 0, \quad \text{for all } q \in \mathcal{P}_T; \tag{7.1}$$

$$\sum_{q'} M(t+1,q') P_{t,q,q'} = M(t,q); \tag{7.2}$$

$$M(t,q) = M(t,q') \quad \text{for all } q, q' \in \mathcal{P}_T \text{ with } q|_{(0,t)} = q'|_{(0,t)}. \tag{7.3}$$

The objective function corresponds to the additive dual in the sense of Proposition 2.1. Constraint (7.3) is an adaptedness condition on $M(t,q)$; constraint (7.1) corresponds to the requirement $M_0 = 0$; and constraint (7.2) is the martingale property.

We can formulate this minimization problem as a linear programming problem by introducing an artificial variable Z:

$$\min \sum_{q \in \mathcal{P}_T} \omega_q Z_q$$

$$\text{s.t. } M(0,q) = 0, \quad \text{for all } q \in \mathcal{P}_T;$$

$$\sum_{q'} M(t+1,q') P_{t,q,q'} = M(t,q); \tag{7.4}$$

$$M(t,q) = M(t,q') \quad \text{for all } q, q' \in \mathcal{P}_T \text{ and } q|_{(0,t)} = q'|_{(0,t)}; \tag{7.5}$$

$$Z_q \geq h(t,q) - M(t,q) \quad \text{for all } t \text{ and } q \in \mathcal{P}_T. \tag{7.6}$$

We now formulate the dual of this problem. We associate dual variables $s(t,q)$ with the constraints in (7.4), dual variables $u(t,q)$ with the constraints in (7.6) and dual variables $v(t,q,q')$ with those in Eq. (7.5). Using standard linear programming duality, we arrive at the following problem:

$$\max \sum_t \sum_{q \in \mathcal{P}_T} h(t,q) u(t,q)$$

$$\text{s.t. } u(t,q) \geq 0, \quad s(t,q) \text{ free}, \quad t = 0, \ldots, T; \quad q \in \mathcal{P}_T;$$

$$v(t,q,q') \text{ free}, \quad t = 0, \ldots, T; \quad q, q' \in \mathcal{P}_T, q|_{(0,t)} = q'|_{(0,t)};$$

$$\sum_t u(t,q) = \omega_q, \quad q \in \mathcal{P}_T;$$

$$u(t,q) - s(t,q) + P_{t-1,q,q} \times \sum_{q':q'|_{(0,t-1)} = q|_{(0,t-1)}} s(t-1,q')$$

$$+ \sum_{q':q'|_{(0,t)} = q|_{(0,t)}} v(t,q,q') - \sum_{q':q'|_{(0,t)} = q|_{(0,t)}} v(t,q',q) = 0. \tag{7.7}$$

From constraint (7.7), we know that

$$\frac{u(T,q)}{P_{T-1,q,q}} = \frac{u(T,q')}{P_{T-1,q',q'}} \quad \text{for all } q \text{ and } q' \text{ such that } q|_{(0,T-1)} = q|_{(0,T-1)}; \tag{7.8}$$

and for $0 \le t \le T-1$ and any $r, r' \in \mathcal{P}_t$ such that $r|_{(0,t-1)} = r'|_{(0,t-1)}$,

$$\frac{\sum_{i=t}^{T} \sum_{q:q|_{(0,t)}=r} u(i,q)}{P_{t-1,r|_{(0,t-1)},r}} = \frac{\sum_{i=t}^{T} \sum_{q:q|_{(0,t)}=r'} u(i,q)}{P_{t-1,r'|_{(0,t-1)},r'}}. \tag{7.9}$$

Thus, we can eliminate the variables s and v by introducing the constraints (7.8) and (7.9) to obtain the problem

$$\begin{aligned}
\max \sum_{t} \sum_{q \in \mathcal{P}_T} & h(t,q)u(t,q) \\
\text{s.t. } u(t,q) &\ge 0, \quad t = 0, \dots, T, \; q \in \mathcal{P}_T; \\
\sum_{t} u(t,q) &= \omega_q, \quad q \in \mathcal{P}_T; \\
u \text{ satisfies } & (7.8) \text{ and } (7.9) \text{ for all } t.
\end{aligned}$$

We claim that this dual formulation is indeed the discrete counterpart of the optimal stopping problem. To see this, interpret $u(t,q)$, $q \in \mathcal{P}_T$, as the probability of following path q and stopping at time t. The objective function is then the expected payoff upon stopping; the first constraint requires that the probabilities be nonnegative; and the second constraint requires that the stopping probabilities along a path sum to the probability of that path. (This formulation is slightly more general than the original optimal stopping problem in that it allows randomized stopping rules: the conditional probability of stopping given the observed path could, in principle, be between 0 and 1.) The last constraint cares about the adaptedness requirement of the exercise policy.

Furthermore, we can show that the above linear programming can be solved using the idea of backward induction. We skip the details here.

Now consider the multiplicative method, which can be formulated as a minimization problem over variables $\{B(t,q), t = 0, 1, \dots, T; q \in \mathcal{P}_T\}$:

$$\min \sum_{q \in \mathcal{P}_T} \omega_q \max_t \left[\frac{h(t,q)}{B(t,q)} \right] B(T,q) \tag{7.10}$$

$$\text{s.t. } B(0,q) = 1, \quad \text{for all } q \in \mathcal{P}_T; \tag{7.11}$$

$$\sum_{q'} B(t+1,q')P_{t,q,q'} = B(t,q) \quad \text{for all } q \in \mathcal{P}_T; \tag{7.12}$$

$$B(t,q) = B(t,q'), \quad t = 0, \dots, T; \; q, q' \in \mathcal{P}_T, q|_{(0,t)} = q'|_{(0,t)}; \tag{7.13}$$

$$B(t,q) > 0. \tag{7.14}$$

As before we can introduce an artificial variable Z to rewrite the objective as

$$\min \sum_{q \in \mathcal{P}_T} \omega_q Z_q B(T,q)$$

and introduce the constraints

$$Z_q B(t,q) \geq h(t,q), \quad \text{for all } q \in \mathcal{P}_T. \tag{7.15}$$

For any fixed B satisfying the constraints (7.11)–(7.14), the optimization problem becomes a linear programming problem. We can easily write down its dual as follows:

$$\begin{aligned} &\max \quad \sum_t \sum_{q \in \mathcal{P}_T} h(t,q)u(t,q) \\ &\text{s.t. } u(t,q) \geq 0; \\ &\qquad \sum_t u(t,q)B(t,q) = \omega_q B(T,q). \end{aligned} \tag{7.16}$$

It is obvious that this dual problem will give us an upper bound for the value of the primal problem (7.10)–(7.14). Furthermore, we claim that the optimal value of the primal problem will be achieved if we choose B carefully. To formulate our result, define a Lagrangian as follows, with Lagrange multipliers s, u, v, ν:

$$\begin{aligned} L = &\sum_{q \in \mathcal{P}_T} \omega_q Z_q B(T,q) - \sum_t \sum_{q \in \mathcal{P}_T} s(t,q)\Big(B(t,q) - \sum_{q'} B(t+1,q')P_{t,q,q'}\Big) \\ &- \sum_t \sum_{q \in \mathcal{P}_T} \nu(t,q)B(t,q) - \sum_t \sum_{q \in \mathcal{P}_T} u(t,q)\Big(Z_q B(t,q) - h(t,q)\Big). \end{aligned} \tag{7.17}$$

Given B and Z, the KKT conditions (cf. [14]) for s, u, v, ν at B and Z are

$$\begin{aligned} &\nabla_{B,Z} L(B, Z; s, u, v, \nu) = 0; \\ &u(t,q) = 0 \quad \text{for all } t, q \text{ such that } Z_q B(t,q) > h(t,q); \\ &u(t,q) \geq 0 \quad \text{for all } t, q; \\ &u(t,q)\left(Z_q B(t,q) - h(t,q)\right) = 0; \\ &\nu(t,q) = 0 \quad \text{for all } t, q \text{ such that } B(t,q) > 0; \\ &\nu(t,q) \geq 0; \\ &\nu(t,q)B(t,q) = 0. \end{aligned}$$

Proposition 7.1 *Suppose that B^* and Z^* solve the primal problem (7.10)–(7.14) and s^*, u^*, v^*, ν^* satisfy the KKT conditions at B^* and Z^*. Then u^* solves the dual problem (7.16), and the optimal value of this dual problem is equal to the optimal value of the primal problem.*

Proof Suppose that s^*, u^*, v^*, ν^* satisfy the KKT conditions at B^*, Z^*. Then $\nabla_{Z*} L = 0$. This implies $\sum_t u^*(t,q)B^*(t,q) = \omega_q B^*(T,q)$. Accordingly, u^* is a feasible solution to the dual when $B = B^*$. On the other hand, for any feasible solution u,

$$\sum_t \sum_{q \in \mathcal{P}_T} h(t,q)u^*(t,q) = L(B^*, Z^*; s^*, u^*, v^*, \nu^*)$$
$$= \sum_{q \in \mathcal{P}_T} Z_q^* \omega_q B^*(T,q) = \sum_{q \in \mathcal{P}_T} Z_q^* \sum_t u(t,q) B^*(t,q)$$

where the first equality follows from the KKT conditions and the second equality holds because the second, third and fourth terms in Eq. (7.17) are all zero. In addition, Eq. (7.15) gives

$$\sum_{q \in \mathcal{P}_T} Z_q^* \sum_t u(t,q) B^*(t,q) = \sum_{q \in \mathcal{P}_T} \sum_t u(t,q) Z_q^* B^*(t,q)$$
$$\geq \sum_{q \in \mathcal{P}_T} \sum_t h(t,q) u(t,q).$$

Thus, u^* is an optimal solution when $B = B^*$ and the optimal value is the same as that of the primal problem. □

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