A NON-ZERO-SUM GAME APPROACH TO CONVERTIBLE BONDS: TAX BENEFIT, BANKRUPTCY COST AND EARLY/LATE CALLS

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Convertible bonds are hybrid securities that embody the characteristics of both straight bonds and equities. The conflicts of interest between bondholders and shareholders affect the security prices significantly. In this paper, we investigate how to use a non-zero-sum game framework to model the interaction between bondholders and shareholders and to evaluate the bond accordingly. Mathematically, this problem can be reduced to a system of variational inequalities and we explicitly derive the Nash equilibrium to the game. Our model shows that credit risk and tax benefit have considerable impacts on the optimal strategies of both parties. The shareholder may issue a call when the debt is in-the-money or out-of-the-money. This is consistent with the empirical findings of "late and early calls" (Ingersoll (1977), Mikkelson (1981), Cowan et al. (1993), Asquith (1995)). In addition, the optimal call policy under our model offers an explanation for certain stylized patterns

The authors are grateful for valuable comments from the anonymous referee, the associate editor, and Prof. Ernst Eberlein, the editor. We thank Paul Glasserman, YK Kwok, and Xunyu Zhou for their insightful suggestions. We are also grateful to many other people who offered insights into this work, including seminar participants at Stanford University, University of Oxford, Hong Kong University of Sciences and Technology, City University of Hong Kong, Peking University, and Tongji University, and conference participants at INFORMS 2009 and 2010 annual meetings. The first and third authors are supported by the Hong Kong Research Council through the General Research Fund Scheme (Grant No. CUHK411108). The second author is supported by the NUS RMI grant (No. R-146-000-124-720/646).

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related to the returns of company assets and stocks on call.

Keywords: convertible bonds, stochastic game, Nash equilibrium, tax benefit, credit risk, late and early calls, variational inequalities

1 INTRODUCTION

Convertible bonds are hybrid securities that have the characteristics of both straight bonds and equities. The bondholder receives coupons periodically and is entitled to exchange the security at her discretion for part of the issuing company's equity. How many shares of common stock one bond can be converted for is pre-specified through a conversion ratio at its issuance. A typical convertible bond also contains a callable feature — the issuer retains the right to call the debt back. Upon calling, the company offers a price, which is also specified in the bond contract in advance, to the bondholder and forces her to either surrender the security for that price or convert immediately.

Convertible bonds are quite popular as fund-raising tools among smaller and more speculative companies. Because they lack stable credit histories, the companies have to pay high interest to their debt holders if they choose to raise funds through straight bonds. Meanwhile, their stocks are usually undervalued because the capital market is uncertain about prospective of their business. Convertible bonds might help to achieve financing with lower coupon payments, which is justified by the conversion right entitled to the bondholders. When the business turns out to be successful, the bondholders will opt to convert to equity voluntarily or compulsorily. This in turn will strengthen the company's capital base. However, the original shareholders of the company will suffer from a dilution after conversion. From the perspective of investors, convertible bonds are also attractive to some extent. They offer equity-like returns and put a bond-floor protection against the downside risk when the business of the issuing company turns sour.

In this paper, we investigate how to price convertible bonds. According to the preceding discussion, the interaction between bondholders and shareholders will affect the bond price significantly. If the bondholders convert earlier than the call announcement issued by the company, then the shareholders lose a chance to force the bondholders to surrender to their interest; if the company calls first, then the bondholders might have no way to act optimally. Hence, any rational pricing model should incorporate the interaction between the two parties. We use a game theoretic approach to tackle this problem.

1.1 Literature Review: A Tale of Two Puzzles

The pioneering work on convertible bond pricing dates back to Brennan and Schwartz (1977, 1980) and Ingersoll (1977a). These authors initiate a structural approach to analyzing the optimal call and conversion rules and evaluating convertibles. The key idea is to regard the bond as a contingent claim on the company's assets. They argue that a company should announce a call if and only if the conversion value — the equity value convertible bonds can be exchanged for — equals the call price.

However, later empirical studies do not support this conclusion. Ingersoll (1977b) finds that a majority of companies under examination (170 out of 179) significantly deviate from the theoretical "optimal" call policy. The median company does not issue a call until the conversion value is 43.9% in excess of the call price. This finding is also confirmed by a series of papers such as those of Constantinides and Grundy (1987), Asquith (1995), and so on. This phenomenon is well known in the literature as an in-the-money call or late call puzzle. More recent research, including studies by Cowan et al. (1993) and Sarkar (2003), presents empirical evidence that shows a few convertibles are called when the conversion value is significantly smaller than the call price, which is known as an out-of-the-money call or early call. The challenge lies in determining how to reconcile the discrepancy between the two puzzles in practice and the optimal policy in theory.

The second group of stylized facts we consider in this paper is related to returns of the stock of the issuing company at the call announcement. Mikkelson (1981) reports that the average daily stock returns on the announcement day and one day before were around -1% for all 113 in-the-money calls tested, in contrast to the small returns of the market portfolio during the same period. This finding raises an interesting question: what motivates these companies to make a capital structure decision that reduces shareholders' wealth? Cowan et al. (1993) document positive and statistically significant common stock price reactions to the announcement of out-of-the-money calls.

Extensive attempts have been made to explain these two puzzles. A great deal of empirical evidence reveals that tax shields and credit risk play an important role behind the scenes in the two puzzles (see, e.g., Mikkelson (1981), Asquith and Mullins (1991), Campbell et al. (1991), Jalan and Barone-Adesi (1995), and Sarkar (2003)). The interest payments of a company to its debt holders are tax-deductible expenses under the current corporate tax codes. This induces the company not to call the debt back even if the conversion value of the bond exceeds its call price. When the company calls, loss of the tax shield will decrease its after-tax value and yield a negative return on the securities of the company, as suggested by Mikkelson (1981). In addition, Rosengren (1993) and Indro et al. (1999), among others, point out that credit risk significantly affects the pricing of convertible bonds in general. The impending threat of bankruptcy prompts companies to call earlier.

Some other explanations are also available in the literature. To name a few, Ingersoll (1977b), Asquith and Mullins (1991), Asquith (1995), Altintig and Butler (2005), and Dai and Kwok (2005) attribute the in-the-money call phenomenon to the call notice period, a 30-day window in which the issuing company allows the bondholders to ponder their decision. Harris and Raviv (1985) and Kim and Kallberg (1998) suggest that the reason for in-the-money calls and negative security returns may be rooted in the asymmetric status of market participants and shareholders in their ability to access the company's asset information. Cowan et al. (1993) explain that the positive reaction on stock returns for out-of-the-money calling occurs because managers receive favorable private information about the value of the firm. Dunn and Eades (1984) think that the call delay is caused by passive investors and argue that an in-the-money call benefits the company if enough investors are expected to delay their voluntary conversions.

1.2 Contribution of Our Paper

In this paper, we develop a two-person game model to incorporate the interaction between the shareholders and the bondholders of an issuing company. In light of the empirical studies mentioned in the last subsection, we highlight a trade-off of two major concerns: tax deduction on interest payments and the losses due to credit risk. On one hand, the tax benefit entices companies to borrow from bondholders, which explains why they make in-the-money calls; on the other hand, too much debt will give rise to a significant possibility of bankruptcy in the future and the concern of costly post-bankruptcy reorganization procedures prompts out-of-the-money calls to mitigate the impending credit risk facing the company. Encouraged by this intuition, we consider the effects of the combination of a tax shield and bankruptcy costs on the strategies and pricing of convertible bonds. Our model is capable of generating both in-the-money and out-of-the-money call phenomena. Furthermore, the special structure of the optimal call policy under the model yields an explanation to the above-mentioned patterns on the security returns at calling.

Mathematically, the model can be formulated as a game involving two coupled optimal stopping problems. One salient feature of our model, compared with other existing ones in convertible bond modeling, is that it consists of a non-zero-sum stochastic game. With the help of the theory of variational inequality systems, we explicitly solve Nash equilibria for the game. Closed-form pricing formulae for both convertible bonds and common stocks are then obtained and the corresponding optimal call, bankruptcy and conversion strategies are specified explicitly. The results provide a rigorous mathematical framework to accommodate the empirical evidence in the last subsection.

The papers of Sîrbu et al. (2004) and Sîrbu and Shreve (2006) are closely related with ours. They discuss how to use a game model to price convertible bonds. However, due to the absence of tax effects, their setting is zero-sum: what the shareholders gain is what the bondholders lose. In their model, the shareholders will never call in-the-money. Bielecki et al. (2008) consider a general defaultable game-option formulation of convertible bonds under an abstract semimartingale market model. Kallsen and Kühn (2005) use a framework of zero-sum game contingent claims to study convertible bonds and introduce a mathematically rigorous concept of no arbitrage price for this kind of derivatives. However, both papers ignore tax effects and adopts a pricing framework of game option discussed in Kifer (2002). In this paper we deal with a non-zero-sum game.

We should acknowledge that many other factors can influence the optimal strategies related to convertible bonds. The purpose of this paper is to emphasize the impact of the trade-off of tax and bankruptcy costs and focus on the mathematical modeling of the problem, especially the application of game theory to convertible bond pricing. We leave the direction of introducing other factors — for instance, those pointed out by the empirical literature cited in Section 1.1 — for future investigation.

1.3 Some Other Literature: Reduced Form and Hazard Process Approaches

Most of the aforementioned literature can be classified under the structural approach of viewing convertible bonds as contingent claims on the company's assets. The main criticism of this approach is that the company asset value is not directly observable. Practitioners would like to build up models that can be calibrated to liquid benchmark securities. Some studies thus suggest another approach: to decompose the security into fixed income and equity components and then to discount the associated cash flows in each component at different rates. Early papers in this area include McConnell and Schwartz (1986), Cheung and Nelken (1994), Ho and Pteffer (1996), Tsiveriotis and Fernandes (1998), and Yigitbasioglu (2002). More recently, some researchers have introduced the effect of defaults on equity to this approach, stimulated by the progress of the intensity-based reduced-form modeling in the study of general credit risk.

One can refer to the work of Takahashi et al. (2001), Davis and Lischka (2002), Ayache et al. (2003), Andersen and Buffum (2003), and Kovalov and Linetsky (2006) for further discussion.

Bielecki et al. (2009a) introduce a hazard process model for credit risk and discuss the valuation and hedging of defaultable game options using doubly reflected backward stochastic differential equations. Based on this work, Bielecki et al. (2009b) focus on convertible bond pricing in the presence of credit default swaps.

The remainder of this paper is organized as follows. We specify our model in Section 2. Section 3 reduces the problem to a variational inequalities formulation and presents some preliminary results. A complete description on the Nash equilibria is included in Section 4. The numerical experiments in Section 5 demonstrate sensitivity analysis on various parameters. All the technical issues arising in the text are deferred to the Appendix.

2 THE MODEL

2.1 The Market Primitives and Asset Process

Let us begin with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is given an exogenous standard Brownian motion W_t. Define $\{\mathcal{F}_t, 0 \leq t < +\infty\}$ to be the augmentation under $\mathbb{P}$ of the filtration generated by W. Since this paper aims to study the relationship between convertible bond value and the issuing company value, we shall take the structural approach to establish our model.

It is well known that the conventional structural approach to modeling credit risk often starts with an assumption that the assets of the company are tradeable, so that we can construct the risk-neutral probability measure $\mathbb{Q}$. Under it, the discounted total value of the company is governed by a martingale process. Such an assumption is stringent from the practical viewpoint. However, recent developments in the literature provide a possibility to circumvent this technical difficulty for the structural approach. Among them, Goldstein et al. (2001) choose a rational-expectation general equilibrium framework to obtain the process of the unlevered value of the company. They assume that the company produces a cash flow continuously as follows:

$$\frac{d\delta_t}{\delta_t} = \mu dt + \sigma dW_t, \tag{2.1}$$

where μ and σ are constants. All the investors in the market are supposed to have a powerfunction utility given by

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^{+\infty} e^{-\beta t}\frac{\epsilon_t^{1-\gamma}}{1-\gamma}dt\right] \tag{2.2}$$

when the investor chooses to consume ϵ_t at time t, where $\gamma \in (0,1)$ is the coefficient of relative risk aversion. In addition, there exists a risk-free money account in the economy that all investors can trade on. The price per share of the account satisfies

$$dB_t = r_t B_t dt, \quad B_0 = 1,$$

where r_t is the stochastic risk-free interest rate.

Following Goldstein and Zapatero (1996), Goldstein et al. (2001) establish the economy equilibrium in which each investor maximizes its utility and the market is clear. They show that the equilibrium risk-free interest rate r_t should be a constant r. Define a new probability measure $\mathbb{Q}$ such that for each $t \geq 0$ and $A \in \mathcal{F}_t$,

$$\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\sigma\gamma W_t - \frac{1}{2}\sigma^2\gamma^2 t\right)\cdot 1_A\right].$$

Then, in equilibrium the market value of a claim on the entire cash flow $\{\delta_t\}$ equals

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[\int_t^{+\infty} e^{-r(u-t)}\delta_u du \Big| \delta_t\right] = \frac{\delta_t}{r-(\mu-\sigma^2\gamma)}.$$

Denote $\delta := \delta_t/V_t$. Then,

$$\frac{dV_t}{V_t} = (r-\delta)dt + \sigma dW_t^{\mathbb{Q}}, \tag{2.3}$$

where $W_t^{\mathbb{Q}} = W_t + \gamma\sigma t$ is a standard Brownian motion under $\mathbb{Q}$.

Suppose that the company is financed by full equity initially. Thus, V_t should be the market value of all its equity and we can regard it as the unlevered asset value of the company. Many models in the line of the structural approach, such as Merton (1974), Black and Cox (1976), Leland (1994), and Leland and Toft (1996), impose (2.3) as a starting point to study defaultable bonds. Here Goldstein et al. (2001) establish it from more fundamental economic primitives. Broadie, Chernov, and Sundaresan (2007) take a similar approach to discuss the effect of bankruptcy reorganization on debt and equity values. Duffie (2001) offers an expository review on various valuation models of corporate securities based on this framework.

Several other approaches are suggested in the literature to address the issue of tradeability of the company's assets. For instance, Ericsson and Reneby (2002) use postulations that at least one of the company's securities, such as its common stock, is traded and the market is complete. Buffet (2000) starts from other fundamental economic variables such as manufacturing costs and profits to derive the risk-neutral representation of defaultable bonds. They reach the dynamic of (2.3) similarly. One may refer to Bielecki and Rutkowski (2002) for a related discussion.

2.2 Debt Structure and Endogenous Default

Now, suppose that the company raises funds by issuing a single perpetual convertible bond and it will not change this capital structure until the moment of call, bankruptcy, or voluntary conversion. Assume that the selection of capital structure has no impact on the profitability of the company, i.e., it will not change the dynamic of $\{\delta_t\}$ in $\mathbb{P}$.

The bond pays out a stream of coupon flow to its holder continuously. Denote the total face value to be P and the coupon rate to be c. Therefore, the bondholder will receive coupon payments amounting to $\$cPdt$ in every time interval $(t, t + dt)$ up to the first time the bond is converted/called or the company is in default. Furthermore, the bondholder has a right to convert the security for some amount of common shares at her discretion. The conversion factor λ, $0 < \lambda < 1$, is defined as what percentage of the company value the bond can be exchanged for. For example, if the company is worth $\$V$ at conversion, then the bondholder will obtain λV after converting. Meanwhile, the convertible bond is subject to redemption calls issued by the company at a preset strike price $\$K$. When calling, the bondholder must opt to surrender the security for $\$K$ or to exercise the conversion right immediately by force. Surely, the bond value equals $\max\{K, \lambda V\}$ if the company value is V when a call happens.

For the purpose of modeling tax benefits, we suppose that the corporate tax rate is κ and the company enjoys tax exemption by serving its coupon payments. It can claim a tax credit of $\kappa cPdt$ from the government for the total interest payment due, $cPdt$, in $(t, t+dt)$. We incorporate this tax benefit in the model by simply assuming that the effective coupon payment for the company is $(1 - \kappa)cPdt$. Such a capital structure specification is quite standard among the credit risk literature. Leland (1994) and Leland and Toft (1996) consider the optimal leverage level and pricing of straight corporate bonds under this assumption; Hilberink and Rogers (2002) and Chen and Kou (2009) incorporate jump risk to the same capital structure to explain non-zero credit spreads of short-term bonds.

How to introduce an appropriate bankruptcy procedure, compromising tractability and reality, plays a crucial role in our modeling. Following Leland (1994) and Leland and Toft (1996), we consider endogenous defaults, i.e., the stockholder, or the management of the company on behalf of him, decides the timing of bankruptcy. This model specification can be viewed as being consistent with the legal practice in the real world[1] [2]. In the default event, the company will be liquidated and a portion ρ of the asset value is lost. The bondholder will recover from what remains. We assume that $1-\rho \geq \lambda$ in our model. This assumption is to avoid a situation in which the bondholder always converts at the ex-default moment, which means that the bankruptcy never happens.

2.3 A Non-Zero-Sum Game Between Bondholder and Shareholder

We follow a game-theoretic approach to model the conflict of interest between the bondholder and the shareholder. According to the model description in the last two subsections, the bondholder chooses when to convert and the shareholder has freedom to select both bankruptcy and call times. They will behave to maximize the values of their own holdings. To ensure simplicity, we ignore the issue of asymmetric information by assuming that both parties have equal access to the information about the company. Suppose that neither the bondholder nor the shareholder is allowed to peer into the future. Let

$$\mathcal{T} = \{\tau \geq 0 : \tau \text{ is a stopping time adaptive with respect to } \{\mathcal{F}_t\}\}.$$

Denote the conversion, bankruptcy, and call time by τ_{con}, τ_b, and τ_{cal}, respectively. All of them should be elements in $\mathcal{T}$.

Given τ_{con}, τ_b, and τ_{cal}, we can view both the convertible bond and equity as contingent claims on the cash flow the company can generate in the future. Goldstein and Zapatero (1996) show that any contingent claim on the cash flow of δ_t can be priced by discounting its associated expected cash flows under $\mathbb{Q}$. Some discussion is needed to fix the cash flows of both bond and equity at and before all these stopping times.

First, consider the situation at τ_{con}, τ_b, or τ_{cal}. If $\tau_{con} < \tau_b \wedge \tau_{cal}$, then the conversion from the bondholder will leave $(1-\lambda)V_{\tau_{con}}$ to the shareholder. However, if $\tau_{con} > \tau_b \wedge \tau_{cal}$, there exist three possibilities:

$$V_{\tau_b \wedge \tau_{cal}} \leq K, \quad K < V_{\tau_b \wedge \tau_{cal}} \leq K/\lambda, \quad \text{or} \quad V_{\tau_b \wedge \tau_{cal}} > K/\lambda.$$

[1] As observed by Uhrig-Homberg (2005), the US bankruptcy code allows companies to file for bankruptcy proactively even if they are not insolvent. For some other developed economies such as Canada, Germany, and Japan, compulsory liquidation should meet certain preconditions, e.g., the company cannot repay its creditors. Although both debtor and creditor can initiate a formal bankruptcy process, it is only the debtor who can declare inability to repay its debt obligation.

[2] Sometimes bond contracts entitle the holder a protective covenant, the right to force liquidation when the company is in financial distress. We leave the research on how different covenants will affect the bond pricing for future investigation.

In the first possibility, the declaration — whether it is a bankruptcy or call — leaves nothing to the shareholder because the company will not have sufficient funds to pay the call price to the bondholder. When the company value $V > K$, a rational shareholder will not declare bankruptcy because doing so will lead to zero payoff to him and in contrast, he will get $V - \max\{K, \lambda V\}$ by a call. This implies that $\tau_b \wedge \tau_{cal} = \tau_{cal}$ in the latter two possibilities. Then the payoffs of the shareholder under the latter two scenarios should be given by $V_{\tau_b \wedge \tau_{cal}} - K$ and $(1-\lambda)V_{\tau_b \wedge \tau_{cal}}$, respectively. In light of this discussion, we introduce two functions to describe the payoffs of equity and bond at $\tau_b \wedge \tau_{cal}$ if $\tau_{con} > \tau_b \wedge \tau_{cal}$. Let

$$h(V) = \min((V-K)^+, (1-\lambda)V) = \begin{cases} 0, & V < K; \\ V - K, & K \le V < K/\lambda; \\ (1-\lambda)\, V, & V \ge K/\lambda; \end{cases}$$

and

$$g(V) = \begin{cases} (1-\rho)V, & V < K; \\ V - h(V), & V \ge K. \end{cases}$$

These two reflect how much the shareholder and bondholder can obtain respectively when the shareholder takes actions first.

Before $\tau_b \wedge \tau_{cal} \wedge \tau_{con}$, the bondholder keeps receiving coupons at a rate of Pc until one of the capital-structure-change events occurs. The shareholder has an obligation to serve the coupon payments to the debt holder continuously. Recall that the company generates a cash flow amount of $\delta V_t dt$ by its operation at time t. The remaining cash flow after coupon obligation is then $(\delta V_t - (1-\kappa)cP)dt$ and assumed to be distributed to the shareholder as dividends. The quantity $\delta V_t - (1-\kappa)cP$ may be negative. In this case, we assume additional new equity is issued to finance this deficit[3]. Under our endogenous-default assumption, the shareholder is allowed to stop purchasing the newly issued equity to fulfill due debt obligation. In this case, he simply announces bankruptcy and gives up to the bondholder the right to all future cash flows.

Denote $D(V; \tau_b, \tau_{cal}; \tau_{con})$ and $E(V; \tau_b, \tau_{cal}; \tau_{con})$ to be the respective bond and equity values when the company is worth V and the policies τ_{con}, τ_b and τ_{cal} are fixed. Therefore, we have

$$\begin{aligned} E(V; \tau_b, \tau_{cal}; \tau_{con}) \quad &= \quad \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau_{con} \wedge \tau_b \wedge \tau_{cal}} e^{-rt}(\delta V_t - (1-\kappa)cP)dt + e^{-r(\tau_b \wedge \tau_{cal})} h(V_{\tau_b \wedge \tau_{cal}}) \mathbf{1}_{\{\tau_{con} \ge \tau_b \wedge \tau_{cal}\}} \\ &\quad + e^{-r\tau_{con}}(1-\lambda)V_{\tau_{con}} \mathbf{1}_{\{\tau_{con} < \tau_b \wedge \tau_{cal}\}} \big| V_0 = V\Big], \end{aligned} \tag{2.4}$$

[3] This is also a standard assumption stipulated by Leland (1994) and Leland and Toft (1996). In practice, it can be realized through rights issue. That is, a company can raise capital under a secondary market offering. With the issued rights, the existing shareholders have the privilege to buy a specified number of new shares from the company at a specified price within a specified time.

and

$$D(V;\tau_b,\tau_{cal};\tau_{con}) \;=\; \mathbb{E}^{\mathbb{Q}}[\int_0^{\tau_{con}\wedge\tau_b\wedge\tau_{cal}} e^{-rt}cPdt + e^{-r\tau_{con}}\cdot\lambda V_{\tau_{con}}\cdot\mathbf{1}_{\{\tau_{con}<\tau_b\wedge\tau_{cal}\}}$$
$$+e^{-r(\tau_b\wedge\tau_{cal})}g(V_{\tau_b\wedge\tau_{cal}})\mathbf{1}_{\{\tau_{con}\geq\tau_b\wedge\tau_{cal}\}}|V_0=V]. \tag{2.5}$$

The bondholder and the shareholder wish to maximize the values of their respective holdings. This creates a two-person game such that

$$\tau^*_{con} = \arg\max_{\tau_{con}\in\mathcal{T}} D(V;\tau^*_b,\tau^*_{cal};\tau_{con}) \tag{2.6}$$

and

$$(\tau^*_b,\tau^*_{cal}) = \arg\max_{\tau_b,\tau_{cal}\in\mathcal{T}} E(V;\tau_b,\tau_{cal};\tau^*_{con}). \tag{2.7}$$

It is worth pointing out that the game is of non-zero-sum property. Given the unlevered company value V at time 0, the sum of the market values of the equity and bond is

$$E+D = V + \mathbb{E}^{\mathbb{Q}}\left[\int_0^{\tau_b\wedge\tau_{cal}\wedge\tau_{con}} e^{-rt}\kappa cPdt\right] - \mathbb{E}^{\mathbb{Q}}[e^{-r\tau_b}\rho V_{\tau_b}\cdot\mathbf{1}_{\{\tau_b<\tau_{cal}\wedge\tau_{con}\}}] \tag{2.8}$$

for any given τ_{con}, τ_b, and τ_{cal}. The right-hand side of the above equality is not a constant. This is consistent with the classic trade-off theory in the field of capital structure selection and can be viewed as an extended version of the well-known Modigliani-Miller theorem in the presence of corporate taxes and bankruptcy costs (see Brealey, Myers, and Allen (2008) or Frank and Goyal (2008)). The second term on the right-hand side of (2.8) reflects the effect of a tax shield on earnings when the company raises debts. However, too much debt amplifies the threat of default, which will incur the deadweight costs of bankruptcy on the company, as shown by the third term on the right-hand side of (2.8).

From now on, we drop the superscript of $\mathbb{Q}$ from the expectation symbol in order to simplify the notations. All the expectations in the remaining part of the paper should be understood as being taken under the risk-neutral probability measure $\mathbb{Q}$.

3 VARIATIONAL INEQUALITIES FORMULATION

Let us now turn to solving the game (2.6-2.7) for Nash equilibria. Mathematically, the problem can be regarded as two coupled optimal stopping problems. This observation leads us to reduce it down to a system of variational inequalities. We will present some preliminary results in this section about the structure of optimal policies by analyzing the inequalities and solve the game completely in the next section. Some new notations are needed to facilitate the presentation. Let

$$U = \left\{ f : [0,+\infty) \to [0,+\infty) \;\middle|\; \begin{array}{l} f(0)=0,\ f \text{ continuous. There are two disjoint finite sets } N_f^1, N_f^2 \\ \text{such that } f \in C^1([0,+\infty)\backslash N_f^1) \cap C^2([0,+\infty)\backslash (N_f^1 \cup N_f^2)). \text{ The first} \\ \text{derivative } f' \text{ is bounded on } [0,+\infty)\backslash N_f^1.\ f'(a\pm) = \lim_{v\to a\pm} f'(v) \\ \text{and } f''(b\pm) = \lim_{v\to b\pm} f''(v) \text{ exist and are finite for } a \in N_f^1 \text{ and} \\ b \in N_f^1 \cup N_f^2. \end{array} \right\}.$$

Define an operator $\mathcal{L}$ such that it maps any function $f \in U$ into a new function such that

$$\mathcal{L}f(v) = -\frac{1}{2}\sigma^2 v^2 \frac{d^2 f}{dv^2}(v) - (r-\delta)v\frac{df}{dv}(v) + rf(v)$$

for any $v \in [0,+\infty)\backslash(N_f^1 \cup N_f^2)$. It is a delicate issue to define $\mathcal{L}f$ at $v \in N_f^1 \cup N_f^2$ because the first or second derivatives of f do not exist on those points. In the paper we adopt the following convention such that[4]

$$\mathcal{L}f(v) := \begin{cases} -\infty, & \text{if } v \in N_f^1 \text{ and } f'(v+) > f'(v-); \\ +\infty, & \text{if } v \in N_f^1 \text{ and } f'(v+) < f'(v-); \end{cases}$$

and $\mathcal{L}f(v) := \mathcal{L}f(v-)$ for $v \in N_f^2\backslash N_f^1$.

Emulate the work of Bensoussan and Friedman (1977) to use a system of variational inequalities to describe the bond and equity value functions in an equilibrium of the game. We would like to motivate readers with the financial interpretation of those inequalities before proceeding to give rigorous justification in Theorem 3.1. Suppose that d and e represent the optimal bond and equity value functions, respectively. First, it is easy to see that the bond value equals λV when the bondholder picks $\tau_{con} = 0$ as her conversion strategy. Due to the sub-optimality of this strategy, $d(V) \geq \lambda V$ for all V. On the shareholder side, the sub-optimality of $\tau_{cal} = 0$ and $\tau_b = 0$ will lead to a conclusion that $e(V) \geq h(V)$ for all V. These two observations require that the functions d and e should satisfy:

1. $d(V) \geq \lambda V$ and $e(V) \geq h(V)$ for all $V \geq 0$.

The second condition is to define the payoff of one party when the other party chooses to stop. For this purpose, with the given functions e and d, define

$$\mathcal{S}_E := \mathrm{cl}(\{V \in [0,+\infty) : e(V) = h(V),\ \mathcal{L}e(V) > (\delta V - (1-\kappa)Pc)\})$$

[4]The intuition of this convention is that when $f'(v+) > f'(v-)$, $f''(v)$ should be a positive Dirac delta, i.e., $+\infty$ and therefore $\mathcal{L}f(v) = -\infty$; when $f'(v+) < f'(v-)$, $f''(v)$ should be a negative Dirac delta, i.e., $-\infty$ and $\mathcal{L}f(v) = +\infty$.

and

$$\mathcal{S}_D := \mathrm{cl}(\{V \in [0, +\infty) : d(V) = \lambda V,\ \mathcal{L}d(V) > Pc\}),$$

where $\mathrm{cl}(A)$ means the closure of a set $A \subset [0, +\infty)$. These two sets indicate all company values at which the shareholder should call or default and the bondholder should convert, respectively[5]. Intuitively, if the bondholder chooses to convert at time 0 when the company asset value is V, then $(1-\lambda)V$ will be left to the shareholder. On the other hand, the payoff to the bondholder should be $g(V)$ when the shareholder declares bankruptcy or call to stop the game. Therefore, the following conditions should hold for d and e:

2. If $V \in \mathcal{S}_D$, then $e(V) = (1-\lambda)V$.

3. If $V \in \mathcal{S}_E$, then $d(V) = g(V)$.

As long as the shareholder does not interrupt the game by a call or bankruptcy declaration, the bondholder faces an optimal stopping problem to maximize (2.5) with a proper τ_{con}. On the other hand, the shareholder solves an optimal stopping problem when the bondholder does not convert. We need to add this intuition to our system of variational inequalities as well. Hence, when $V \notin \mathcal{S}_E$, the following variational inequality is imposed to characterize the optimality of d:

4. $N_d^1 \subset \mathcal{S}_E$. On the set $\mathcal{S}_E^c := [0, +\infty) \backslash \mathcal{S}_E$, the function d satisfies

$$\min\{d(V) - \lambda V,\ \mathcal{L}d(V) - cP\} = 0.$$

Function e should be governed by another variational inequality when $V \notin \mathcal{S}_D$. Note that the payoff function of the equity, h, is not smooth at K/λ. Thus, we cannot expect the function e is always differentiable at this point. Salminen (1985) and Dayanik and Karatzas (2003) prove that the left derivative of the value function of a general optimal stopping problem with lower obstacle is always larger than its right derivative. Inspired by their observation, we introduce

5. $N_e^1 \subset \mathcal{S}_D \cup \{K/\lambda\}$. $e'(K/\lambda-) \geq e'(K/\lambda+)$ and the equality holds if and only if $K/\lambda \in \mathcal{S}_E^c$. On the set $\mathcal{S}_D^c := [0, +\infty) \backslash \mathcal{S}_D$, the function e satisfies

$$\min\{e(V) - h(V),\ \mathcal{L}e(V) - (\delta V - (1-\kappa)cP)\} = 0.$$

[5]It is worth pointing out that the condition $e(V) = h(V)$ alone is not sufficient to define the voluntary stopping region for the shareholder. One can easily see this from the following observation: the conversion from the bondholder also can force $e(V) = h(V)$ to hold. We add the condition of $\mathcal{L}e(V) > (\delta V - (1-\kappa)Pc)$ to distinguish a voluntary stop from a compulsory one. The same comments apply for $d(V)$ too.

Finally, we assume that

6. $\mathcal{S}_D \cap \mathcal{S}_E = \emptyset$.

The intention of Condition 6 is to avoid simultaneous voluntary stop requests from the bondholder and shareholder. We find that $\tau^*_{cal} = \tau^*_{con} = 0$ constitutes a trivial Nash equilibrium for the game (2.6-2.7) when the initial company asset value $V > K$. However, such equilibrium is less interesting from the perspective of understanding the optimal strategies of the bondholder and shareholder: both players choose to act immediately just because the other one does so. We introduce Condition 6 to preclude such trivial solutions.

The following theorem tells us that we can solve for Nash equilibria of the game (2.6-2.7) once we obtain functions d and e through the variational inequalities 1-6.

Theorem 3.1 *Suppose that there exists a pair of functions $\{d^*, e^*\} \in U$ that satisfy Condition 1-6. Define*

$$\tau^*_{con} = \inf\{t \geq 0 : V_t \in \mathcal{S}_D\},\ \tau^*_b = \inf\{t \geq 0 : V_t \in \mathcal{S}_E \cap [0, K]\},$$

and

$$\tau^*_{cal} = \inf\{t \geq 0 : V_t \in \mathcal{S}_E \cap (K, +\infty)\}.$$

Then, these stopping times constitute a Nash equilibrium for the game (2.6-2.7). Furthermore,

$$d^*(V) = D(V; \tau^*_b, \tau^*_{cal}; \tau^*_{con}) \text{ and } e^*(V) = E(V; \tau^*_b, \tau^*_{cal}; \tau^*_{con}),$$

where D and E are defined as in (2.4) and (2.5).

It is possible to characterize the structures of the aforementioned stopping regions even without solving the system 1-6 explicitly. Related results are summarized in the following theorem. They turn out to be very helpful for the analysis in the next section.

Theorem 3.2 *Suppose that two functions e^* and d^* in U solve the system of variational inequalities 1-6. The following conclusions hold:*
(i) There exists a unique $V^*_{con} \in [(cP)/(\delta\lambda), +\infty)$ such that $\mathcal{S}_D = [V^*_{con}, +\infty)$.
(ii) There exists a unique $V^*_b \in (0, \min\{K,\ (1-\kappa)cP/\delta\}]$ such that $\mathcal{S}_E \cap [0, K] = [0, V^*_b]$.
(iii) A necessary condition of $\mathcal{S}_E \cap (K, +\infty) \neq \emptyset$ is that $K \leq (1-\kappa)Pc/\delta$.
(iv) Moreover, if $\mathcal{S}_E \cap (K, +\infty) \neq \emptyset$, then $K/\lambda \in \mathcal{S}_E \cap (K, +\infty)$ and there exist unique $V^*_{cal,1} \in (K, K/\lambda]$ and $V^*_{cal,2} \in [K/\lambda, (1-\kappa)cP/(\lambda\delta)]$ such that

$$\mathcal{S}_E \cap (K, +\infty) = [V^*_{cal,1}, V^*_{cal,2}].$$

We can interpret the meaning of Theorem 3.2 as follows. Conclusion (i) implies that the bondholder should convert when the company value increases to a sufficiently high level. Conclusion (ii) states that a default will occur if the company value goes too low. The call strategy of the shareholder is more subtle, depending on the magnitude of K. As shown by conclusion (iii), $\mathcal{S}_E \cap (K, +\infty) = \emptyset$ for a large K, i.e., the shareholder should not call at all during the life of the bond if the call is too expensive. This makes financial sense because he has to pay a high call price in exchange for the bond security in this case. The theorem also states that K/λ must be contained in $\mathcal{S}_E \cap (K, +\infty)$ if it is not empty. When a call is issued at $V = K/\lambda$, the conversion value of the bond is λV, equal to its call price; it is an at-the-money call. In this sense, our model can be viewed as an extension of Brennan and Schwartz (1977, 1980) and Ingersoll (1970a) in the presence of tax benefit and credit risk. Finally, it is worth pointing out that all three stopping regions are disjointed.

4 NASH EQUILIBRIUM

In this section, we will construct the Nash equilibrium of the game (2.6-2.7). The analysis is based on the system of variational inequalities presented in the previous section. We introduce some new notations here. For any three real numbers such that $0 < b \leq v \leq d$, let ς be the first passage time of V_t across double boundaries $V = b$ and $V = d$, i.e.,

$$\varsigma = \inf\{t \geq 0 : V_t \leq b \text{ or } V_t \geq d\}.$$

Define functions p and q to be the present values of two Arrow-Debreu securities, paying one dollar on the events of $V_\varsigma = b$ and $V_\varsigma = d$, respectively. In other words,

$$p(v; b, d) = \mathbb{E}[e^{-r\varsigma}\mathbf{1}_{\{V_\varsigma = b\}}|V_0 = v] \quad \text{and} \quad q(v; b, d) = \mathbb{E}[e^{-r\varsigma}\mathbf{1}_{\{V_\varsigma = d\}}|V_0 = v].$$

Under the specification of geometric Brownian motion (2.3), both of them admit closed-form expressions (cf. Formula 3.0.5, Borodin and Salminen (2002), p. 627):

$$p(v; b, d) = \frac{d^{\beta+\gamma} - v^{\beta+\gamma}}{d^{\beta+\gamma} - b^{\beta+\gamma}} \left(\frac{b}{v}\right)^{\gamma} \quad \text{and} \quad q(v; b, d) = \frac{v^{\beta+\gamma} - b^{\beta+\gamma}}{d^{\beta+\gamma} - b^{\beta+\gamma}} \left(\frac{d}{v}\right)^{\gamma},$$

where two parameters β and γ are given by

$$\beta = \frac{-(r - \delta - \sigma^2/2) + \Delta}{\sigma^2}, \quad \gamma = \frac{(r - \delta - \sigma^2/2) + \Delta}{\sigma^2} \tag{4.1}$$

and $\Delta = \sqrt{(r - \delta - \sigma^2/2)^2 + 2r\sigma^2}$.

According to Theorem 3.2, there are only two possibilities: the shareholder never calls the debt back when K is sufficiently large and he may issue a call for small K. The corresponding equilibria will be specified in Subsections 4.1 and 4.2, respectively.

4.1 No Voluntary Calls

When K is large, $\mathcal{S}_E \cap (K, +\infty)$ will be an empty set, which indicates that the shareholder never declares a call decision proactively. In this case, we can show the following theorem:

Theorem 4.1 *There exists a critical value K_1. When $K \geq K_1$, we can find unique V_b^* and V_{con}^* such that in a Nash equilibrium, the bondholder should convert at*

$$\tau_{con}^* = \inf\{t \geq 0 : V_t \geq V_{con}^*\}$$

and the shareholder never calls and should announce bankruptcy at the moment

$$\tau_b^* = \inf\{t \geq 0 : V_t \leq V_b^*\}.$$

Furthermore, the optimal equity and bond values are given by

$$(E^*(V), D^*(V)) = \begin{cases} (0, (1-\rho)V) & \text{if } V \leq V_b^*; \\ (E_1(V; V_b^*, V_{con}^*), D_1(V; V_b^*, V_{con}^*)) & \text{if } V_b^* < V \leq V_{con}^*; \\ ((1-\lambda)V, \lambda V), & \text{if } V \geq V_{con}^*, \end{cases}$$

where

$$\begin{aligned} E_1(V; V_b^*, V_{con}^*) &= V - \frac{(1-\kappa)Pc}{r} + \left(\frac{(1-\kappa)Pc}{r} - V_b^*\right) p(V; V_b^*, V_{con}^*) \\ &\quad + \left(\frac{(1-\kappa)Pc}{r} - \lambda V_{con}^*\right) q(V; V_b^*, V_{con}^*) \end{aligned}$$

and

$$D_1(V; V_b^*, V_{con}^*) = \frac{Pc}{r} + \left((1-\rho)V_b^* - \frac{Pc}{r}\right) p(V; V_b^*, V_{con}^*) + \left(\lambda V_{con}^* - \frac{Pc}{r}\right) q(V; V_b^*, V_{con}^*).$$

Parameters K_1, V_b^*, and V_{con}^* in Theorem 4.1 can be determined semi-explicitly. The discussion on K_1 is deferred to Appendix D, and we describe how to determine V_b^* and V_{con}^* briefly as follows. Figure 1 visualizes the relationship of the optimal stop regions for both parties. For any $V \in (V_b^*, V_{con}^*) = \mathcal{S}_E^c \cap \mathcal{S}_D^c$, the game will not be stopped and then the equity and bond value functions should satisfy the ODEs

$$\mathcal{L}E^*(V) = \delta V - (1-\kappa)Pc \quad \text{and} \quad \mathcal{L}D^*(V) = Pc,$$

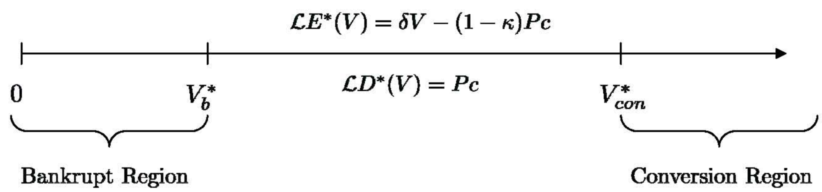

Figure 4.1: The optimal conversion and bankruptcy regions when $K > K_1$. The shareholder declares bankruptcy on $\mathcal{S}_E = [0, V_b^*]$ and the bondholder converts on $\mathcal{S}_D = [V_{con}^*, +\infty)$.

respectively, according to the variational inequality system 1-6. Both equations yield closed-form solutions, which are provided by Appendix A:

$$D^*(V) = \frac{Pc}{r} + c_1 V^{\beta} + c_2 V^{-\gamma} \quad \text{and} \quad E^*(V) = V - \frac{(1-\kappa)Pc}{r} + c_3 V^{\beta} + c_4 V^{-\gamma}$$

where β and γ are defined by (4.1). We can fix the four constants $c_i, 1 \le i \le 4$, making use of the boundary conditions

$$\begin{cases} D^*(V_{con}^*) = \lambda V_{con}^*; \\ E^*(V_{con}^*) = (1-\lambda) V_{con}^* \end{cases} \quad \text{and} \quad \begin{cases} D^*(V_b^*) = (1-\rho) V_b^*; \\ E^*(V_b^*) = 0. \end{cases}$$

It is straightforward to verify that the ODEs with the above boundary conditions result in the functions E_1 and D_1 in Theorem 4.1.

The optimal boundaries V_b^* and V_{con}^* can be determined through the variational system too. According to Conditions 4 and 5, E^* should be differentiable on $\mathcal{S}_D^c = [0, V_{con}^*)$ and D^* should be differentiable on $\mathcal{S}_E^c = (V_b^*, +\infty)$. This amounts to requiring

$$\frac{dE^*}{dV}(V)|_{V=V_b^*} = \frac{dE_1^*}{dV}(V; V_b^*, V_{con}^*)|_{V=V_b^*+} = 0, \tag{4.2}$$

where 0 is the left derivative of E^* at V_b^* because $E^*(V) = 0$ for all $V \in [0, V_b^*)$, and

$$\frac{dD^*}{dV}(V)|_{V=V_{con}^*} = \frac{dD_1^*}{dV}(V; V_b^*, V_{con}^*)|_{V=V_{con}^*-} = \lambda, \tag{4.3}$$

where λ is the right derivative of D^* at V_{con}^* because $D^*(V) = \lambda V$ for all $V \in [V_{con}^*, +\infty)$. From (4.2) and (4.3) we can solve uniquely for V_b^* and V_{con}^*, as proved by Lemma D.1 in Appendix D. Here we have an instance of the smooth pasting condition, which is common in optimal stopping.

The financial explanation of the pricing formulas in Theorem 4.1 is very clear. If the bankruptcy and conversion never occur during the whole life of the company, the present value of its total debt obligation at time 0 will be

$$\int_0^{+\infty} (1-\kappa)Pce^{-rt}dt = \frac{(1-\kappa)Pc}{r}$$

in the presence of the corporate tax exemption. Accordingly, the equity value will be $V-(1-\kappa)Pc/r$ at time 0, which is the first term of E_1^*. Take the effects of conversion and bankruptcy into account. At the moment the conversion happens, the company's capital structure changes and it is released from the obligation of a continuous debt payment flow, the value of which is worth $(1-\kappa)Pc/r$. At the same time, the shareholder has to shift λV_{con}^* to the bondholder. The net equity value change for the shareholder at converting is then

$$\frac{(1-\kappa)Pc}{r} - \lambda V_{con}^*.$$

When the bankruptcy occurs, the shareholder loses the total asset value due to the reorganization, although he does not need to serve the debt obligation any longer. Hence, the net equity value change at that time will be

$$\frac{(1-\kappa)Pc}{r} - V_b^*.$$

Recall the probabilistic meaning of p and q. The last two terms in the expression of $E_1(V; V_b^*, V_{con}^*)$ exactly reflect the present values of these two changes. A similar observation applies to the bond value.

Our bankruptcy condition (4.2) can be further analyzed by computing the expected appreciation of equity value around the bankruptcy trigger. A heuristic derivation in Appendix E reveals that at V_b^*, the change in value of equity just equals the additional cash flow that must be provided by the shareholder to keep the company solvent. When $V_t < V_b^*$, the equity appreciation would be less than the contribution required from the shareholder to let the company operate. The shareholder surely chooses to stop buying the issued new equity and the company will be bankrupt immediately due to the cash shortage. As observed by Uhrig-Homberg (2005), companies go bankrupt mainly for one of the following two reasons: either the available cash flow is insufficient to meet due payments to creditors or the companies' liabilities exceed their assets. In this sense, our endogenous bankruptcy model is a reflection of the former reason.

4.2 Early and Late Calls

In this subsection, we consider the cases with cheap call prices, or more specifically the cases in which $K \leq K_1$. For such K, the shareholder will keep calling the debt back as an option, i.e., $\mathcal{S}_E \cap (K, +\infty) \neq \emptyset$. According to Theorem 3.2, there exist two critical points $V^*_{cal,1} \leq K/\lambda \leq V^*_{cal,2}$ so that $\mathcal{S}_E \cap (K, +\infty) = [V^*_{cal,1}, V^*_{cal,2}]$. Meanwhile, the shareholder will declare bankruptcy if the company value is lower than V^*_b and the bondholder's conversion region is specified by $[V^*_{con}, +\infty)$. Both sets of $[0, V^*_b]$ and $[V^*_{con}, +\infty)$ do not have any overlaps with $[V^*_{cal,1}, V^*_{cal,2}]$. Figure 2 illustrates the relationship of all three stopping regions.

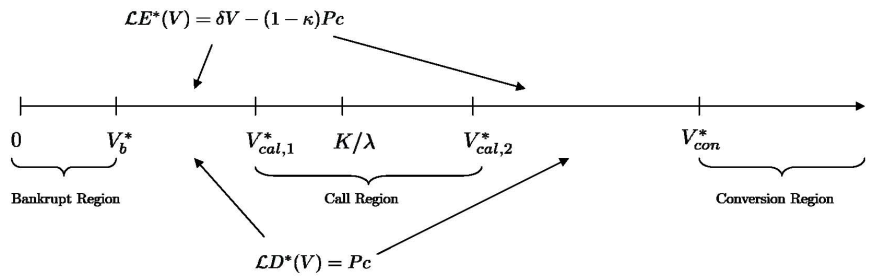

Figure 4.2: The optimal conversion and bankruptcy regions when $K > K_1$. The shareholder declares bankruptcy on $\mathcal{S}_E \cap [0, K] = [0, V^*_b]$ and makes a call on $\mathcal{S}_E \cap (K, +\infty) = [V^*_{cal,1}, V^*_{cal,2}]$. The bondholder converts on $\mathcal{S}_D = [V^*_{con}, +\infty)$.

The following theorem synthesizes a Nash equilibrium in the case of $K \leq K_1$ from the aforementioned stopping regions. Using the smooth pasting principle and the general solutions to the ODEs once again, we can obtain the values of all endpoints of the stopping regions. The details are deferred until after the theorem.

Theorem 4.2 *When $K < K_1$, a Nash equilibrium to the game (2.6-2.7) is formed if the bondholder converts her security at the moment*

$$\tau^*_{con} = \inf\{t \geq 0 : V_t \geq V^*_{con}\}$$

and the shareholder declares bankruptcy and call at

$$\tau^*_b = \inf\{t \geq 0 : V_t \leq V^*_b\} \quad \textit{and} \quad \tau^*_{cal} = \inf\{t \geq 0 : V_t \in [V^*_{cal,1}, V^*_{cal,2}]\},$$

respectively.

Under this equilibrium, the equity and bond value functions should be given by

$$(E^*(V), D^*(V)) = \begin{cases} (0, (1-\rho)V), & \text{if } V \leq V^*_b; \\ (E_2(V; V^*_b, V^*_{cal,1}), D_2(V; V^*_b, V^*_{cal,1})), & \text{if } V^*_b < V \leq V^*_{cal,1}; \\ (V - K, K), & \text{if } V^*_{cal,1} < V \leq K/\lambda; \\ ((1-\lambda)V, \lambda V), & \text{if } K/\lambda < V \leq V^*_{cal,2}; \\ (E_3(V; V^*_{cal,2}, V^*_{con}), D_3(V; V^*_{cal,2}, V^*_{con})), & \text{if } V^*_{cal,2} < V \leq V^*_{con}; \\ ((1-\lambda)V, \lambda V), & \text{if } V > V^*_{con}, \end{cases}$$

where

$$
\begin{aligned}
E_2(V; V_b^*, V_{cal,1}^*) &= V - \frac{(1-\kappa)Pc}{r} + \left(\frac{(1-\kappa)Pc}{r} - V_b^*\right) p(V; V_b^*, V_{cal,1}^*) \\
&\quad + \left(\frac{(1-\kappa)Pc}{r} - K\right) q(V; V_b^*, V_{cal,1}^*) \\
D_2(V; V_b^*, V_{cal,1}^*) &= \frac{Pc}{r} + \left((1-\rho)V_b^* - \frac{Pc}{r}\right) p(V; V_b^*, V_{cal,1}^*) + \left(K - \frac{Pc}{r}\right) q(V; V_b^*, V_{cal,1}^*)
\end{aligned}
$$

and

$$
\begin{aligned}
E_3(V; V_{cal,2}^*, V_{con}^*) &= V - \frac{(1-\kappa)Pc}{r} + \left(\frac{(1-\kappa)Pc}{r} - \lambda V_{cal,2}^*\right) p(V; V_{cal,2}^*, V_{con}^*) \\
&\quad + \left(\frac{(1-\kappa)Pc}{r} - \lambda V_{con}^*\right) q(V; V_{cal,2}^*, V_{con}^*) \\
D_3(V; V_{cal,2}^*, V_{con}^*) &= \frac{Pc}{r} + \left(\lambda V_{cal,2}^* - \frac{Pc}{r}\right) p(V; V_{cal,2}^*, V_{con}^*) + \left(\lambda V_{con}^* - \frac{Pc}{r}\right) q(V; V_{cal,2}^*, V_{con}^*).
\end{aligned}
$$

In addition, the endpoints of the call region $[V_{cal,1}^*, V_{cal,2}^*]$ degenerate to K/λ for intermediate-size call price K. More precisely, there exist another two critical points $K_2, K_3 < K_1$. When $K_2 \le K \le K_1$, we have $V_{cal,1}^* = K/\lambda$; when $K < K_2$, $V_{cal,1}^* < K/\lambda$. Similarly, when $K_3 \le K \le K_1$, $V_{cal,2}^* = K/\lambda$, and when $K < K_3$, $V_{cal,2}^* > K/\lambda$.

We can determine K_2, K_3, V_b^*, $V_{cal,1}^*$, $V_{cal,2}^*$, and V_{con}^* semi-explicitly too. The discussion on how to determine K_2 and K_3 appears in Appendix D. Some comments are made here to guide readers to find these endpoints of all stopping regions.

In the two disjoint intervals $(V_b^*, V_{cal,1}^*)$ and $(V_{cal,2}^*, V_{con}^*)$, both parties do not take actions to stop the game. Therefore, the bond and equity value functions should solve the following ODEs

$$\mathcal{L}D^*(V) = Pc \quad \text{and} \quad \mathcal{L}E^*(V) = \delta V - (1-\kappa)Pc,$$

respectively. We need some boundary conditions to fix the solutions. Take the interval $(V_b^*, V_{cal,1}^*)$, for instance. As $D^*(V) = (1-\rho)V$, $E^*(V) = 0$ for $V \le V_b^*$ and $D^*(V) = K$, $E^*(V) = V - K$ for $V \in [V_{con,1}^*, K/\lambda]$, the continuous property of D^* and E^* requires that

$$\begin{cases} D^*(V_b^*) = (1-\rho)V_b^* \\ E^*(V_b^*) = 0 \end{cases} \quad \text{and} \quad \begin{cases} D^*(V_{cal,1}^*) = K \\ E^*(V_{cal,1}^*) = V_{cal,1}^* - K. \end{cases}$$

We can easily show that $D^*(V) = D_2(V; V_b^*, V_{cal,1}^*)$ and $E^*(V) = E_2(V; V_b^*, V_{cal,1}^*)$ are the solutions. Similar procedures yield the solutions to D^* and E^* in the interval $(V_{cal,2}^*, V_{con}^*)$.

When $K < K_2$ or $K < K_3$, invoke the smooth requirement in the variational inequality system once again to find the optimal boundaries V_b^*, $V_{cal,1}^*$, $V_{cal,2}^*$, and V_{con}^*. Note that $E^*(V) = 0$ for $V \leq V_b^*$ and $E^*(V) = V - K$ for $V \in [V_{cal,1}^*, K/\lambda]$. According to Conditions 4 and 5 in the variational inequality system, E^* should be smooth at $V = V_b^*$ and $V = V_{cal,1}^*$, which requires

$$(4.4) \qquad \frac{dE_2^*}{dV}(V; V_b^*, V_{cal,1}^*)|_{V=V_b^*+} = 0 \quad \text{and} \quad \frac{dE_2^*}{dV}(V; V_b^*, V_{cal,1}^*)|_{V=V_{cal,1}^*-} = 1.$$

Lemma D.3(i) proves that the equations in (4.4) admit unique solutions V_b^* and $V_{cal,1}^*$. Similarly, when $K < K_3$, the following two equations can be used to determine $V_{cal,2}^*$ and V_{con}^*:

$$(4.5) \quad \frac{dE_3^*}{dV}(V; V_{cal,2}^*, V_{con}^*)|_{V=V_{cal,2}^*+} = 1 - \lambda \quad \text{and} \quad \frac{dD_3^*}{dV}(V; V_{cal,2}^*, V_{con}^*)|_{V=V_{con}^*-} = \lambda.$$

Lemma D.3(ii) shows the existence and uniqueness of the solutions.

For such K that $K_2 \leq K \leq K_1$ or $K_3 \leq K \leq K_1$, the endpoints of the interval $[V_{cal,1}^*, V_{cal,2}^*]$ will degenerate to K/λ. By Theorem 4.2, if $K_2 \leq K \leq K_1$, $V_{cal,1}^* = K/\lambda$. In this case, we use

$$\frac{dE_2^*}{dV}(V; V_b^*, K/\lambda)|_{V=V_b^*+} = 0$$

to solve for the optimal bankrupt boundary V_b^*. When $K_3 \leq K \leq K_1$, $V_{cal,2}^* = K/\lambda$ and the following

$$\frac{dD_3^*}{dV}(V; K/\lambda, V_{con}^*)|_{V=V_{con}^*-} = \lambda$$

is used to determine the conversion boundary V_{con}^*.

Some conclusions of Theorem 4.2 are of special interest to us: our model is capable of generating both in-the-money and out-of-the-money calls. Consider the case of $K < K_2$ and assume that the initial asset value of the company V_0 is within the interval $(V_b^*, V_{cal,1}^*)$. Under the equilibrium, the shareholder should declare a call decision the first time that the asset value surges up to $V_{cal,1}^*$. When he calls, the conversion value of the bond equals $\lambda V_{cal,1}^*$, less than K. This call is out of the money. For cases with $K < K_3$, if the company starts with $V_0 > V_{cal,2}^*$, then the shareholder will call the debt back as soon as the company asset value hits $V_{cal,2}^*$. The conversion value of the bond at the call is then $\lambda V_{cal,2}^*$. The call will be in the money because the bond's conversion value exceeds K.

Our model can also explain the stylized patterns on the return rates at the call announcements. According to Theorem 4.2, out-of-the-money calls are triggered only when V_t up-crosses

$V^*_{cal,1}$. Hence, the return rate of V_t at the out-of-the-money calls should be positive. As to in-the-money calls, they happen when V_t down-crosses a level $V^*_{cal,2}$, which will lead to a negative return on the asset value. Numerical experiments in Section 5 reveal that the equity return rate behaves alike around the call date: it is positive when the call is out of the money and negative if the call is in the money. These conclusions are consistent with the empirical findings mentioned in the introduction.

At the end of this section, we emphasize two points about our model. The first is that the tax effect is crucial to generate the phenomenon of in-the-money calls. Such calls never happen in absence of tax benefits, i.e., when $\kappa = 0$. We can argue for it easily. By (2.8), the bond and equity values in an equilibrium should satisfy that $E^* + D^* \leq V$. In contrast, Condition 1 in the variational inequality system implies that $E^*(V) \geq (1-\lambda)V$ and $D^*(V) \geq \lambda V$ for any V such that $V \geq K/\lambda$. Hence, $E^*(V) = (1-\lambda)V$ and $D^*(V) = \lambda V$ on the set $\{V : \lambda V \geq K\}$. Hence, the company should call the debt back immediately when V_t hits K/λ and the call is at the money. This conclusion coincides with that of Brennan and Schwartz (1977, 1980) and Ingersoll (1970a). They show that the optimal call policy for a company in a frictionless market should be to call at $\inf\{t \geq 0 : \lambda V_t \geq K\}$.

The second point is about the uniqueness of the Nash equilibrium. As we can see from the development of Sections 3 and 4, D^* and E^* should be the only functions in class U that satisfy the variational inequality system 1-6. Theorem 3.2 and Appendix D prove that the stopping regions induced by a solution to the system 1-6 should be identical. Because the solutions can be determined by the corresponding stopping regions, we know D^* and E^* should be the unique functions satisfying 1-6.

5 NUMERICAL RESULTS

In this section, we use numerical experiments to demonstrate the impacts of various parameters on the equity and bond values and the optimal call policy. Table 1 summarizes the parameters we use in the base case. In addition, we assume that one year is equal to 252 trading days.

5.1 To Call or Not to Call: The Impact of K

In the base case given by Table 1, we can calculate to obtain that $K_1 = 87.51$, $K_2 = 55.73$, and $K_3 = 53.90$. Figure 5.1 plots a graph of the bond value function against the company asset value V in a case that $K = 100$. There is no voluntary call and the shareholder announces a default the first time that V_t drops down to $V^*_b = 36.43$. The bondholder opts to convert at

Macroeconomic parameters: $r = 8\%$, $\kappa = 35\%$
Company-specific parameters: $\delta = 6\%$, $\sigma = 22\%$, $\rho = 50\%$
Bond contract parameters: $c = 7\%$, $\lambda = 20\%$, $P = 100$

Table 5.1: Basic parameters for numerical illustration. The risk-free rate $r = 8\%$ is close to the average historical treasury rate during 1973-1998, and the corporate tax rate $\kappa = 35\%$ is chosen in line with Leland and Toft (1996). We set the paying-out ratio at $\delta = 6\%$, which is consistent with the average coupon and dividend payments in the United States during 1973-1998 (Huang and Huang (2003)). The diffusion volatility $\sigma = 0.22$, which is reported as the average asset volatility for companies with credit rating A to Baa by Schaefer and Strebulaev (2007). The recovery ratio in the default is assumed to be 50%. The bond pays coupons at a rate of $c = 7\%$. It is slightly lower than the risk-free interest rate. We choose it to reflect the low coupon payment feature of convertible bonds. The conversion ratio and the bond face value are assumed to be 20% and \$100 respectively.

$V^*_{con} = 914.62$. From this figure, we can see that the convertible bond behaves more like an equity security when the company asset V is large. This is because the bondholder has more incentive to convert when the company value increases. When V is close to V^*_b, the influence of bankruptcy becomes more significant. The convertible bond is more like a regular defaultable bond in this region.

Figure 5.2 illustrates the bond value function in a case with a lower call price. Let $K = 50$. Four endpoints, V^*_b, $V^*_{cal,1}$, $V^*_{cal,2}$, and V^*_{con}, divide the whole range of the company asset value into five segments. If the initial company value falls between V^*_b and $V^*_{cal,1}$, then the shareholder will call the debt back when V_t crosses $V^*_{cal,1}$ for the first time. Note that $V^*_{cal,1} = 97.90 < K/\lambda = 250$. This call must occur out of the money. However, if the company starts somewhere between $V^*_{cal,2}$ and V^*_{con}, then the debt-call will be in the money because it occurs at $V^*_{cal,2}$ and $V^*_{cal,2} = 269.51 > K/\lambda$.

5.2 Comparative Statics

This subsection reports the effects of variation in selected parameters on the optimal strategies of both parties and the convertible bond value. The parameters include the risk-free interest rate r, the bond coupon rate c, the paying-out rate δ, and the corporate tax rate κ. Table 5.2 displays the changes of default, conversion, and call barriers in response to the parameter changes. To facilitate interpretation of the results in the table, we consider two companies in the following discussion. The parameters of both companies are given in Table 1, except for the initial asset values. Company A starts with $V_0 = 300$ and Company B starts from $V_0 = 90$.

Effect of risk-free interest rate. When r increases, we can see that the optimal call region $[V^*_{cal,1}, V^*_{cal,2}]$ shrinks, converging toward $K/\lambda = 250$, the call barrier predicted by the classical

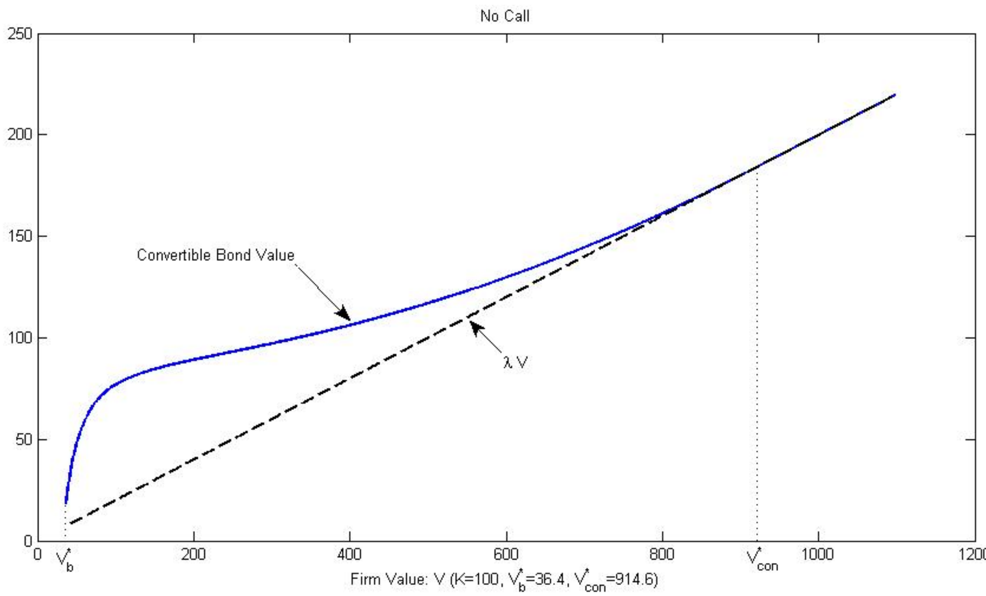

Figure 5.1: The convertible bond value in a case with a high call price. The default barrier $V_b^* = 36.43$ and the conversion barrier $V_{con}^* = 914.62$. The shareholder will never call the debt voluntarily.

literature. Under all r, the initial asset value of Company A falls in $(V_{cal,2}^*, V_{con}^*)$, a region in which only in-the-money calls are possible. For larger r, the call barrier $V_{cal,2}^*$ is farther away from V_0. Hence V_t takes longer to hit the barrier. This implies that, when r is high and all else being constant, the company tends to delay the call. This observation applies for Company B as well. The call for this company will be out of the money. As we raise r, $V_{cal,1}^*$ increases and Company B will wait longer until it issues a call announcement. The economic intuition of this observation is fairly apparent: a high interest rate environment means that the coupon payment of the company is relatively low. This makes the convertible bond a more attractive financing tool to the company, which will cause the shareholder to delay the call.

A higher r also implies a lower default barrier V_b^* and a smaller conversion barrier V_{con}^*, as shown in Table 5.2. This is also what we expect. Relatively low coupon payments in the settings of high r encourage the bondholder to convert to equity sooner, because staying in bond to receive coupons is not attractive. From the shareholder's perspective, lower coupon payments means less debt obligation. Thus, the shareholder tends to postpone default by pushing the barrier down.

Effect of coupon rate. The bond coupon rate c affects the optimal strategies in a way totally

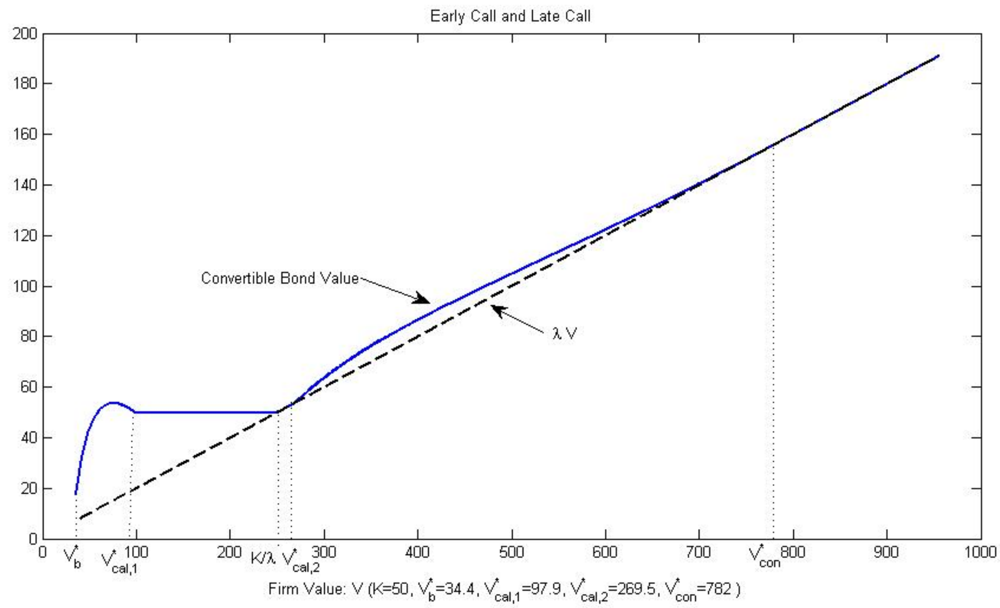

Figure 5.2: The convertible bond value in a case with a lower call price. $V_b^* = 35.44$, $V^*_{cal,1} = 97.90$, $V^*_{cal,2} = 269.51$, and $V^*_{con} = 782.00$. Once the initial company value falls in the interval $(V^*_{cal,1}, K/\lambda)$, the shareholder will call the debt immediately to force the bondholder to surrender her security. Hence, the bond value equals $K = 50$ in this interval, which is reflected by the horizontal straight line between $V^*_{cal,1}$ and K/λ. In the interval $(K/\lambda, V_{cal,2})$, the bondholder chooses to convert in response to the call from the shareholder. Hence, the bond value and its conversion value λV coincide.

opposite to the risk free interest rate. When c increases, the optimal call region $[V^*_{cal,1}, V^*_{cal,2}]$ is enlarged and both companies tend to call in a shorter period after the issuance of the bond. High coupon payments prompt a call decision because the convertible bond becomes an expensive fund-raising tool for the company. Moreover, when c is large, the shareholder will also adopt a higher V_b^* to interrupt coupon service to the bondholder earlier, while the bondholder will be attracted to holding the bond for a longer time, which leads to a higher V^*_{con}.

Effect of paying-out ratio. Given the coupon rate is unchanged, the effect of a higher paying-out ratio is to augment the dividends paid to the shareholder and in turn reduce the bond value. Hence, with a high δ setting, the shareholder will have less incentive to eliminate the bondholder from the game because the existence of the bond does not shift too much wealth away from the company. This intuition is consistent with the numerical outcomes in Table 5.2. Regardless of whether $V_0 = 300$ or 90, the distance between the call barriers and V_0 tends to be larger as δ increases. In other words, the call will be delayed if δ is high. The effect of δ on the default and conversion policies is similar to that of r: a high δ tempts the bondholder to convert sooner

	V_b^*	$V_{cal,1}^*$	$V_{cal,2}^*$	V_{con}^*
r				
0.07	36.49	80.49	271.20	795.59
0.08	35.44	97.90	269.51	782.00
0.09	33.98	250.00	268.14	768.97
c				
0.06	30.96	250.00	250.00	656.96
0.07	35.44	97.90	269.51	782.00
0.08	38.49	74.11	308.01	893.72
δ				
0.05	36.61	92.71	318.83	928.80
0.06	35.44	97.90	269.51	782.00
0.07	34.17	104.33	250.00	664.61
κ				
0.15	40.84	65.28	423.86	675.69
0.25	38.68	73.22	339.04	733.19
0.35	35.44	97.90	269.51	782.00
0.45	30.58	250.00	250.00	795.60

Table 5.2: Effects of various parameters on the optimal strategies. The defaulting parameter used is $K = 50$. We vary the parameter of interest each time and keep all other parameters the same as the base case in Table 1.

and the shareholder to declare bankruptcy later.

Effect of tax rate. In our model, the tax shield is an important factor to encourage the shareholder to borrow. Therefore, we expect that a high corporate tax will induce the company to put off the call announcement. Table 5.2 illustrates that $V_{cal,1}^*$ and $V_{cal,2}^*$ are increasing and decreasing functions of κ, respectively. Hence, the convertible bond should be called at a later stage if κ is large.

In summary, the foregoing numerical experiments indicate that delayed calls should be associated with low coupon rate, high corporate tax, high paying-out ratio and high risk-free interest rate. These implications are supported by empirical tests conducted by Sarkar (2003).

Table 5.3 presents a sensitivity analysis of the value of an in-the-money convertible bond with respect to the risk-free interest rate, coupon rate, paying-out rate, and corporate tax rate. The analysis shows that the bond value is positively related to the coupon rate, tax rate and conversion ratio, and negatively related to the interest rate and payout rate. The former group of factors determines the cash inflows for the bondholder. Thus, higher values in those factors will boost the security value. The latter two factors reduce the bond value as they rise. High

Interest rate r	0.06	0.07	0.08	0.09
Bond value	105.60	105.25	104.88	104.49
Coupon rate c	0.06	0.07	0.08	0.09
Bond value	101.88	104.88	107.12	108.57
Paying out rate δ	0.05	0.06	0.07	0.08
Bond value	106.62	104.88	102.39	100.45
Tax rate κ	0.15	0.25	0.35	0.45
Bond value	100.67	102.42	104.88	105.74

Table 5.3: Effects of various parameters on the convertible bond value. The defaulting parameters used are $K = 50$ and $V_0 = 500$. We vary r, c, δ, and κ in each row and keep all other parameters the same as those in Table 1.

risk-free interest will discount the future cash flow of the bond more, which generates a lower present value. A high paying-out ratio implies a high dividend payment to the shareholder, which will shift the wealth away from the bondholder.

5.3 Negative and Positive Equity Returns

This subsection illustrates that our model can generate negative stock returns at an in-the-money call and positive stock returns at an out-of-the-money call.

We use Monte Carlo simulation method to simulate the equity value changes for a specific company around the debt-call date. More precisely, consider the parameters in Table 1 and Company A. Simulate daily sample paths for V_t, starting with $V_0 = 300$ and following the model (2.3). The equity value each day is obtained if we substitute the simulated V_t into the equity function E^*. According to our calculation, this V_0 falls in the interval between $V^*_{cal,2} = 269.51$ and $V^*_{con} = 782$. When a call occurs, it must be in the money. We choose the discrete time unit to be 1 trading day (i.e., 1/252 year) to simulate the call and conversion time. Figure 5.3 shows a typical realization of such a path in a time window from 60 days before the call to 60 days after. The daily returns of the company's stock are not significant at all (less than 0.5%) except for the day on which a call announcement is issued. The daily return on calling drops almost 2%.

Figure 5.4 shows the daily equity returns in a 121-day time window centering on the day on which an out-of-the-money call is issued. We consider Company B with $V_0 = 90$. There is a significant positive stock return on the call day, larger than 3.5%. However, the returns on the remaining days are less than 1%.

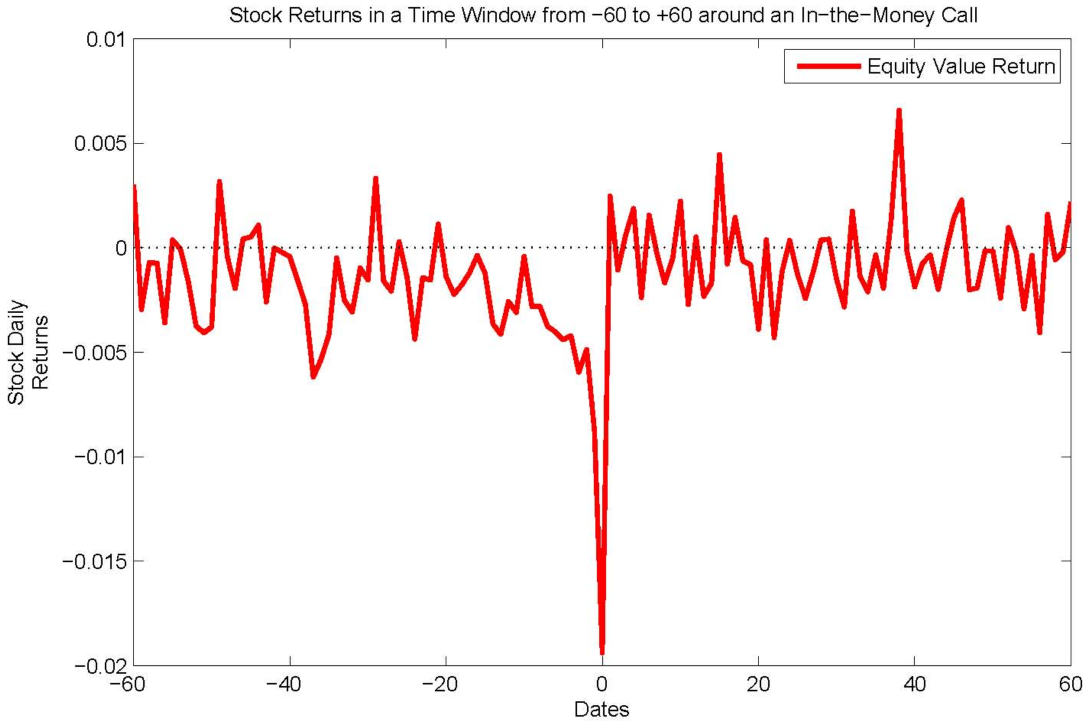

Figure 5.3: Daily returns of equity in a time window from 60 days before the call to 60 days after. We use the default parameters in Table 1 and let $V_0 = 300$. $K = 50$. The call boundary is $V^*_{cal,2} = 269.51$ and the conversion boundary is $V^*_{con} = 782$. We simulate 100 sample paths in which a call happens and plot the average.

6 CONCLUSION

We have established a non-zero-sum game framework to study the pricing problem of callable convertible bonds. The impact of a trade-off between a tax shield and bankruptcy costs is highlighted. Taking this trade-off into account will significantly change the strategies of the bondholder and shareholder when compared with the zero-sum setting in Sîrbu et al. (2004) and Sîrbu and Shreve (2006). In the presence of tax benefits and bankruptcy cost, the shareholder may call the debt in the money or out of the money. The corresponding equity returns on the call day exhibit patterns consistent with the well-documented empirical results.

One direction of our future work is to introduce other factors that might accentuate the effect of the aforementioned trade-off. For instance, the indentures of many convertible bonds prohibit the issuers from calling for a certain period. Our model can be extended to cover this prohibition by viewing the problem as a two-stage sequential game. The first stage is the

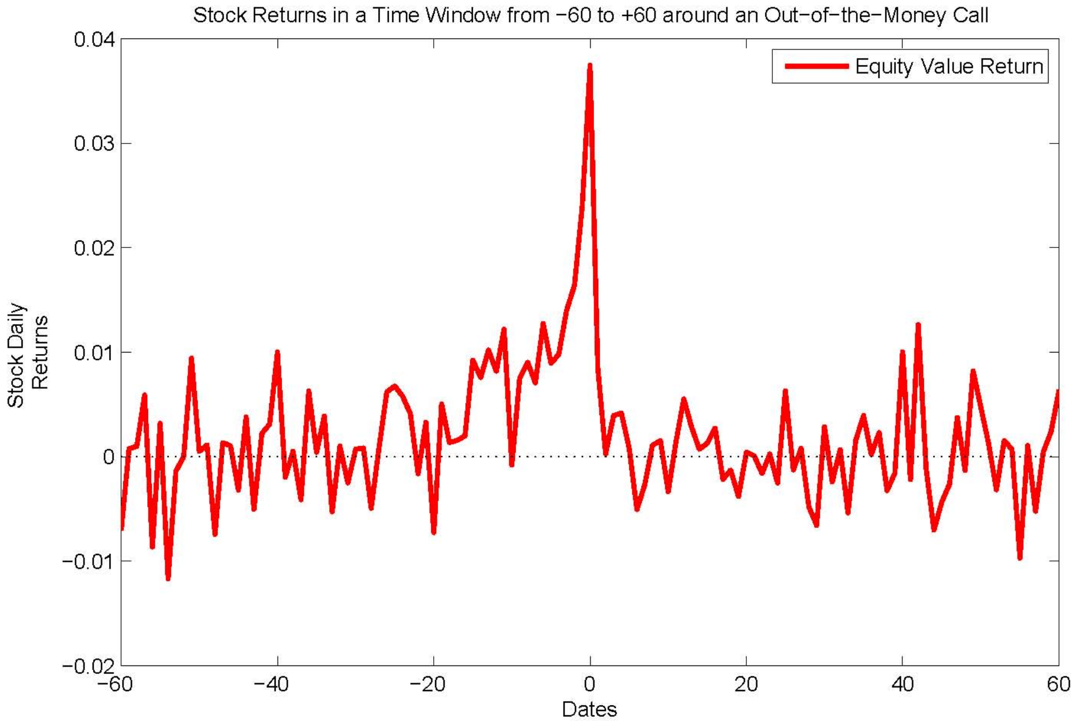

Figure 5.4: Daily returns of equity in a time window from 60 days before the call to 60 days after. We use the default parameters in Table 1 and let $V_0 = 90$. The call price is 50. The call boundary is $V^*_{cal,1} = 97.90$ and the default boundary is $V^*_b = 35.44$. We simulate 100 sample paths and draw the average.

call protection period, in which the two parties interact with each other by choosing optimal conversion and default policies. The analysis in this paper constitutes the second stage. Another possible extension would be to incorporate the asymmetric information access of the bondholder and shareholder. In reality, bond investors cannot observe the company's asset directly and suffer from imperfect accounting information (see, e.g., Duffie and Lando (2001)). A game framework with imperfect information would be an appropriate model under this setting.

APPENDIX

A THE EULER-CAUCHY ODE

Consider two second-order non-homogeneous ODEs such as

$$\mathcal{L}D(v) = -\frac{1}{2}\sigma^2 v^2 \frac{d^2}{dv^2}D(v) - (r-\delta)v\frac{d}{dv}D(v) + rD(v) = Pc$$

and

$$\mathcal{L}E(v) = -\frac{1}{2}\sigma^2 v^2 \frac{d^2}{dv^2}E(v) - (r-\delta)v\frac{d}{dv}E(v) + rE(v) = \delta v - (1-\kappa)Pc.$$

Explicit general solutions to both equations are known (Zwillinger (1997), p. 120). They are given by

$$D(v) = \frac{Pc}{r} + c_1 v^{\beta} + c_2 v^{-\gamma} \quad \text{and} \quad E(v) = v - \frac{(1-\kappa)Pc}{r} + c_3 v^{\beta} + c_4 v^{-\gamma}$$

respectively, where $c_i, 1 \le i \le 4$ are constants to be determined and β and γ are specified by (4.1).

B PROOF OF THEOREM 3.1

Lemma B.1 *If functions d^* and e^* satisfy the conditions in Theorem 3.1, then*

$$\mathcal{L}d^*(V) = Pc \quad \textit{and} \quad \mathcal{L}e^*(V) = \delta V - (1-\kappa)Pc$$

for all $V \in \mathcal{S}_D^c \cap \mathcal{S}_E^c$.

Proof. Given $V \in \mathcal{S}_D^c \cap \mathcal{S}_E^c$, we know that $V \notin N_d^1$ and

$$\min\{d^*(V) - \lambda V, \mathcal{L}d^*(V) - Pc\} = 0 \tag{B.1}$$

from Condition 4. If $\mathcal{L}d^*(V) > Pc$, then we know from (B.1) that $d^*(V) = \lambda V$. This contradicts the assumption that $V \in \mathcal{S}_D^c$. Hence, $\mathcal{L}d^*(V) = Pc$. Following similar arguments, we can show $\mathcal{L}e^*(V) = \delta V - (1-\kappa)Pc$ is also true. □

Proof of Theorem 3.1. Let τ_b and τ_{cal} be any stopping times in $\mathcal{T}$. First, we shall prove that, for any $V \ge 0$,

$$e^*(V) \ge E(V; \tau_b, \tau_{cal}; \tau_{con}^*).$$

Note that $e^* \in U$. It must be the difference of two convex functions. Applying the generalized Ito's formula for convex functions (see, e.g., Theorem 3.7.1, Karatzas and Shreve (1991)),

$$\begin{aligned} e^{-r(\tau_b \wedge \tau_{cal} \wedge \tau_{con}^*)} e^*(V_{\tau_b \wedge \tau_{cal} \wedge \tau_{con}^*}) &= e^*(V) + \int_0^{\tau_b \wedge \tau_{cal} \wedge \tau_{con}^*} (e^*(V_u))' \cdot e^{-ru} \sigma V_u dW_u \\ - \int_0^{\tau_b \wedge \tau_{cal} \wedge \tau_{con}^*} e^{-ru} \mathcal{L}e^*(V_u) du &+ \sum_{a \in N_{e^*}^1} ((e^*(a+))' - (e^*(a-))') \int_0^{\tau_b \wedge \tau_{cal} \wedge \tau_{con}^*} e^{-ru} dL_u(a), \end{aligned}$$

(B.2)

where $\{L_u(a), u \geq 0\}$ is the (nondecreasing) local time process of $\{V_u, u \geq 0\}$ at a.

For any $u < \tau^*_{con}$, $V_u \notin \mathcal{S}_D$. According to Condition 5, e^* is C^1 on $\mathcal{S}^c_D \backslash \{K/\lambda\}$. Therefore, $L_{\tau^*_{con}}(a) = 0$ for all $a \in N^1_{e^*} \backslash \{K/\lambda\}$. We then have

$$\begin{aligned} &\sum_{a \in N^1_{e^*}} ((e^*(a+))' - (e^*(a-))') \int_0^{\tau_b \wedge \tau_{cal} \wedge \tau^*_{con}} e^{-ru} dL_u(a) \\ = \;& ((e^*(K/\lambda+))' - (e^*(K/\lambda-))') \int_0^{\tau_b \wedge \tau_{cal} \wedge \tau^*_{con}} e^{-ru} dL_u(K/\lambda) \leq 0. \end{aligned}$$

Furthermore, the boundedness of the derivative of e^* implies that

$$\int_0^{\tau_b \wedge \tau_{cal} \wedge \tau^*_{con}} (e^*(V_u))' \cdot e^{-ru} \cdot \sigma V_u dW_u$$

will be a martingale. Take expectations on both sides of (B.2). Some term rearrangement will lead to

$$e^*(V) \geq \mathbb{E}\left[\int_0^{\tau_b \wedge \tau_{cal} \wedge \tau^*_{con}} e^{-ru} \mathcal{L} e^*(V_u) \mathbf{1}_{\{V_u \notin N_{e^*}\}} du + e^{-r(\tau_b \wedge \tau_{cal} \wedge \tau^*_{con})} e^*(V_{\tau_b \wedge \tau_{cal} \wedge \tau^*_{con}}) | V_0 = V\right]. \tag{B.3}$$

From Condition 5, we also know

$$\mathcal{L} e^*(V_u) \geq \delta V_u - (1-\kappa) P c$$

for all $0 \leq u < \tau^*_{con}$. By Conditions 1 and 2,

$$\begin{aligned} e^*(V_{\tau_b \wedge \tau_{cal} \wedge \tau^*_{con}}) &= e^*(V_{\tau^*_{con}}) \mathbf{1}_{\{\tau^*_{con} \leq \tau_b \wedge \tau_{cal}\}} + e^*(V_{\tau_b \wedge \tau_{cal}}) \mathbf{1}_{\{\tau^*_{con} > \tau_b \wedge \tau_{cal}\}} \\ &\geq ((1-\lambda) V_{\tau^*_{con}}) \mathbf{1}_{\{\tau^*_{con} \leq \tau_b \wedge \tau_{cal}\}} + h(V_{\tau_b \wedge \tau_{cal}}) \mathbf{1}_{\{\tau^*_{con} > \tau_b \wedge \tau_{cal}\}}. \end{aligned}$$

Substituting the above inequality into the right hand side of (B.3), we know that $e^*(V) \geq E(V; \tau_b, \tau_{cal}; \tau^*_{con})$. Similarly, we can show that $d^*(V) \geq D(V; \tau^*_b, \tau^*_{cal}; \tau_{con})$ for all $\tau_{con} \in \mathcal{T}$.

Next we shall prove that the optimal equity value is achievable by τ^*_b and τ^*_{cal}, i.e.,

$$e^*(V) = E(V; \tau^*_b, \tau^*_{cal}; \tau^*_{con}).$$

Plug τ^*_b and τ^*_{cal} into equation (B.2). For any $u < \tau^*_b \wedge \tau^*_{cal} \wedge \tau^*_{con}$, $V_u \in \mathcal{S}^c_D \cap \mathcal{S}^c_E$ according to the definitions of τ^*_b, τ^*_{cal} and τ^*_{con}. When $K/\lambda \in \mathcal{S}^c_E$, by Condition 5, we have $(e^*(K/\lambda+))' = (e^*(K/\lambda-))'$; when $K/\lambda \notin \mathcal{S}^c_E$, $L_{\tau^*_b \wedge \tau^*_{cal} \wedge \tau^*_{con}}(K/\lambda) = 0$. Therefore, no matter which scenario it is,

$$((e^*(K/\lambda+))' - (e^*(K/\lambda-))') \int_0^{\tau_b \wedge \tau_{cal} \wedge \tau^*_{con}} e^{-ru} dL_u(K/\lambda) = 0.$$

According to Lemma B.1, $\mathcal{L}d^*(V_u) = Pc$ and $\mathcal{L}e^*(V_u) = \delta V - (1-\kappa)Pc$. Combining all these facts with (B.2),

$$e^*(V) = \mathbb{E}\left[\int_0^{\tau_b^*\wedge\tau_{cal}^*\wedge\tau_{con}^*} e^{-ru}Pcdu + e^{-r(\tau_b^*\wedge\tau_{cal}^*\wedge\tau_{con}^*)}e^*(V_{\tau_b^*\wedge\tau_{cal}^*\wedge\tau_{con}^*})|V_0 = V\right].$$

By the definitions of τ_b^* and τ_{cal}^*,

$$e^*(V_{\tau_b^*\wedge\tau_{cal}^*\wedge\tau_{con}^*}) = ((1-\lambda)V_{\tau_{con}^*})\mathbf{1}_{\{\tau_{con}^*\le\tau_b^*\wedge\tau_{cal}^*\}} + h(V_{\tau_b^*\wedge\tau_{cal}^*})\mathbf{1}_{\{\tau_{con}^*>\tau_b\wedge\tau_{cal}\}}.$$

This finishes the proof of the equity part. The justification of the bond value parallels. $\square$

C PROOF OF THEOREM 3.2

Lemma C.1 *Suppose that two functions d^* and e^* satisfy the conditions in Theorem 3.2. If we define the sets $\mathcal{S}_D$ and $\mathcal{S}_E$ through these two functions, then*

(C.1) $\mathcal{S}_D \subset [cP/(\delta\lambda), +\infty)$,

(C.2) $\mathcal{S}_E \cap [0, K] \subset (0, \min(K, (1-\kappa)cP/\delta)]$,

(C.3) $\mathcal{S}_E \cap (K, K/\lambda) \subset (K, (1-\kappa)cP/(\delta\lambda))$, *if* $\mathcal{S}_E \cap (K, K/\lambda) \neq \emptyset$,

(C.4) $\mathcal{S}_E \cap (K/\lambda, +\infty) \subset (K/\lambda, (1-\kappa)cP/(\lambda\delta)]$, *if* $\mathcal{S}_E \cap (K/\lambda, +\infty) \neq \emptyset$.

Proof. Consider the set $\mathcal{S}_D$ first. From Conditions 4 and 6, we can see that $N_{d^*}^1 \cap \mathcal{S}_D = \emptyset$. For any $V \in \mathcal{S}_D$, there exists a sequence of $\{V_N\}$ such that $V_N \to V$ and V_N satisfies

$$d^*(V_N) = \lambda V_N \text{ and } \mathcal{L}d^*(V_N) > Pc.$$

For each V_N, it must be a local minimum of the function $d^*(v) - \lambda v$ because $d^*(v) \ge \lambda v$ for all $v \ge 0$. Note $V_N \notin N_{d^*}^1$. The smoothness of function d^* at V_N implies that

$$\frac{d}{dv}d^*(v)|_{v=V_N} = \lambda.$$

Furthermore, using the Taylor expansion of the function $d^*(v) - \lambda v$ at V_N, for any $v \le V_N$ there exists a $\theta \in (v, V_N)$ such that

$$0 \le (d^*(v) - \lambda v) - (D^*(V_N) - \lambda V_N) = \frac{d^2}{dv^2}d^*(\theta)\cdot(V_N - v)^2$$

Hence, we have $\frac{d^2}{dv^2}d^*(V_N-) \ge 0$ if letting $v \uparrow V_N$ and $\mathcal{L}d^*(V_N-) \le \delta\lambda V$. From $\mathcal{L}d^*(V_N) > Pc$, we can see $V_N > cP/(\delta\lambda)$. Therefore $V \in [cP/(\delta\lambda), +\infty)$. We have proven (C.1).

The proof of (C.2)-(C.4) is quite similar, and we omit the details in the interest of space. □

Proof of Theorem 3.2. We now prove part (i). First we claim that $\mathcal{S}_D \neq \emptyset$. Suppose not. Lemma C.1 implies $\mathcal{S}_E \subset [0, (1-\kappa)cP/(\lambda\delta)]$. We then infer that $d^*(V)$ satisfies the ODE

$$\mathcal{L}d^*(V) = cP \qquad \text{in } V > (1-\kappa)cP/(\lambda\delta).$$

According to Appendix A, the ODE admits a general solution in the form of

$$d^*(V) = \frac{cP}{r} + c_1 V^{\beta} + c_2 V^{-\gamma},$$

where $\beta > 1$ and $\gamma > 0$. On the other hand, since d^* is an element in U, its derivative should be bounded. Therefore, $c_1 = 0$. As V tends to $+\infty$, d^* converges to cP/r. This contradicts the condition $d^*(V) \geq \lambda V$ for sufficiently large V. Therefore, $\mathcal{S}_D \neq \emptyset$.

Second, we show that if some $V_1 \in \mathcal{S}_D$, then $[V_1, +\infty) \subset \mathcal{S}_D$. Following the arguments leading to the conclusion $\mathcal{S}_D \neq \emptyset$, we can see that there is an unbounded, monotonically increasing sequence $\{\tilde{V}_N\}$ such that $d^*(\tilde{V}_N) = \lambda \tilde{V}_N$ for all N. It suffices to prove that $(V_1, \tilde{V}_N) \subset \mathcal{S}_D$ for any N. We can easily see that $(V_1, \tilde{V}_N) \subset \mathcal{S}_E^c$ from Lemma C.1. Thus, $d^*(V)$ should satisfy the variational inequality problem

$$\text{(C.5)} \qquad \begin{cases} \min\{\mathcal{L}d^*(V) - cP, d^*(V) - \lambda V\} = 0 \text{ in } (V_1, \tilde{V}_N) \\ d^*(V_1) = \lambda V_1, \ d^*(\tilde{V}_N) = \lambda \tilde{V}_N. \end{cases}$$

By (C.1), $V_1 \geq cP/(\delta\lambda)$, thus $V \geq cP/(\delta\lambda)$ for all $V \in (V_1, \tilde{V}_N)$. It indicates that

$$\mathcal{L}(\lambda V) - cP = \delta\lambda V - cP \geq 0 \text{ in } (V_1, \tilde{V}_N).$$

As a result, λV is a supersolution to problem (C.5), i.e., $d^*(V) \leq \lambda V$ in $(V_1, \tilde{V}_N)$. This leads to the desired results $d^*(V) = \lambda V$ and $\mathcal{L}d^*(V) - cP \geq 0$ in $(V_1, \tilde{V}_N)$. Let $V^*_{con} = \inf\{V : V \in \mathcal{S}_D\}$. Then we have $\mathcal{S}_D = [V^*_{con}, +\infty)$ and $V^*_{con} \geq cP/(\delta\lambda)$ because of (C.1).

We now move to the proof of part (ii). The nonemptyness of $\mathcal{S}_E \cap [0, K]$ can be proved in a similar way as what we did for $\mathcal{S}_D$. Indeed, if $\mathcal{S}_E \cap [0, K] = \emptyset$, then we can infer

$$\mathcal{L}e^*(V) = \delta V - (1-\kappa)cP \qquad \text{in } V < \min\{K, (1-\kappa)cP/\delta\}.$$

Again, based on the general solution presented in Appendix A, we have

$$e^*(V) = V - \frac{(1-\kappa)cP}{r} + c_3 V^{\beta} \to -\frac{(1-\kappa)cP}{r} \text{ as } V \to 0,$$

a contradiction! Using a similar argument as in the proof of part (i), we also can show that if some $V_1 \in \mathcal{S}_E \cap [0,K]$, then $[0,V_1) \subset \mathcal{S}_E \cap [0,K]$. Define $V_b^* = \sup\{V : V \in \mathcal{S}_E \cap [0,K]\}$. The bankruptcy region $\mathcal{S}_E \cap [0,K] = [0,V_b^*]$. In addition, $V_b^* \le \min(K,(1-\kappa)cP/\delta)$ because of (C.2).

Part (iii) is clear to see from the proof of Lemma C.1 and it remains to show (iv) when $\mathcal{S}_E \cap (K,+\infty) \neq \emptyset$. In this case, either $\mathcal{S}_E \cap (K, K/\lambda]$ or $\mathcal{S}_E \cap (K/\lambda,+\infty)$ is not empty. We only focus on the first case $\mathcal{S}_E \cap (K,K/\lambda] \neq \emptyset$ and the other case can be treated in a similar way. Suppose $V_1 \in \mathcal{S}_E \cap (K,K/\lambda]$. We might as well assume $V_1 \neq K/\lambda$. To show (iv), it suffices to prove $e^*(V) = V - K$ for all $x \in (V_1, K/\lambda]$. Owing to part (i) and (C.3), we have $V_{con}^* > K/\lambda$ and $e^*(V_{con}^*) = (1-\lambda)V_{con}^* < V_{con}^* - K$. Noticing $e^*(V) \ge V - K$ in $V \in (V_1, K/\lambda]$, we then have that there exists a point $V_2 \in [K/\lambda, V_{con}^*)$ such that $e^*(V_2) = V_2 - K$ by the continuity of e^*. Consider the interval (V_1, V_2), in which $e^*(V)$ should be governed by the variational inequality problem

$$\text{(C.6)} \qquad \begin{cases} \min\{\mathcal{L}e^*(V) - \delta V + (1-\kappa)cP, e^*(V) - h(V)\} = 0, \\ e^*(V_1) = V_1 - K, \ e^*(V_2) = V_2 - K. \end{cases}$$

Thanks to (C.3), we know that $K \le (1-\kappa)cP/\delta$. Thus

$$\mathcal{L}(V-K) - \delta V + (1-\kappa)cP = -rK + (1-\kappa)cP \ge 0$$

which, combined with $V - K \ge h(V)$ in (V_1,V_2), implies that the function $V-K$ is a supersolution to the problem (C.6) in (V_1,V_2). That is, $e^*(V) \le V - K$ for all $V \in (V_1,V_2)$. We then deduce that $e^*(V) = V - K$ for all $V \in (V_1, K/\lambda]$. The proof is complete. □

D DETERMINATION OF CRITICAL POINTS K_1, K_2 AND K_3

Section 4 shows that the actions of both bondholder and the shareholder depend on the value of the call price K. Three critical points K_i, $1 \le i \le 3$, distinguish different equilibrium behaviors of the game. We will demonstrate how to determine these K_i's in this appendix. Most of the arguments used in this appendix rely on basic calculus, which is irrelevant with the main theme of the paper. Therefore, we only provide a sketch here and skip detailed arguments. A more complete discussion can be found in the third author's PhD thesis (Wan (2010)).

Consider the equations (4.2) and (4.3). They involve V_b^* and V_{con}^* as unknowns. The following lemma proves that we can solve these two equations to determine V_b^* and V_{con}^* uniquely.

Lemma D.1 *There exist unique V_b^* and V_{con}^* satisfying (4.2) and (4.3) simultaneously. In addition, the solution $V_b^* \le (1-\kappa)Pc/\delta$ and $V_{con}^* > Pc/(\delta\lambda)$.*

Proof. Note that solving for V_b^* and V_{con}^* from (4.2) and (4.3) is equivalent to solving for V_b^* and V_b^*/V_{con}^*. Substituting the expressions of D_1 and E_1 into (4.2) and (4.3), we can see that V_b^* and V_b^*/V_{con}^* should satisfy

$$V_b^* = \frac{(1-\kappa)Pc}{r} \cdot \frac{\gamma + \beta(V_b^*/V_{con}^*)^{\beta+\gamma} - (\beta+\gamma)(V_b^*/V_{con}^*)^{\beta}}{(\gamma+1) + (\beta-1)(V_b^*/V_{con}^*)^{\beta+\gamma} - \lambda(\beta+\gamma)(V_b^*/V_{con}^*)^{\beta-1}} \tag{D.1}$$

and V_b^*/V_{con}^* should be a solution to the following equation

$$\begin{aligned} 0 &= x(\beta + \gamma x^{\beta+\gamma} - (\beta+\gamma)x^{\gamma})((\gamma+1) + (\beta-1)x^{\beta+\gamma} - \lambda(\beta+\gamma)x^{\beta-1}) \\ &\quad -(1-\kappa)(\gamma + \beta x^{\beta+\gamma} - (\beta+\gamma)x^{\beta})(\lambda(\beta-1) + \lambda(\gamma+1)x^{\beta+\gamma} - (1-\rho)(\beta+\gamma)x^{\gamma+1}). \end{aligned} \tag{D.2}$$

Using basic calculus will reveal that the equation (D.2) admits a unique solution $x_1^* \in (0,1)$. Based on such x_1^*, we can obtain

$$\begin{cases} V_b^* = \frac{(1-\kappa)cP}{r} \frac{\gamma+\beta(x_1^*)^{\beta+\gamma} - (\beta+\gamma)(x_1^*)^{\beta}}{(\gamma+1)+(\beta-1)(x_1^*)^{\beta+\gamma} - \lambda(\beta+\gamma)(x_1^*)^{\beta-1}}, \\ V_{con}^* = \frac{cP}{r} \frac{\beta+\gamma(x_1^*)^{\beta+\gamma} - (\beta+\gamma)(x_1^*)^{\gamma}}{\lambda(\beta-1)+\lambda(\gamma+1)(x_1^*)^{\beta+\gamma} - (1-\rho)(\beta+\gamma)(x_1^*)^{\gamma+1}}. \end{cases}$$

Notice that x_1^* is the unique solution to (D.2). We can prove that $V_b^* \leq (1-\kappa)Pc/\delta$ and $V_{con}^* > Pc/(\lambda\delta)$ from the representation of V_b^* and V_{con}^*. $\square$

Given V_b^* and V_{con}^* solved from (4.2) and (4.3), consider an equation

$$E_1^*(V; V_b^*, V_{con}^*) = (1-\lambda)V. \tag{D.3}$$

The following lemma shows that it admits a unique solution and we can construct the first critical point K_1 based on that solution.

Lemma D.2 *Equation (D.3) has at most a unique solution in the interval (V_b^*, V_{con}^*). Denote it to be k_1 if such solution exists or set $k_1 = V_{con}^*$ otherwise. Let $K_1 = \lambda k_1$. Then, when $K \geq K_1$, we have*

$$E_1^*(V; V_b^*, V_{con}^*) \geq h(V) \quad \textit{and} \quad D_1^*(V; V_b^*, V_{con}^*) \geq \lambda V$$

for any $V \in [V_b^*, V_{con}^*]$, where V_b^* and V_{con}^* are determined by (4.2) and (4.3).

Proof. When $V = V_b^*$ and V_{con}^*, we have

$$E_1(V_b^*; V_b^*, V_{con}^*) = 0 \quad \text{and} \quad E_1(V_{con}^*; V_b^*, V_{con}^*) = (1-\lambda)V_{con}^*.$$

Thus, $V = V_{con}^*$ is one root to the equation (D.3). Furthermore, routine calculation reveals that there is only one root in (V_b^*, V_{con}^*) for this equation if it exists. Let K_1 be the value defined in the theorem. Then one can establish

$$E_1(V; V_b^*, V_{con}^*) \geq \min((V - K_1)^+, (1-\lambda)V)$$

for $V \in (V_b^*, V_{con}^*)$, making use of the definition of K_1. In addition, it is easy to see that

$$\min((V - K_1)^+, (1-\lambda)V) \geq \min((V - K)^+, (1-\lambda)V) =: h(V)$$

when $K \geq K_1$. As to the bond value function, after some tedious calculation, we have

$$\frac{\partial D_1}{\partial V_{con}}(V; V_b^*, V_{con}) \geq 0$$

for any V and V_{con} such that $V < V_{con} \leq V_{con}^*$. This implies that $D_1(V; V_b^*, V_{con})$ is an increasing function with respect to V_{con}. In particular,

$$\lambda V = D_1(V; V_b^*, V) \leq D_1(V; V_b^*, V_{con}^*). \square$$

In Section 4.2, we use equations (4.4) and (4.5) to determine $V_{cal,1}^*$ and $V_{cal,2}^*$. The following lemma validates such claim.

Lemma D.3 *(i). For each $K < (1-\kappa)Pc/r$, equation (4.4) has unique solutions V_b^* and $V_{cal,1}^*$. In addition, the solutions satisfy $V_b^* < \min\{K, (1-\kappa)Pc/\delta\}$ and $V_{cal,1}^* > K$.*
(ii). Equation (4.5) yields unique solutions $V_{cal,2}^*$ and V_{con}^*, where $V_{cal,2}^* < (1-\kappa)Pc/(\lambda\delta)$ and $V_{con}^* > Pc/(\lambda\delta)$.

Proof. (i). Substituting the expressions of E_2 and D_2 into equations (4.4) and doing some transformation, we have that $x = V_b^*/V_{cal,1}^*$ should satisfy

$$K = \frac{(1-\kappa)cP}{r} \cdot \frac{\beta(\gamma+1) - (\beta-1)\gamma x^{\beta+\gamma} - (\beta+\gamma)x^{\gamma}}{\beta(\gamma+1) - (\beta-1)\gamma x^{\beta+\gamma}}.$$

One can show that the above equation has a unique solution $x_2^* \in (0,1)$ if $K < (1-\kappa)cP/r$. Then, based on this solution, V_b^* and $V_{cal,1}^*$ are constructed as follows:

$$\begin{cases} V_b^* = K \cdot \frac{\beta\gamma(1-(x_2^*)^{\beta+\gamma})}{\beta(\gamma+1)-(\beta-1)\gamma(x_2^*)^{\beta+\gamma}-(\beta+\gamma)(x_2^*)^{\gamma}}, \\ V_{cal,1}^* = K \cdot \frac{\beta\gamma(1-(x_2^*)^{\beta+\gamma})}{x_2^*(\beta(\gamma+1)-(\beta-1)\gamma(x_2^*)^{\beta+\gamma}-(\beta+\gamma)(x_2^*)^{\gamma})}. \end{cases}$$

They solve equation (4.4). Furthermore, from the presentation of these solutions we can verify that $V_b^* < \min\{K, (1-\kappa)Pc/\delta\}$ and $V_{cal,1}^* > K$.
(ii). The proof of this part is quite similar to that of the previous part. Manipulating (4.5) will yield a conclusion that if the solutions to (4.5) exist, the ratio $x = V_{cal,2}^*/V_{con}^*$ should satisfy an equation

$$\begin{aligned} & x(\beta + \gamma x^{\beta+\gamma} - (\beta+\gamma)x^{\gamma})((\gamma+1) + (\beta-1)x^{\beta+\gamma} - (\beta+\gamma)x^{\beta-1}) \\ & - (1-\kappa)(\gamma + \beta x^{\beta+\gamma} - (\beta+\gamma)x^{\beta})((\beta-1) + (\gamma+1)x^{\beta+\gamma} - (\beta+\gamma)x^{\gamma+1}) = 0. \end{aligned}$$

On the other hand, the above equation has a unique solution $x_4^* \in (0,1)$. Based on x_4^*, let

$$\begin{cases} V_{cal,2}^* = \frac{(1-\kappa)cP}{\lambda r} \cdot \frac{\gamma+\beta(x_4^*)^{\beta+\gamma} - (\beta+\gamma)(x_4^*)^{\beta}}{(\gamma+1)+(\beta-1)(x_4^*)^{\beta+\gamma} - (\beta+\gamma)(x_4^*)^{\beta-1}}, \\ V_{con}^* = \frac{cP}{\lambda r} \cdot \frac{\beta+\gamma(x_4^*)^{\beta+\gamma} - (\beta+\gamma)(x_4^*)^{\gamma}}{(\beta-1)+(\gamma+1)(x_4^*)^{\beta+\gamma} - (\beta+\gamma)(x_4^*)^{\gamma+1}}. \end{cases}$$

Such $V_{cal,2}^*$ and V_{con}^* solve (4.5). In addition, tedious calculation leads to $V_{cal,2}^* < (1-\kappa)Pc/(\lambda\delta)$ and $V_{con}^* > Pc/(\lambda\delta)$. □

However, $V_{cal,1}^*$ and $V_{cal,2}^*$ obtained through (4.4) and (4.5) may not satisfy the requirement that $V_{cal,1}^* \le K/\lambda \le V_{cal,2}^*$. These inequalities turn out to be true only for small K; that is, there exist critical points K_2 and K_3 such that $V_{cal,1}^* < K/\lambda$ when $K < K_2$ and $V_{cal,2}^* > K/\lambda$ when $K < K_3$.

Turn to the determination of K_2 first. Note that the expression of E_2 contains K. Hence, we can view $V_{cal,1}^*$ obtained via (4.4) as a function of K. The next lemma proves that there exists a unique K to the equation

$$V_{cal,1}^*(K) = K/\lambda. \tag{D.4}$$

Denote this root to be K_2.

Lemma D.4 *Equation (D.4) admits a unique solution K_2 satisfying $K_2 < K_1$. When $K < K_2$, we have $V_{cal,1}^* < K/\lambda$. Furthermore,*

$$E_2(V; V_b^*, V_{cal,1}^*) \ge (V-K)^+ \quad \textit{and} \quad D_2(V; V_b^*, V_{cal,1}^*) \ge \lambda V$$

for all $V \in [V_b^*, V_{cal,1}^*]$, where V_b^* and $V_{cal,1}^*$ are determined by (4.4).

Proof. According to the proof of part (i) in Lemma D.3, we can define x_2^* uniquely for any given K and then use it to obtain $V_{cal,1}^*$. Equation (D.4) is equivalent to $f(x_2^*) = \lambda$, where

$$f(x) = \frac{x(\beta(\gamma+1) - (\beta-1)\gamma x^{\beta+\gamma} - (\beta+\gamma)x^{\gamma})}{\beta\gamma(1-x^{\beta+\gamma})}.$$

On the other hand, it is easy to show that $f(x) = \lambda$ admits a unique solution $x_3^* \in (0,1)$. Letting

$$K = \frac{(1-\kappa)Pc}{r} \cdot \frac{\beta(\gamma+1) - (\beta-1)\gamma(x_3^*)^{\beta+\gamma} - (\beta+\gamma)(x_3^*)^{\gamma}}{\beta(\gamma+1) - (\beta-1)\gamma(x_3^*)^{\beta+\gamma}},$$

we can prove that the $V_{cal,1}^*$ determined by this K will be the unique root to (D.4). This K is exactly the critical point K_2 we look for. In addition, we also can verify that $K_2 < K_1$ and when $K < K_2$, $V_{cal,1}^* < K/\lambda$.

Consider any call price $K < K_2$ and V_b^* and $V_{cal,1}^*$ defined by this K. Suppose that V is any number in the interval $(V_b^*, V_{cal,1}^*)$. Fixing V and $V_{cal,1}^*$, we can show that the partial derivative of E_2 with respect to V_b

$$\frac{\partial E_2}{\partial V_b}(V; V_b, V_{cal,1}^*) < 0$$

for $V_b^* \leq V_b \leq V$. In other words, E_2 is decreasing in V_b for $V_b \geq V_b^*$. In particular, $E_2(V; V_b^*, V_{cal,1}^*) \geq E_2(V; V, V_{cal,1}^*) = 0$. Similarly, the partial derivative of E_2 with respect to $V_{cal,1}$, if we fix V and V_b^*, should satisfy

$$\frac{\partial E_2}{\partial V_{cal,1}}(V; V_b^*, V_{cal,1}) > 0$$

for $V \leq V_{cal,1} \leq V_{cal,1}^*$. This implies $E_2(V; V_b^*, V_{cal,1}^*) \geq E_2(V; V_b^*, V) = V - K$. We have already established $E_2(V; V_b^*, V_{cal,1}^*) \geq (V-K)^+$. In a similar manner, one can show $D_2(V; V_{cal,2}^*, V_{con}^*) \geq \lambda V$. □

Finally we will present how to determine K_3. Note that both D_3 and E_3 are independent of K. Hence $V_{cal,2}^*$ and V_{con}^* obtained through (4.5) do not depend on K either. Let $K_3 = \lambda V_{cal,2}^*$. We have

Lemma D.5 *For any given $K < K_3$, $K/\lambda < V_{cal,2}^*$. In addition,*

$$E_3(V; V_{cal,2}^*, V_{con}^*) \geq (1-\lambda)V \quad \textit{and} \quad D_3(V; V_{cal,2}^*, V_{con}^*) \geq \lambda V$$

if $V \in (V_{cal,2}^*, V_{con}^*)$.

Proof. The proof of this lemma parallels with that of Lemma D.4. We omit the details. □

For $K \in (K_2, K_1)$ or (K_3, K_1), the endpoints of $\mathcal{S}_{EC}$, $V_{cal,1}^*$ and $V_{cal,2}^*$, will degenerate to K/λ respectively. If $V_{cal,1}^* = K/\lambda$, we can use

$$\text{(D.5)} \qquad \frac{dE^*}{dV}(V)|_{V=V_b^*} = \frac{dE_2}{dV}(V_b^*; V_b^*, K/\lambda) = 0$$

to determine the optimal bankrupt boundary V_b^*. If $V_{cal,2}^* = K/\lambda$, we can use

$$\frac{dD^*}{dV}(V)|_{V=V_{con}^*} = \frac{dD_3}{dV}(V_{con}^*; K/\lambda, V_{con}^*) = \lambda \tag{D.6}$$

to find the conversion boundary V_{con}^*. We summarize some related properties of E^* and D^* under such intermediate-sized K in the following lemma for later reference.

Lemma D.6 *(i). When $K \in (K_2, K_1)$, we can find a unique $V_b^* < \min\{K, (1-\kappa)Pc/\delta\}$ satisfying (D.5). Furthermore, once we substitute such V_b^* in the functions E_2 and D_2,*

$$E_2(V; V_b^*, K/\lambda) \geq (V-K)^+ \quad \textit{and} \quad D_2(V; V_b^*, K/\lambda) \geq \lambda V$$

for $V \in [V_b^*, K/\lambda]$.
(ii). Suppose that $K_3 < K_1$. For any $K \in [K_3, K_1)$, equation (D.6) yields a unique solution V_{con}^*. $V_{con}^* > cP/(\delta\lambda)$. In addition,

$$E_3(V; K/\lambda, V_{con}^*) > (1-\lambda)V \quad \textit{and} \quad D_3(V; K/\lambda, V_{con}^*) > \lambda V$$

for any $V \in (K/\lambda, V_{con}^*)$.
(iii). Furthermore, when $\max\{K_2, K_3\} < K < K_1$, $E_2'(K/\lambda-; V_b^*, K/\lambda) > E_3'(K/\lambda+; K/\lambda, V_{con}^*)$.

Proof. (i). Suppose that V_b^* is a root to equation (D.5). Then, the ratio of $x = V_b^*/(K/\lambda)$ solves

$$\frac{K}{\lambda}x((\gamma+1) + (\beta-1)x^{\beta+\gamma} - \lambda(\beta+\gamma)x^{\beta-1}) - \frac{(1-\kappa)cP}{r}(\gamma + \beta x^{\beta+\gamma} - (\beta+\gamma)x^{\beta}) = 0.$$

On the other hand, the above equation has a unique solution such that $x_5^* \in (0,1)$. Hence, letting

$$V_b^* = \frac{(1-\kappa)cP}{r} \cdot \frac{\gamma + \beta(x_5^*)^{\beta+\gamma} - (\beta+\gamma)(x_5^*)^{\beta}}{(\gamma+1) + (\beta-1)(x_5^*)^{\beta+\gamma} - \lambda(\beta+\gamma)(x_5^*)^{\beta-1}},$$

it is the unique solution to (D.5).
(ii). Letting $x = (K/\lambda)/V_{con}^*$, by equation (D.6), x solves

$$\frac{cP}{r}x\left(\beta + \gamma x^{\beta+\gamma} - (\beta+\gamma)x^{\gamma}\right) - K\left(\lambda\left((\beta-1) + (\gamma+1)x^{\beta+\gamma}\right) - (\beta+\gamma)x^{\gamma+1}\right) = 0,$$

which has a unique solution $x_6^* \in (0,1)$ provided $K < cP/\lambda$. And then

$$V_{con}^* = \frac{cP}{\lambda r}\frac{\beta + \gamma(x_6^*)^{\beta+\gamma} - (\beta+\gamma)(x_6^*)^{\gamma}}{(\beta-1) + (\gamma+1)(x_6^*)^{\beta+\gamma} - (\beta+\gamma)(x_6^*)^{\gamma+1}}.$$

The other statements of the Lemma are verified term by term via the similar arguments as the proof of Lemma D.3.
(iii). The verification of this statement comes from the straightforward calculation. □

E PROOF OF THEOREMS 4.1 AND 4.2

Proof of Theorem 4.1. To show this theorem, it is sufficient to prove that the functions E^* and D^* satisfy the system 1-6.

We know that $\mathcal{S}_D = [V_{con}^*, +\infty)$ and $\mathcal{S}_E = [0, V_b^*]$ from the definitions of E^* and D^*. It is easy to see that Condition 6 is met. On $\mathcal{S}_E^c$, $D^*(V)$ satisfies either

$$\mathcal{L}D^*(V) = Pc, \ D^*(V) > \lambda V \quad \text{for } V \in (V_b^*, V_{con}^*)$$

or

$$\mathcal{L}D^*(V) \geq Pc, \ D^*(V) = \lambda V \quad \text{for } V \in (V_{con}^*, +\infty).$$

Therefore, Condition 4 holds for D^*. Similarly, we can check that E^* should satisfy Condition 5.

Lemma D.2 in the last appendix shows that $E^*(V) \geq h(V)$ and $D^*(V) \geq \lambda V$ for all $V \geq 0$. We have Condition 1. The verification of Conditions 2 and 3 is trivial. So far, we have shown that E^* and D^* satisfy the variational inequality system. According to Theorem 3.1, they constitutes the Nash equilibrium of the game. □

Proof of Theorem 4.2. We can use the lemmas in Appendix D to verify that the functions E^* and D^* defined in the theorem satisfy the system 1-6. The details are omitted in the interest of space since they are very similar as the arguments in the proof of Theorem 4.1. □

Finally, we present a heuristic derivation to analyze the meaning of (4.2). Applying Ito's lemma to E^*,

$$dE^*(V_t) = \frac{\partial E^*}{\partial V} dV_t + \frac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 E^*}{\partial V^2} dt = \frac{\partial E_1}{\partial V} dV_t + \frac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 E_1}{\partial V^2} dt.$$

According to (4.2),

$$\frac{\partial E_1}{\partial V}(V_b^*) = 0$$

and the expression of E_1 yields that

$$\frac{1}{2}\sigma^2 V^2 \frac{\partial^2 E_1}{\partial V^2}(V)\Big|_{V=V_b^*} = (1-\kappa)Pc - \delta V_b^*.$$

Consequently,

$$\mathbb{E}[dE^*(V_t)|V_0 = V_b^*] = [(1-\kappa)Pc - \delta V_b^*]dt.$$

The left hand side of the above equality can be interpreted as the expected appreciation in the equity value at the default boundary $V_t = V_b^*$ if the shareholder puts off the default to a moment later. The right hand side is the additional cash flow required from him to keep the company solvent, which is the difference between the after-tax coupon payment and the cash flow available for paying out by liquidating a portion of the company's asset. From this equality, we can see that the smooth pasting condition (4.2) implies that at $V = V_b^*$, equity appreciation just equals the amount of cash flow that must be provided by the shareholder to meet the debt obligation. Hence, he should choose to announce a default when $V_t = V_b^*$ because it will not be an attractive option any longer to continue contributing new capital to make the company run.

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