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Robust Risk Quantification via Shock Propagation in Financial Networks

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Abstract. Given limited network information, we consider robust risk quantification under the Eisenberg–Noe model for financial networks. To be more specific, motivated by the fact that the structure of the interbank network is not completely known in practice, we propose a robust optimization approach to obtain worst-case default probabilities and associated capital requirements for a specific group of banks (e.g., systemically important financial institutions) under network information uncertainty. Using this tool, we analyze the effects of various incomplete network information structures on these worst-case quantities and provide regulatory insights into the collection of actionable network information. All claims are numerically illustrated using data from the European banking system.

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Keywords: risk quantification • financial network • robust optimization • information uncertainty

1. Introduction

Along with the evolution of interconnectedness in financial systems, the importance of financial institutions' exposure to losses arising from other institutions' defaults has increased significantly in recent years. The more banks are exposed to such losses, the more likely they are to suffer from the domino effects of financial failures. As seen in the financial crisis of 2008, large financial institutions may face severe financial stress in quick succession. Another example is the Greek sovereign debt crisis in 2010 when some European countries failed to make scheduled debt payments to their creditors such as the International Monetary Fund, resulting in significant stress on many European banks (Guerrieri et al. 2012).

Such repeating financial crises have provoked heated academic debates on how to measure, mitigate, and manage the aforementioned risk with a focus on the impact of the financial system's network topology. Above all, Eisenberg and Noe (2001) have played a key role in those discussions. Specifically, the authors provide a clearing mechanism that settles payment obligations of financial institutions based on a fixed-point characterization. This mechanism effectively describes default cascades triggered by interbank liabilities in financial networks, leading to many subsequent studies. To name a few, Elsinger et al. (2006) develop an empirical approach to assess systemic risk, Capponi et al. (2016) analyze the effect of liability concentration on systemic losses, and Cifuentes et al. (2005) and Rogers and Veraart (2013) propose extended models incorporating asset fire sales and bankruptcy costs, respectively.[1] Liu and Staum (2010) and Feinstein et al. (2018) conduct sensitivity analyses of clearing payments, and Barucca et al. (2020) study a network-based valuation model for interbank claims. See Birge et al. (2018) for a good review of this strand of the literature. Interested readers can also consult Gai and Kapadia (2010) and Elliott et al. (2014) for other models of default contagion in the networks.

Nonetheless, only a few papers use the Eisenberg–Noe model to investigate the impact of random losses in banks' assets on their solvency, which is of practical importance to regulators. For example, Chen et al. (2016) find a lower bound of the probability that a shock to a single bank leads to other banks' default. Khabazian and Peng (2019) provide a lower bound of the probability of bankruptcy in the system when all banks receive normally distributed shocks. However, these works are not applicable to the case of multivariate shocks following a general class of probability distributions.

Another limitation of the model is that it requires full information on interbank exposures in the network, which is rarely available in practice as commonly noted in three well-known surveys in this context (Capponi 2016, Glasserman and Young 2016, Benoit et al. 2017). Instead, various types of partial information can be collected. For example, each bank's aggregate interbank exposure can be generally known from the balance sheet. The Basel III framework requires banks to report large interbank exposures, defined as exposures greater than or equal to 10% of their Tier 1 capital (BCBS 2020a, c). Also, the Basel Committee annually investigates the interbank assets and liabilities of the so-called systemically important financial institutions (SIFIs), but not all banks are examined.

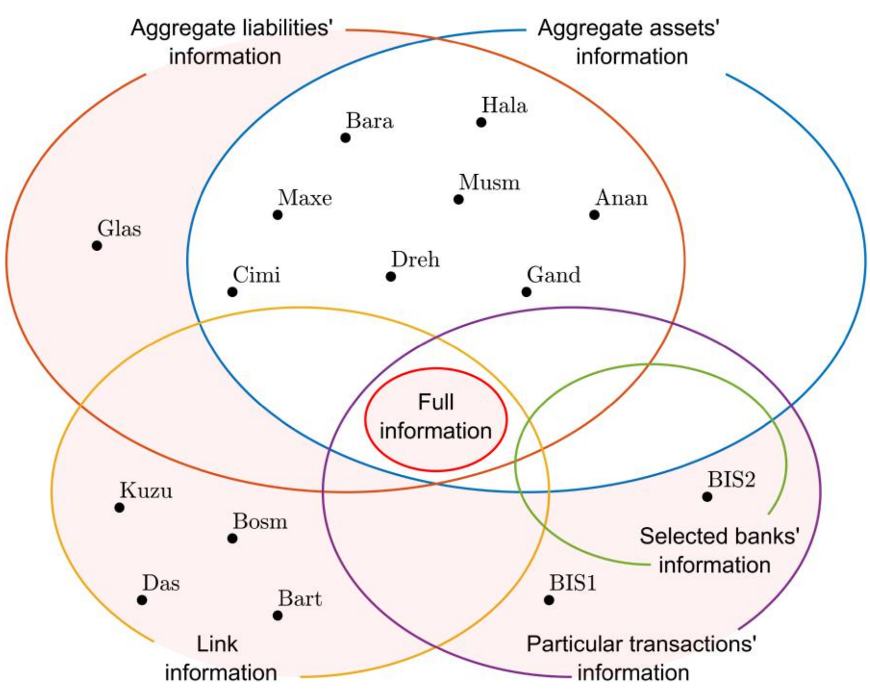
To address the issue of incomplete network information, many studies have developed network reconstruction methods using the aggregate information of interbank assets and liabilities (e.g., Upper and Worms 2004, Baral and Fique 2012, Drehmann and Tarashev 2013, Halaj and Kok 2013, Musmeci et al. 2013, Anand et al. 2015, Cimini et al. 2015, Gandy and Veraart 2017). However, because these methods do not consider random shocks to financial institutions, they do not provide a clear answer to risk quantification (in particular, to the estimation of risk capital). Moreover, Anand et al. (2018) find that none of those methods are generally superior to the other methods and that the results highly depend on the jurisdiction of the data and the performance measure. See Squartini et al. (2018) for a systematic review of recent network reconstruction methods.

Several other works focus on connectedness between banks, which we call link information, to understand each bank's risk contribution. For example, Kuzubaş et al. (2014), Das (2016), Bosma et al. (2019), and Bartesaghi et al. (2020) use the link information to rank the banks according to their systemic importance and identify SIFIs through the notion of network centrality. Still, the effects of random shocks are not discussed in those works.

To our knowledge, Glasserman and Young (2015) is so far the only paper that considers both random shocks and partial network information. They use the aggregate liabilities' information to obtain upper bounds of banks' default probability and expected loss, assuming that random shocks follow a particular type of distribution. However, little is known of a risk quantification method that can accommodate various types of incomplete network information and random shocks.

Motivated by these limitations, this paper studies robust risk quantification based on multivariate random shocks to financial institutions and incomplete network information under the Eisenberg–Noe model. In particular, we provide a comprehensive framework for estimating the worst-case default probabilities under different kinds of network information. In Figure 1, we illustrate the network information structures applicable to our

Figure 1. (Color online) The Network Information Structures that this Work Covers



Notes. The shaded area represents the network information that we address. The existing literature indicated here includes Upper and Worms (2004) (Maxe), Baral and Fique (2012) (Bara), Drehmann and Tarashev (2013) (Dreh), Halaj and Kok (2013) (Hala), Musmeci et al. (2013) (Musm), Kuzubaş et al. (2014) (Kuzu), Anand et al. (2015) (Anan), Cimini et al. (2015) (Cimi), Glasserman and Young (2015) (Glas), Das (2016) (Das), Gandy and Veraart (2017) (Gand), Bosma et al. (2019) (Bosm), Bartesaghi et al. (2020) (Bart), BCBS (2020a) (BIS1), and BCBS (2020c) (BIS2).

framework and classify the aforementioned studies based on the partial information they consider. We particularly focus on the default event of a specific set of banks inspired by the fact that regulators pay close attention to the solvency of SIFIs. Indeed, because the collapse of SIFIs could lead to a financial crisis, the Basel Committee developed higher loss absorbency requirements for those institutions to secure the financial system by preventing their bankruptcy (BCBS 2020b). The detailed contributions of this work are as follows.

1. Mixed-integer linear program (MILP) formulation for worst-case default probabilities. We first characterize a bank's default event with respect to a shock vector, assuming that full network information is available. This is then leveraged to formulate a mixed-integer linear program that identifies the worst-case default event of one or more banks in a specific set, taking into account all possible network configurations under limited network information. Our approach facilitates the unbiased estimation of the worst-case default probability and outperforms existing robust optimization methods that provide more conservative solutions or require computationally intractable procedures.

2. Worst-case risk capital. We compute the minimum capital amount required to keep all SIFIs solvent with a certain probability to protect the financial system from falling into a crisis, which we call *worst-case risk capital*.[2] Specifically, we formulate a chance-constrained optimization problem in which the worst-case default probability of SIFIs is controlled by a cash injection strategy. The difficulty of solving the problem is alleviated by the shock propagation mechanism and sample average approximation (SAA).

3. Information sensitivity. We proceed to apply our results to various types of limited network information based on real-world data. For information to be gathered efficiently, we make the following suggestions on regulatory policies.

a. When it comes to regulating some selected banks (e.g., SIFIs), gathering only those banks' information would be one of the best options for regulators to take. Information from the other banks is found to have little impact on improving the worst-case quantities associated with the target banks. Given that data collection can be costly, this result helps effectively reduce the amount of information that regulators need to collect.

b. Another effective option is to collect information on large exposures because they are likely to be the main source of potential shock contagion. Our numerical analysis indicates that this information leads to small gaps between the associated worst-case quantities and the true quantities. This alternative, however, calls for a judicious choice of the threshold for large exposures to balance the cost of gathering information with its effectiveness.

c. In the case, it is difficult to require banks to report their full data; collecting only link information without exact amounts is a sensible alternative, provided that the network has low density. Our study demonstrates that the worst-case quantities under this kind of information are close to the true quantities for sparse networks.[3]

d. It is unlikely that relying on only aggregate information or a part of target banks' information will help. We observe a significant gap between the true quantities and the worst-case quantities under such information.

We note that the observations in contributions (3a) and (3b) share a common thread with Musmeci et al. (2013) and Cimini et al. (2015) because they numerically show the robustness of systemic risk estimates to partial network information. However, in contrast to those works, this paper considers random shocks for risk quantification and investigates how the difference between true and worst-case risk quantities varies according to various types of limited network information.

The remainder of the paper is organized as follows. Section 2 introduces the underlying model and describes the problem formulation. In Section 3, we characterize the region for a shock vector to drive a specific bank to default using the notion of shock propagation. In Section 4, we obtain the worst-case default probabilities, which change according to the level of available information. In Section 5, we apply our results to the computation of the worst-case risk capital under incomplete network information. In Section 6, we verify the practical applicability of our results via numerical experiments using real-world data. Finally, Section 7 concludes the paper. All proofs and the source code used for numerical experiments can be found in the e-companion.

2. Problem Formulation

We begin this section by introducing the basic notations in this paper. All vectors are column vectors, denoted by bold symbols (e.g., $\mathbf{u} = (u_1, \ldots, u_d)^\top \in \mathbb{R}^d$). The Euclidean norm of a vector $\mathbf{u}$ is denoted by $\|\mathbf{u}\|$. We use $\mathbf{1}$, $\mathbf{0}$, and $\mathbf{I}$ for vectors of ones and zeros and the identity matrix in a suitable dimension, respectively. For any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, $\mathbf{u} \le \mathbf{v}$ means an entry-wise inequality, $[\mathbf{u}, \mathbf{v}] = \{\mathbf{x} \in \mathbb{R}^d \,|\, \mathbf{u} \le \mathbf{x} \le \mathbf{v}\}$, $\mathbf{u} \wedge \mathbf{v} = (\min\{u_1, v_1\}, \ldots, \min\{u_d, v_d\})^\top$, and $\mathbf{u}^+ = (\max\{u_1, 0\}, \ldots, \max\{u_d, 0\})^\top$. For any matrix $\mathbf{M}$, we denote by $\mathbf{M}_{-i}$ the matrix obtained by eliminating its ith column and row (similarly defined for vectors). For any two index sets $\mathcal{I}, \mathcal{J} \subset \{1, \ldots, d\}$, $\mathbf{v}_{\mathcal{I}}$ is the vector obtained by restricting the entries of the vector $\mathbf{v} \in \mathbb{R}^d$ to $\mathcal{I}$, and $\mathbf{M}_{\mathcal{I},\mathcal{J}}$ is the matrix obtained by restricting the components of the $d \times d$ matrix $\mathbf{M}$ to $\mathcal{I} \times \mathcal{J}$. If $\mathcal{I} = \mathcal{J}$, we simply write $\mathbf{M}_{\mathcal{I}}$.

2.1. The Model and Basic Assumptions

In this paper, we consider an n-bank financial system, where banks are indexed by $1,\dots,n$, and adopt an extended framework of Eisenberg and Noe (2001). The extension comes from first, the existence of liabilities to entities outside the financial network and second, the inclusion of random shocks to the external asset values and bankruptcy costs. This modeling framework is proposed by Glasserman and Young (2015), and it consists of the following ingredients.

- For each i,j, $\overline{p}_{ij}$ is the payment obligation from bank i to bank j, with $\overline{p}_{ii}=0$.
- For each i, $c_i, b_i \geq 0$ are the external assets and the external liabilities of bank i, respectively.
- Each bank's balance sheet is given by
  1. the asset side: $c_i + \sum_{j\neq i}\overline{p}_{ji}$;
  2. the liability side: $\overline{p}_i := b_i + \sum_{j\neq i}\overline{p}_{ij}$; and
  3. the initial net worth (book value): $w_i := c_i + \sum_{j\neq i}\overline{p}_{ji} - \overline{p}_i$.
- The proportion of bank i's obligation to bank j is defined by $a_{ij} := (\overline{p}_{ij}/\overline{p}_i)\mathbf{1}_{\{\overline{p}_i>0\}}$.
- A random variable X_i is a shock to the external asset c_i.
- The constant $\eta \geq 0$ is a multiplier for bankruptcy costs, which will be introduced shortly.

After the shock, bank i's external asset and its net worth become $c_i - X_i$ and $w_i - X_i$, respectively. Here, $\mathbf{1}_{\{\overline{p}_i>0\}}$ yields one if $\overline{p}_i > 0$ and zero otherwise. We denote the matrix of the relative liabilities by $\mathbf{A} = (a_{ij})$ and the sum of the ith row of $\mathbf{A}$ by β_i. The quantity β_i is the so-called *financial connectivity* of bank i, measuring its reliance on funding sources inside the financial system. In this paper, we denote by $\mathbf{x} = (x_1,\dots,x_n)$ a realization of the random shock vector $\mathbf{X} = (X_1,\dots,X_n)$.

In this framework, bank i defaults if bank i fails to pay its full liabilities $\overline{p}_{ij}$ and b_i (i.e., if the asset side of its balance sheet is not large enough to keep it solvent). Then, its assets are further reduced by *bankruptcy costs* proportional to its shortfall in payments, given by

$$\eta\left\{\overline{p}_i - \left(c_i - x_i + \sum_{j\neq i} p_j a_{ji}\right)\right\}, \tag{1}$$

where p_j denotes the total debt payment of bank j. After deducting the bankruptcy costs, its residual assets are distributed to its creditors according to the pro rata allocation rule, where interbank liabilities $\overline{p}_{ij}$ and external liabilities b_i have the equal priority.

Based on the mentioned features of this framework, the celebrated notion of the clearing payment vector $\mathbf{p}(\mathbf{x}) \in \mathbb{R}^n_+$ is defined as a solution to the following implicit equation:

$$p_i = \overline{p}_i \wedge \left\{c_i - x_i + \sum_{j\neq i} p_j a_{ji} - \eta\left(\overline{p}_i - c_i + x_i - \sum_{j\neq i} p_j a_{ji}\right)\right\}^+, \quad i = 1,\dots,n. \tag{2}$$

Note that the term in the braces represents the amount of remaining assets after accounting for bankruptcy costs. Thus, (2) implies that each bank either meets its payment obligation or distributes all its remaining assets to the creditors. We note that the solvency condition for bank i is equivalent to $p_i(\mathbf{x}) = \overline{p}_i$; the default of bank i happens only when $p_i(\mathbf{x}) < \overline{p}_i$.

To facilitate our analysis, we impose the following modeling assumptions in the paper.

Assumption 1.

a. *The initial net worth is positive and bounded by the external asset (i.e.,* $0 < w_i \leq c_i$ *for each* i*).*

b. *There does not exist any subset* $\mathcal{S}$ *of banks such that* $a_{ij} = a_{ji} = 0$ *for all* $(i,j) \in \mathcal{S}\times\mathcal{S}^c$.

c. $0 \leq \eta < (\max_i \beta_i)^{-1} - 1$.

d. *For each* i, $x_i \leq c_i - \eta\overline{p}_i/(1+\eta)$.

Item (a) allows us to focus on the impact of a shock vector on the system stability, and it also indicates that interbank assets are smaller than total liabilities. Assumption 1(b) implies that the financial network is irreducible because if such set $\mathcal{S}$ exists, then banks in $\mathcal{S}$ can be regarded as being outside the system and considered separately. Condition (c) makes the function $\mathbf{p}(\mathbf{x})$ unique (Glasserman and Young 2015), and item (d) prevents bankruptcy costs in (1) from exceeding the total assets so that taking the positive part in (2) becomes unnecessary.[4]

2.2. The Main Problem

In this paper, under the framework in Section 2.1, we focus on the probability that at least one bank in a specific set $\mathcal{T}$ defaults: that is,

$$\mathsf{P}\left(\mathbf{X} \in \bigcup_{i\in\mathcal{T}} \mathsf{D}_i\right), \tag{3}$$

where D_i is a set of shock vectors that make bank i default. Because of the shock propagation, for each $i \in \mathcal{T}$, the default event D_i of bank i results not only from the loss in the bank's external assets but also, from its exposure to the loss in interbank transactions.

If we know full network information $\mathbf{A}$ and the distribution of $\mathbf{X}$, then the default probability (3) can be easily estimated via Monte Carlo simulation; for l simulated shocks $\mathbf{x}^1,\dots,\mathbf{x}^l$, by solving (2), one can find clearing payments $\mathbf{p}(\mathbf{x}^1),\dots,\mathbf{p}(\mathbf{x}^l)$, based on which the frequency of the target event is measured. Alternatively, Ahn and Kim (2018) introduce an efficient computational method for the probability (3) using conditional Monte Carlo and importance sampling. However, the public, banks, or regulators often face a lack of full network information and have only partial information about financial networks. This poses a huge challenge to accurately assess the default probability (3) because of possible misspecification of the target financial network. We provide a simple example showing that the network misspecification leads to a misestimation of the default probability.

Example 1. Consider a five-bank financial network where $w_i = 2$, $\overline{p}_i = 8$ for $i = 1, \ldots, 5$, $\eta = 0.1$, and the matrix $\mathbf{A}$ of relative liabilities is given by $a_{ij} = 0.2$ for all $i \neq j$. Note that $\beta_1 = \cdots = \beta_5 = 0.8$. However, if only the aggregate information $(\beta_1, \ldots, \beta_5)$ is available, one may incorrectly specify the network structure as, for example, a ring network or a star network. Table 1 provides four possible misspecified networks given the aggregate information. Assuming that the random shocks $X_1, \ldots, X_5$ are independent and identically distributed and follow a lognormal distribution with parameters $\mu = 0$ and $\sigma \in [0.2, 0.5]$, we demonstrate in Figure 2 that the default probabilities of bank 5 under the misspecified networks could deviate significantly from its true default probability. The star network and semicomplete network in Table 1 result in the smallest and largest default probabilities for bank 5, respectively, and the other networks in the table lead to values in between. The case of multiple target banks will be discussed in Section 4.

Given such network uncertainties that make the exact computation of (3) intractable, one would need to explore its worst-case version while utilizing all incomplete but available network information. Hence, given each bank's total liabilities $(\overline{p}_i)$, external liabilities (b_i), equities (w_i), and partial network information, we aim at investigating the *worst-case default probability*:

$$\mathsf{P}(\mathbf{X} \in \mathsf{D}_{\mathcal{A},\mathcal{T}}). \tag{4}$$

Here, $\mathcal{A}$ is the set of all possible matrices of relative liabilities given the partial network information, which we call the *network uncertainty set* hereafter. The set $\mathsf{D}_{\mathcal{A},\mathcal{T}} := \cup_{\mathbf{A}\in\mathcal{A}} \cup_{i\in\mathcal{T}} \mathsf{D}_i$ is a collection of shock vectors that might cause at least one bank in $\mathcal{T}$ to default in the worst case. We often suppress the subscript $\mathcal{T}$ if it is clear from the context. By definition, (4) bounds (3) from above, and they are equal when the full network information is known (i.e., when $\mathcal{A}$ is a singleton).

Remark 1. Given a lack of knowledge on the true network structure, for practical purposes, one might consider estimating the probability (3) with a reduced-form approach, such as a one-factor Gaussian copula model. This relies on each bank's marginal default probability and the correlation coefficients between the common market factor and idiosyncratic factors without considering network effects. However, if network effects are not accounted for, each bank's marginal probability will depend only on losses in external assets, leading to the underestimation of the target probability.[5]

3. Shock Propagation

To facilitate the derivation of the set $\mathsf{D}_{\mathcal{A},\mathcal{T}}$, we characterize the solvency condition via the analysis of shock propagation in the following lemma.

Lemma 1. *For each shock realization $\mathbf{x}$, bank i is solvent if and only if $\Phi_i(\mathbf{x}) \leq w_i$, where*

$$\Phi_i(\mathbf{x}) := x_i + \max_{\boldsymbol{\zeta}\in\mathsf{Q}_i} \boldsymbol{\zeta}^\top(\mathbf{x}_{-i} - \mathbf{w}_{-i}), \tag{5}$$

$\mathsf{Q}_i := \{\mathbf{v}_{-i} \in \mathbb{R}_+^{n-1} \mid \mathbf{v}_{\mathcal{I}} = (1+\eta)(\mathbf{I} - (1+\eta)\mathbf{A}_{\mathcal{I}})^{-1}\mathbf{a}^i_{\mathcal{I}},\ \mathbf{v}_{\mathcal{I}^c} = \mathbf{0},$ $\mathcal{I} \subset \{1, 2, \ldots, n\} \setminus \{i\}\}$, *$\mathbf{a}^i$ is the ith column of $\mathbf{A}$, and w_i is bank i's initial net worth defined in Section 2.1 for each $i \in \{1, 2, \ldots, n\}$.*

For each shock realization $\mathbf{x}$, $\Phi_i(\mathbf{x})$ in (5) can be viewed as the total shock to bank i, which aggregates the *direct* shock to bank i (i.e., the first term on the right-hand side of (5)) and the *indirect* shock (i.e., the second term on the right-hand side of (5)) that indicates shock propagation from other banks to bank i. Based on the lemma, the set D_i defined in Section 2.2 can be represented as $\mathsf{D}_i = \{\mathbf{x} \in [\mathbf{0}, \mathbf{c}] \mid \Phi_i(\mathbf{x}) > w_i\}$, which is the set of shock vectors that cause the total shock to bank i to exceed its net worth.[6]

To understand the role of the vector $\boldsymbol{\zeta}$ in (5), we observe that for each shock realization $\mathbf{x}$ and for each i, there exists some $\mathcal{I}$ in $\{1, 2, \ldots, n\} \setminus \{i\}$ such that

$$\Phi_i(\mathbf{x}) = x_i + (\mathbf{a}^i_{\mathcal{I}})^\top((\mathbf{I} - (1+\eta)\mathbf{A}_{\mathcal{I}})^{-1})^\top((1+\eta)(\mathbf{x}_{\mathcal{I}} - \mathbf{w}_{\mathcal{I}})), \tag{6}$$

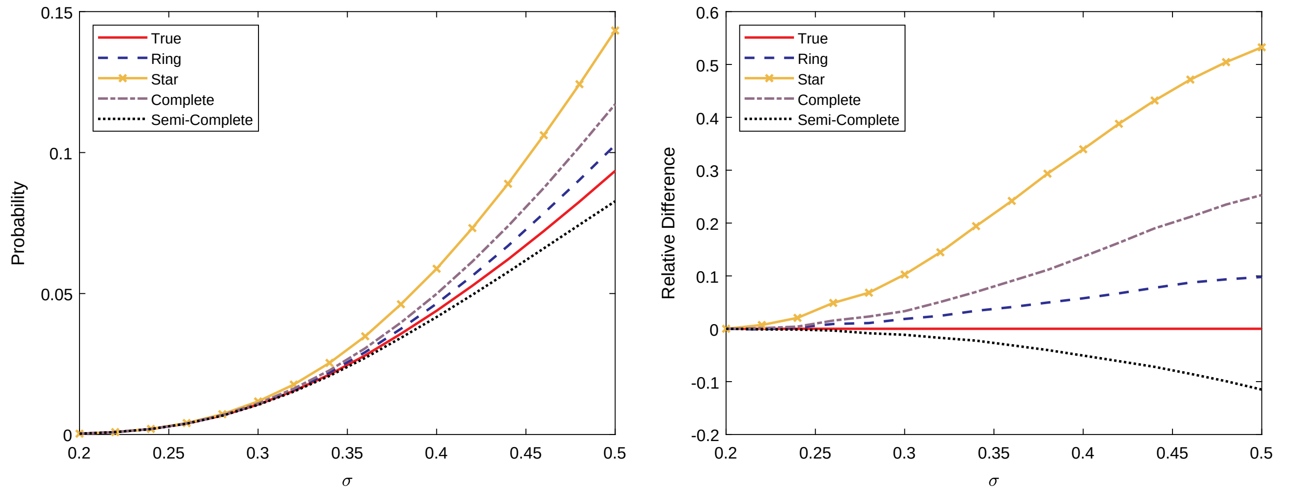
where the set $\mathcal{I}$ represents the set of insolvent banks. This implies that excess shocks of banks in $\mathcal{I}$ (i.e., $\mathbf{x}_{\mathcal{I}} - \mathbf{w}_{\mathcal{I}}$) are increased by a factor $(1+\eta)$ and further amplified while circulating within the set $\mathcal{I}$ because $(\mathbf{I} - (1+\eta)\mathbf{A}_{\mathcal{I}})^{-1} = \mathbf{I} + (1+\eta)\mathbf{A}_{\mathcal{I}} + (1+\eta)^2\mathbf{A}^2_{\mathcal{I}} + \cdots$. Then, the resulting shocks are transferred to bank i through $\mathbf{a}^i_{\mathcal{I}}$. Note that the banks in the set $\mathcal{I}^c$ do not affect bank i. Accordingly, the vector $\boldsymbol{\zeta}$ in (5) captures the banks involved in shock propagation to bank i and the extent of

Table 1. Possible Examples of Misspecified Networks in Example 1

Network type	Ring	Star	Complete	Semicomplete
Relative liability matrix	$\begin{bmatrix} 0 & .8 & 0 & 0 & 0 \\ 0 & 0 & .8 & 0 & 0 \\ 0 & 0 & 0 & .8 & 0 \\ 0 & 0 & 0 & 0 & .8 \\ .8 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 0 & .8 \\ 0 & 0 & 0 & 0 & .8 \\ 0 & 0 & 0 & 0 & .8 \\ 0 & 0 & 0 & 0 & .8 \\ .2 & .2 & .2 & .2 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & .1 & .1 & .1 & .5 \\ .1 & 0 & .1 & .1 & .5 \\ .1 & .1 & 0 & .1 & .5 \\ .1 & .1 & .1 & 0 & .5 \\ .2 & .2 & .2 & .2 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & .4 & .2 & .2 & 0 \\ .2 & 0 & .4 & .2 & 0 \\ .2 & .2 & 0 & .4 & 0 \\ .4 & .2 & .2 & 0 & 0 \\ .2 & .2 & .2 & .2 & 0 \end{bmatrix}$

Notes. The leading zeros are omitted in the matrices for ease of exposition. The naming convention of the networks is consistent with the literature except for the semicomplete network, whose name stems from the fact that its subnetwork consisting of banks 1–4 is a complete network.

Figure 2. (Color online) Bank 5's Default Probabilities Under Different Network Structures in Example 1



Notes. The left panel illustrates the bank's default probabilities with different σ under the true network and the misspecified networks in Table 1. The right panel describes the relative differences between the default probabilities under the misspecified networks and under the true network. The Monte Carlo method with 10^5 replications is used for the estimation of the probabilities.

the propagated shocks, and thus, it can be interpreted as the impact of other banks' excess shocks on bank i.[7]

Before concluding this section, we highlight that for each shock realization $\mathbf{x}$, the condition that no bank in $\mathcal{T}$ defaults is given by $\Phi_i(\mathbf{x}) \le w_i$ for each $i \in \mathcal{T}$, where $\Phi_i(\mathbf{x})$ can be rewritten as

$$\Phi_i(\mathbf{x}) = \max\ x_i + \sum_{j\in\mathcal{T}^c} \zeta_j(x_j - w_j)$$
$$\text{s.t. } \zeta_j \le (1+\eta)\left(a_{ji} + \sum_{k\in\mathcal{T}^c} a_{jk}\zeta_k\right) \quad \forall j \in \mathcal{T}^c,$$
$$\zeta_j \ge 0 \quad \forall j \in \mathcal{T}^c, \tag{7}$$

and for each $j \in \mathcal{T}^c$, the optimal solution ζ_j^* of (7) coincides with that of (5). Equation (7) holds because bank i in $\mathcal{T}$ is not affected by the other banks in $\mathcal{T}$ (i.e., $\zeta_j^* = 0$ for all $j \in \mathcal{T} \setminus \{i\}$ because of their solvency), and the vectors in Q_i are the extreme points of the feasible set of the linear program. This result will be particularly useful in the next section for characterizing the worst-case total shock to multiple target banks under network information uncertainty.

4. Robust Quantification of Default Probabilities

In this section, we discuss tractable quantification of (4) under incomplete network information. Because $\mathsf{D}_i = \{\mathbf{x} \in [\mathbf{0}, \mathbf{c}] \mid \Phi_i(\mathbf{x}) > w_i\}$ by Section 3, for fixed sets $\mathcal{A}$ and $\mathcal{T}$, one can derive that

$$\mathsf{D}_{\mathcal{A},\mathcal{T}} = \left\{\mathbf{x} \in [\mathbf{0}, \mathbf{c}] \,\middle|\, \max_{i\in\mathcal{T}}\{\overline{\Phi}_i(\mathbf{x}) - w_i\} > 0\right\},$$

where $\overline{\Phi}_i(\mathbf{x}) := \max_{\mathbf{A}\in\mathcal{A}} \Phi_i(\mathbf{x})$ means the worst-case total shock to bank i for each shock realization $\mathbf{x}$. A naive Monte Carlo estimation of the worst-case default probability (4) requires solving a bilevel optimization problem for each shock realization $\mathbf{x}$; the inner layer solves (7) to find $\Phi_i(\mathbf{x})$ given $\mathbf{A}$, and the outer layer maximizes $\Phi_i(\mathbf{x})$ over the set $\mathcal{A}$. Such an optimization problem is, however, difficult to solve in general. Hence, we first propose a tractable formulation of $\max_{\mathbf{A}\in\mathcal{A}} \Phi_i(\mathbf{x})$ that facilitates the computation of (4) and then apply it to various examples of partial information.

4.1. Main Result

We assume that total liabilities $\overline{p}_1, \ldots, \overline{p}_n$ and external liabilities $b_1, \ldots, b_n$ are given because their information is often available to the public in practice, which implies that $\beta_j = (\overline{p}_j - b_j)/\overline{p}_j$ is known for all j. Also, individual interbank transactions are assumed to be partially (or not) observable. In particular, the amount of the liabilities $\overline{p}_{jk}$ of bank j to bank k may be exactly known for some j, k. In some other cases, their positive lower bounds may be known instead of exact amounts; for example, if the liabilities between banks j and k are only known to exist, they would be at least as large as a very small amount (e.g., a dollar).

Let $\mathcal{K}$ denote the set of indices (j, k) such that the amount of $\overline{p}_{jk}$ is exactly known. Similarly, we define $\tilde{\mathcal{K}}$ as the set of indices (j, k) such that only a positive lower bound of $\overline{p}_{jk}$ is known. Then, considering all aforementioned cases, we construct the following network uncertainty set:

$$\mathcal{A} = \left\{\tilde{\mathbf{A}} = (\tilde{a}_{jk}) \in \mathbb{R}_+^{n\times n} \,\middle|\, \tilde{a}_{jk} = a_{jk} \ \ \forall (j,k) \in \mathcal{K}, \ \tilde{a}_{jk} \ge a_{jk} \ \ \forall (j,k) \in \tilde{\mathcal{K}}, \ \sum_{k=1}^{n} \tilde{a}_{jk} = \beta_j \ \ \forall j\right\}, \tag{8}$$

where $\{a_{jk}\}_{(j,k)\in\mathcal{K}}$ are the known relative liabilities, and $\{a_{jk}\}_{(j,k)\in\tilde{\mathcal{K}}}$ are positive constants representing the known lower bounds of the relative liabilities corresponding to $\tilde{\mathcal{K}}$. As we shall see in Section 4.2, given the available network information, $\mathcal{K}$, $\tilde{\mathcal{K}}$, and $\{a_{jk}\}_{(j,k)\in\mathcal{K}\cup\tilde{\mathcal{K}}}$ can be properly determined so that the network uncertainty set $\mathcal{A}$ in (8) represents the set of all possible network configurations under that network information.

To facilitate discussions, we classify banks in $\mathcal{T}^c$ according to the availability of their network information. For fixed $i\in\mathcal{T}$, we denote by $\mathcal{G}_1^i := \{j\in\mathcal{T}^c \mid (j,k)\in\mathcal{K}\ \ \forall k\in\{i\}\cup\mathcal{T}^c\}$ the set of banks whose liabilities to bank i and banks in $\mathcal{T}^c$ are *all* known, by $\mathcal{G}_2^i := \{j\in\mathcal{T}^c \mid (j,i)\notin\mathcal{K}\}$ the set of banks whose liabilities to bank i are *not* known, and by $\mathcal{G}_3^i := \mathcal{T}^c\setminus(\mathcal{G}_1^i\cup\mathcal{G}_2^i)$ the set of banks whose liabilities to bank i are known but whose liabilities to some banks in $\mathcal{T}^c$ are unknown. Clearly, the three sets are disjoint, there is no priority among them, and $\mathcal{T}^c = \mathcal{G}_1^i\cup\mathcal{G}_2^i\cup\mathcal{G}_3^i$. Based on these sets, we establish our main result on $\overline{\Phi}_i(\mathbf{x})$ by applying the worst-case scenario to the potential shock propagation from bank j in $\mathcal{T}^c$ to bank I (i.e., the right-hand side of the constraints in the linear program (7)) considering the network uncertainty set $\mathcal{A}$ of relative liabilities.

Recall that ζ_j in (5) and (7) measures the impact of bank j's excess shocks on bank i. Observe that for each $j\in\mathcal{G}_1^i$, there is no uncertainty in its liabilities a_{ji} and $\{a_{jk}\}_{k\in\mathcal{T}^c}$ (see Figure 3(a) in a four-bank system with $\mathcal{T}=\{4\}$ and $\mathcal{G}_1^4=\{1\}$). Thus, the corresponding constraints in (7) remain unchanged as follows:

$$0\le\zeta_j\le(1+\eta)\left(a_{ji}+\sum_{k\in\mathcal{T}^c}a_{jk}\zeta_k\right)\quad\text{for } j\in\mathcal{G}_1^i. \tag{9}$$

By contrast, if the direct link from bank j to bank i is not known (i.e., $j\in\mathcal{G}_2^i$), one may intuit that the worst-possible network would concentrate all unknown liabilities of bank j on that direct link; Figure 3(c) shows the worst-case allocation of bank 2's liabilities when we consider a four-bank system with $\mathcal{T}=\{4\}$ and $\mathcal{G}_2^4=\{2\}$ (Figure 3(b)). Let $\tilde{\beta}_j := \beta_j-\sum_{\{k\mid(j,k)\in\mathcal{K}\cup\tilde{\mathcal{K}}\}}a_{jk}$, which represents the sum of unknown relative liabilities of bank j. Then, the intuition implies that the worst-case version of the constraints in (7) could be written as

$$0\le\zeta_j\le(1+\eta)\left(\tilde{\beta}_j+\sum_{k\in\mathcal{K}_j\cup\tilde{\mathcal{K}}_j}a_{jk}\zeta_k\right)\quad\text{for } j\in\mathcal{G}_2^i, \tag{10}$$

where $\mathcal{K}_j := \{k\in\mathcal{T}^c \mid (j,k)\in\mathcal{K}\}$ and $\tilde{\mathcal{K}}_j := \{k\in\mathcal{T}^c \mid (j,k)\in\tilde{\mathcal{K}}\}$.

On the other hand, if $j\in\mathcal{G}_3^i$, because the direct link a_{ji} is known, the worst-possible network would allocate all unknown liabilities of bank j to the link to a bank in $\mathcal{T}^c\setminus\mathcal{K}_j$ that has the greatest impact on bank i (i.e., $\arg\max_{l\in\mathcal{T}^c\setminus\mathcal{K}_j}\zeta_l$). For example, in a four-bank system with $\mathcal{T}=\{4\}$ and $\mathcal{G}_3^4=\{3\}$ (see Figure 3(d)), bank 3's total unknown liabilities are allocated to its link to bank 1 if the impact of bank 1 on bank 4 is greater than that of bank 2 on bank 4 (i.e., $\zeta_1>\zeta_2$) (see Figure 3(e)); otherwise, they are allocated to its link to bank 2 (see Figure 3(f)). Hence, the associated worst-case constraints may correspond to

$$0\le\zeta_j\le(1+\eta)\left(a_{ji}+\sum_{k\in\mathcal{K}_j\cup\tilde{\mathcal{K}}_j}a_{jk}\zeta_k+\tilde{\beta}_j\left(\max_{l\in\mathcal{T}^c\setminus\mathcal{K}_j}\zeta_l\right)\right)\ \text{for } j\in\mathcal{G}_3^i. \tag{11}$$

The right-hand sides in (10) and (11) represent the worst-possible shock propagation from bank j to bank i. It is worth noting that banks in $\mathcal{T}$ are minimally liable because no unknown liabilities are allocated to the links between them.

These considerations of worst-possible network structures help develop the tractable formulation of $\overline{\Phi}_i(\mathbf{x})$ by maximizing the total shock to bank I (i.e., $x_i+\sum_{j\in\mathcal{T}^c}\zeta_j(x_j-w_j)$ subject to the constraints on $\{\zeta_j\}_{j\in\mathcal{T}^c}$, which is formalized in the following theorem).

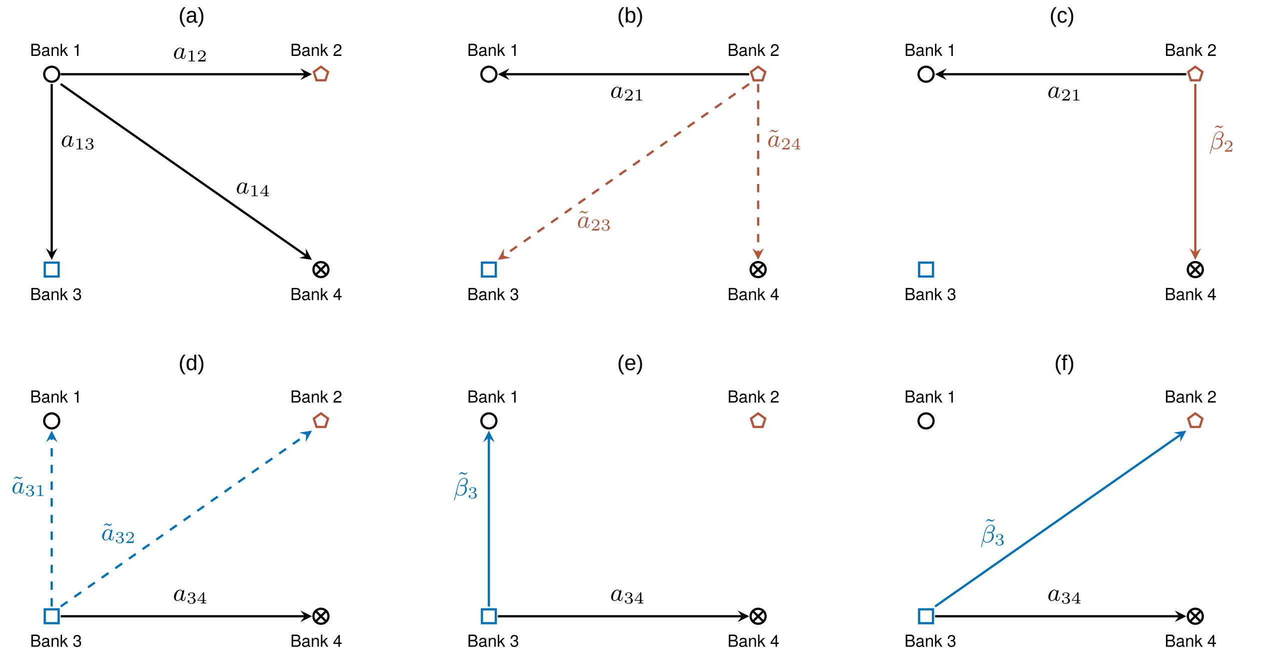
Theorem 1. *Assume that every bank in $\mathcal{T}$ is solvent and that the network uncertainty set $\mathcal{A}$ satisfies (8). Then, for all $\mathbf{x}\in[\mathbf{0},\mathbf{c}]$ and for each $i\in\mathcal{T}$, $\overline{\Phi}_i(\mathbf{x})$ is the maximum of $x_i+\sum_{j\in\mathcal{T}^c}\zeta_j(x_j-w_j)$ among all impacts $\{\zeta_j\}_{j\in\mathcal{T}^c}$ of banks in $\mathcal{T}^c$ on bank i satisfying the worst-case shock propagation Constraints (9), (10), and (11). Further, the nonlinear Constraints (11) can be written as the following equivalent linear constraints:*

$$\begin{cases} 0\le\zeta_j\le(1+\eta)\left(a_{ji}+\displaystyle\sum_{k\in\mathcal{K}_j\cup\tilde{\mathcal{K}}_j}a_{jk}\zeta_k+\tilde{\beta}_j\zeta_l+1-z_{jl}\right) & \\ & \textit{for } j\in\mathcal{G}_3^i,\ l\in\mathcal{T}^c\setminus\mathcal{K}_j; \\ \displaystyle\sum_{l\in\mathcal{T}^c\setminus\mathcal{K}_j}z_{jl}=1 & \textit{for } j\in\mathcal{G}_3^i; \\ z_{jl}\in\{0,1\} & \textit{for } j\in\mathcal{G}_3^i,\ l\in\mathcal{T}^c\setminus\mathcal{K}_j, \end{cases} \tag{12}$$

where for each $j\in\mathcal{T}^c$, the binary variable z_{jl} determines whether bank l has the greatest impact on bank i among banks in $\mathcal{T}^c\setminus\mathcal{K}_j$.

The maximization problem in Theorem 1, constructed by replacing (11) with (12), is an MILP that is much easier to solve than the original formulation $\max_{\mathbf{A}\in\mathcal{A}}\Phi_i(\mathbf{x})$. Thus, it greatly facilitates the computation of $\mathsf{P}(\mathbf{X}\in\mathsf{D}_{\mathcal{A}})$. Specifically, the Monte Carlo estimate for $\mathsf{P}(\mathbf{X}\in\mathsf{D}_{\mathcal{A}})$ can be computed by solving the MILP for each $i\in\mathcal{T}$ and checking if $\max_{i\in\mathcal{T}}\{\overline{\Phi}_i(\mathbf{x})-w_i\}>0$. Although solving the MILP $|\mathcal{T}|$ times may seem time consuming, this is not a critical issue because the set $\mathcal{T}$ would often be a collection of SIFIs, and the number of SIFIs is limited in

Figure 3. (Color online) An Illustration of the Worst-Possible Networks with a Four-Bank System When $\mathcal{T} = \{4\}$



Notes. We assume that all relative liabilities are known except for a_{23}, a_{24}, a_{31}, and a_{32}. Black solid arrows represent known liabilities (a_{ij}), and dashed arrows denote unknown liabilities ($\tilde{a}_{ij}$). Red and blue solid arrows are the liabilities of banks 2 and 3, respectively, in the worst-possible network. (a) Bank 1's liabilities. (b) Bank 2's liabilities (unknown). (c) Bank 2's liabilities (worst case). (d) Bank 3's liabilities (unknown). (e) Bank 3's liabilities (worst case). (f) Bank 3's liabilities (worst case).

practice. Also, we can relieve the computational burden by not solving the MILP for some shock realizations. Because $\max_{i\in\mathcal{T}}\{x_i - w_i\} > 0$ implies $\max_{i\in\mathcal{T}}\{\overline{\Phi}_i(\mathbf{x}) - w_i\} > 0$, it is enough to solve the MILP only when $x_i \le w_i$ for all $i \in \mathcal{T}$. Later, we will see some cases that further improve the time efficiency of the algorithm by transforming it into linear programs or by obtaining $\overline{\Phi}_i(\mathbf{x})$ even without solving an optimization problem.

Remark 2. For the robust quantification of (3), one may consider two alternative approaches. The first method is to use the robust counterpart of the linear programming version of (2): $\max \mathbf{1}^\top \mathbf{p}$ s.t. $(\mathbf{I} - (1+\eta)\mathbf{A}^\top)\mathbf{p} \le (1+\eta)(\mathbf{c} - \mathbf{x}) - \eta\overline{\mathbf{p}}$, $\mathbf{0} \le \mathbf{p} \le \overline{\mathbf{p}}$, $\forall \mathbf{A} \in \mathcal{A}$, where the result of Soyster (1976) can be applied. However, the solution turns out to be more conservative than ours. The second approach is to identify the worst possible network configuration in $\mathcal{A}$ that yields the largest default probability (i.e., $\max_{\mathbf{A}\in\mathcal{A}} \mathsf{P}(\mathbf{X} \in \cup_{i\in\mathcal{T}} \mathsf{D}_i)$) using sample average approximation. However, this requires solving a large-scale bilevel nonlinear optimization problem that is computationally intractable. In contrast, our method relies on solving a single-level MILP in Theorem 1, which is smaller in size and easier to solve. A numerical performance comparison is presented in the appendix.

Remark 3. Given the network uncertainty set $\mathcal{A}$, one might alternatively consider the worst-case "systemic default" probability (WSDP). Although there is no formal definition of systemic default, it is often understood as the default of all banks in the system (Tasca et al. 2014, Battiston et al. 2016, Roukny et al. 2018). In this case, the WSDP can be easily characterized under our model as long as $\mathbf{X}$ is a continuous random vector. First, a simple calculation shows that for each $\mathbf{x}$ and $\mathbf{A}$, systemic default occurs if and only if $\mathbf{s}(\mathbf{x}; \mathbf{A}) > \mathbf{0}$, where $\mathbf{s}(\mathbf{x}; \mathbf{A}) := (1+\eta)(\mathbf{I} - (1+\eta)\mathbf{A}^\top)^{-1}(\mathbf{x} - \mathbf{w})$ denotes the vector of each bank's payment shortfall. Hence, similar to (4), the WSDP can be defined by $\mathsf{P}(\mathbf{X} \in \tilde{\mathsf{D}}_{\mathcal{A}})$, where $\tilde{\mathsf{D}}_{\mathcal{A}} := \cup_{\mathbf{A}\in\mathcal{A}}\{\mathbf{x} \,|\, \mathbf{s}(\mathbf{x}; \mathbf{A}) > \mathbf{0}\}$ is the set of shock vectors that could cause all banks to default in the worst case. Then, by Farkas' lemma, it is not difficult to show that $\mathsf{P}(\mathbf{X} \in \tilde{\mathsf{D}}_{\mathcal{A}}) = \mathsf{P}(\Psi(\mathbf{X}) = 0)$, where $\Psi(\mathbf{x}) := \max\{(\mathbf{w} - \mathbf{x})^\top \boldsymbol{\xi} \,|\, (\mathbf{I} - (1+\eta)\mathbf{A})\boldsymbol{\xi} \ge \mathbf{0}$ for all $\mathbf{A} \in \mathcal{A}\}$. Note that the optimization problem for $\Psi(\mathbf{x})$ can be simply converted into a tractable linear program via a standard approach to robust linear optimization with polyhedral uncertainty (Bertsimas et al. 2011). Accordingly, the WSDP can be estimated in a manner similar to the estimation of $\mathsf{P}(\mathbf{X} \in \mathsf{D}_{\mathcal{A}})$ discussed. See Section EC.1.3 in the e-companion for technical details.

4.2. Applications to Various Network Information

This subsection provides several examples of partial network information to illustrate the usefulness of the MILP formulation in Theorem 1 from the perspectives of individual banks and regulators.

4.2.1. Individual Bank's Information. Suppose that bank 1 wants to estimate the worst-case default probability (4) of its counterparty bank n (i.e., $\mathsf{P}(\mathbf{X} \in \mathsf{D}_{\mathcal{A},\{n\}}) = \mathsf{P}(\overline{\Phi}_n(\mathbf{X}) > w_n)$) using all the information bank 1 has (i.e., $\mathcal{K} = \cup_{j=1}^{n} \{(1,j),(j,1),(j,j)\}$ and $\tilde{\mathcal{K}} = \emptyset$). In practice, bank 1 would compute such a probability conditional on itself remaining solvent prior to bank n's bankruptcy. In this case, solving the MILP results in

$$\overline{\Phi}_n(\mathbf{x}) = x_n + (1+\eta)\sum_{j=2}^{n-1}(\beta_j - a_{j1})(x_j - w_j)^+ \tag{13}$$

because $\mathcal{G}_1^n = \mathcal{G}_3^n = \emptyset$ and $\mathcal{G}_2^n = \{2,\dots,n-1\}$. This corresponds to the fact that, from bank 1's perspective, the worst-case shock propagation to bank n occurs when all unknown liabilities of banks 2 to $n-1$ are associated with bank n. In this worst-case network structure, the condition that bank 1 is solvent before bank n's default is given by $x_1 + (1+\eta)\sum_{j=2}^{n-1} a_{j1}(x_j - w_j)^+ \le w_1$, and hence, we can estimate the said probability by sampling $\mathbf{X}$ from the distribution satisfying this condition. Note that the increase or decrease in the amount of bank 1's lending to bank n does not affect (4) because $\overline{\Phi}_n(\mathbf{x})$ and w_n remain unchanged irrespective of these actions. However, if bank 1 allows bank n to roll over the loan, the probability will be reduced because it has the same effect as temporarily increasing w_n while keeping $\overline{\Phi}_n(\mathbf{x})$ unchanged.

4.2.2. Multiple Banks' Information. Suppose that regulators want to estimate banks' default probabilities but have limited information obtained only from some of the banks. Let $\mathcal{S}$ denote the set of banks that provide their own information to the regulators. Then, we have $\mathcal{K} = (\cup_{i\in\mathcal{S}, j=1,\dots,n}\{(i,j),(j,i)\}) \cup (\cup_{j\in\mathcal{S}^c}\{(j,j)\})$ and $\tilde{\mathcal{K}} = \emptyset$. Given this situation, the MILP in Theorem 1 can be used for any $\mathcal{S},\mathcal{T} \subset \{1,\dots,n\}$. We present two particular cases of $\mathcal{S}$ and $\mathcal{T}$ that would be of interest to regulators. We find that in those cases, the MILP can be greatly simplified.

- The regulators might want to estimate the worst-case default probability (4) of the banks whose information is not available. Thus, let us assume that the target banks' information is not available (i.e., $\mathcal{T} \subset \mathcal{S}^c$). Then, we have $\mathcal{G}_1^i = \mathcal{S}$, $\mathcal{G}_2^i = \mathcal{T}^c \setminus \mathcal{S}$, and $\mathcal{G}_3^i = \emptyset$ for each $i \in \mathcal{T}$, and thus, one can show that the MILP can be recast as the following linear program:

$$\begin{aligned} \max\ & x_i + \mathbf{u}^\top(\mathbf{x}_{\mathcal{T}^c} - \mathbf{w}_{\mathcal{T}^c}) \\ \text{s.t.}\ & \left((1+\eta)^{-1}\mathbf{I} - \tilde{\mathbf{A}}\right)\mathbf{u} \le \mathbf{q},\ \mathbf{u} \in \mathbb{R}_+^{|\mathcal{T}^c|}, \end{aligned} \tag{14}$$

where

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{\mathcal{S}} & \mathbf{A}_{\mathcal{S},\mathcal{T}^c\setminus\mathcal{S}} \\ \mathbf{A}_{\mathcal{T}^c\setminus\mathcal{S},\mathcal{S}} & \mathbf{0} \end{bmatrix} \text{ and } \mathbf{q} = \begin{bmatrix} \mathbf{a}_{\mathcal{S}}^i \\ \boldsymbol{\beta}_{\mathcal{T}^c\setminus\mathcal{S}} - \mathbf{A}_{\mathcal{T}^c\setminus\mathcal{S},\mathcal{S}}\mathbf{1} \end{bmatrix}.$$

- Given incomplete network information, even if the regulators observe the target banks' information (i.e., $\mathcal{T} = \mathcal{S}$), the exact computation of their default probability (3) may not be feasible. In this case, we have $\mathcal{G}_1^i = \mathcal{G}_2^i = \emptyset$ and $\mathcal{G}_3^i = \mathcal{T}^c$ for each $i \in \mathcal{T}$, and hence, the MILP is equivalent to the following simple MILP formulation for $i \in \mathcal{T}$:

$$\begin{aligned} \max\ & x_i + \mathbf{u}^\top(\mathbf{x}_{\mathcal{T}^c} - \mathbf{w}_{\mathcal{T}^c}) \\ \text{s.t.}\ & (1+\eta)^{-1}\mathbf{u}\mathbf{1}^\top - \tilde{\boldsymbol{\beta}}_{\mathcal{T}^c}\mathbf{u}^\top + \mathbf{Z} \le (\mathbf{a}_{\mathcal{T}^c}^i + \mathbf{1})\mathbf{1}^\top, \\ & \mathbf{Z}\mathbf{1} = \mathbf{1},\ \operatorname{diag}\mathbf{Z} = \mathbf{0}, \\ & \mathbf{Z} \in \{0,1\}^{|\mathcal{T}^c|\times|\mathcal{T}^c|},\ \mathbf{u} \in \mathbb{R}_+^{|\mathcal{T}^c|}, \end{aligned}$$

where we simplify the notation by adding a redundant constraint $\operatorname{diag}\mathbf{Z} = \mathbf{0}$.

4.2.3. Large Exposures' Information. As discussed in Section 1, according to the Basel III framework (BCBS 2020a), a bank's large exposures greater than or equal to 10% of the bank's capital are likely to be reported to regulators. We can deal with such a case in the MILP formulation by letting $\mathcal{K}$ be the set of interbank transactions corresponding to large exposures and $\tilde{\mathcal{K}}$ be an empty set. We will see in Section 6 and Section EC.4 in the e-companion that this information contains significant interbank liabilities that could largely affect network stability in practice, and thus, the worst-case default probabilities under this information are likely to be close to the true values.

4.2.4. Link Information. Recall that link information refers to the information on whether each interbank link exists or not. We assume that such link information is available, but the exact amount of interbank liabilities is not observed. Let $\mathcal{K}$ and $\tilde{\mathcal{K}} = \mathcal{K}^c$ denote the set of nonexisting links and the set of existing links, respectively. Then, for some $\epsilon > 0$, the uncertainty set $\mathcal{A}$ can be given by

$$\mathcal{A} = \left\{ \tilde{\mathbf{A}} \in \mathbb{R}_+^{n\times n} \,\middle|\, \sum_{k=1}^{n} \tilde{a}_{jk} = \beta_j\ \ \forall j,\ \tilde{a}_{jk} = 0\ \ \forall (j,k) \in \mathcal{K}, \ \tilde{a}_{jk} \ge \epsilon\ \ \forall (j,k) \notin \mathcal{K} \right\}. \tag{15}$$

Clearly, the uncertainty set $\mathcal{A}$ in (15) gets smaller as the set $\mathcal{K}$ becomes larger. This implies that the lower the network density, the closer the worst-case probability (4) under the link information is to the true probability (3). Note that the corresponding MILP problem is not as simple as the equivalent formulations in the previous cases, but it is still easy to formulate and solve.

In the following theorems, we further observe the closeness between the probabilities (3) and (4) given the link information (15) in the presence of regularly varying shocks and lognormal shocks.[8]

Theorem 2. *Suppose that $\mathcal{A}$ satisfies (15) for fixed $\mathcal{K}$ and $\epsilon > 0$ and that X_i be a nonnegative random variable having a regularly varying distribution with index $\rho_i > 1$ for each i (i.e., for all $t > 0$, $\lim_{x\to\infty} f_i(tx)/f_i(x) = t^{-\rho_i}$, where $f_i(\cdot)$ is* *the density function of X_i). Let $\mathbf{X}^m = \mathbf{X}/m$ for $m = 1, 2, \ldots,$ and let $\rho_* = \min_{i \in \mathcal{H} \cup \mathcal{T}} \rho_i$, where $\mathcal{H}$ is the set of banks that have a directed path to a bank in $\mathcal{T}$. Assume that for large m, $X_1^m, \ldots, X_n^m$ are independent and constrained to be in $[\mathbf{0}, \mathbf{c}]$ almost surely. Then, for any $\mathbf{A} \in \mathcal{A}$,*

$$\lim_{m\to\infty} \frac{1}{\log m} \log \mathsf{P}(\mathbf{X}^m \in \mathsf{D}_{\mathcal{A}}) = \lim_{m\to\infty} \frac{1}{\log m} \log \mathsf{P}\left(\mathbf{X}^m \in \bigcup_{i\in\mathcal{T}} \mathsf{D}_i\right) = -\rho_* + 1. \tag{16}$$

Theorem 3. *Suppose that $\mathcal{A}$ satisfies (15) for fixed $\mathcal{K}$ and $\epsilon > 0$ and that X_i follow a lognormal distribution with parameters μ_i and σ_i for each i. Let $\mathbf{X}^m = \mathbf{X}/m$ for $m = 1, 2, \ldots,$ and let $\sigma_* = \max_{i \in \mathcal{H} \cup \mathcal{T}} \sigma_i$, where the set $\mathcal{H}$ is defined as in Theorem 2. Assume that for large m, $X_1^m, \ldots, X_n^m$ are independent and constrained to be in $[\mathbf{0}, \mathbf{c}]$ almost surely. Then, for any $\mathbf{A} \in \mathcal{A}$,*

$$\lim_{m\to\infty} \frac{1}{(\log m)^2} \log \mathsf{P}(\mathbf{X}^m \in \mathsf{D}_{\mathcal{A}}) = \lim_{m\to\infty} \frac{1}{(\log m)^2} \log \mathsf{P}\left(\mathbf{X}^m \in \bigcup_{i\in\mathcal{T}} \mathsf{D}_i\right) = -\frac{1}{2\sigma_*^2}.$$

The theorems commonly indicate that under the link information (15), the probabilities (3) and (4) are asymptotically equivalent as the shock size gets smaller in the sense that for any $\mathbf{A} \in \mathcal{A}$,

$$\lim_{m\to\infty} \frac{\log \mathsf{P}(\mathbf{X}^m \in \cup_{i\in\mathcal{T}} \mathsf{D}_i)}{\log \mathsf{P}(\mathbf{X}^m \in \mathsf{D}_{\mathcal{A}})} = 1.$$

This not only highlights the importance of the link information but also, allows us to use (4) as a proxy of (3) when it comes to small but heavy-tailed shocks. The set $\mathcal{H}$ can be identified using the link information (15) and represents the set of banks that may affect the banks in $\mathcal{T}$.[9] Also, $(-\rho_i)$ and σ_i indicate the heavy tailedness of the shock X_i. Thus, among the shocks that could affect the banks in $\mathcal{T}$, the one with the heaviest tail has the most powerful influence on the probabilities (3) and (4). See Section EC.2 in the e-companion for more discussions on the two theorems.

4.2.5. Aggregate Information.

Suppose that individual interbank liabilities are not observed at all and that only the aggregate information $(\beta_1, \ldots, \beta_n)$ is available, which is the case in Glasserman and Young (2015, 2016). Then, $\mathcal{K} = \{(1,1), (2,2), \ldots, (n,n)\}$ and $\tilde{\mathcal{K}} = \emptyset$. In this case, it is easy to see that $\overline{\Phi}_i(\mathbf{x}) = x_i + (1+\eta)\boldsymbol{\beta}_{\mathcal{T}^c}^\top(\mathbf{x}_{\mathcal{T}^c} - \mathbf{w}_{\mathcal{T}^c})^+$ for $i \in \mathcal{T}$, and the worst-case default probability becomes

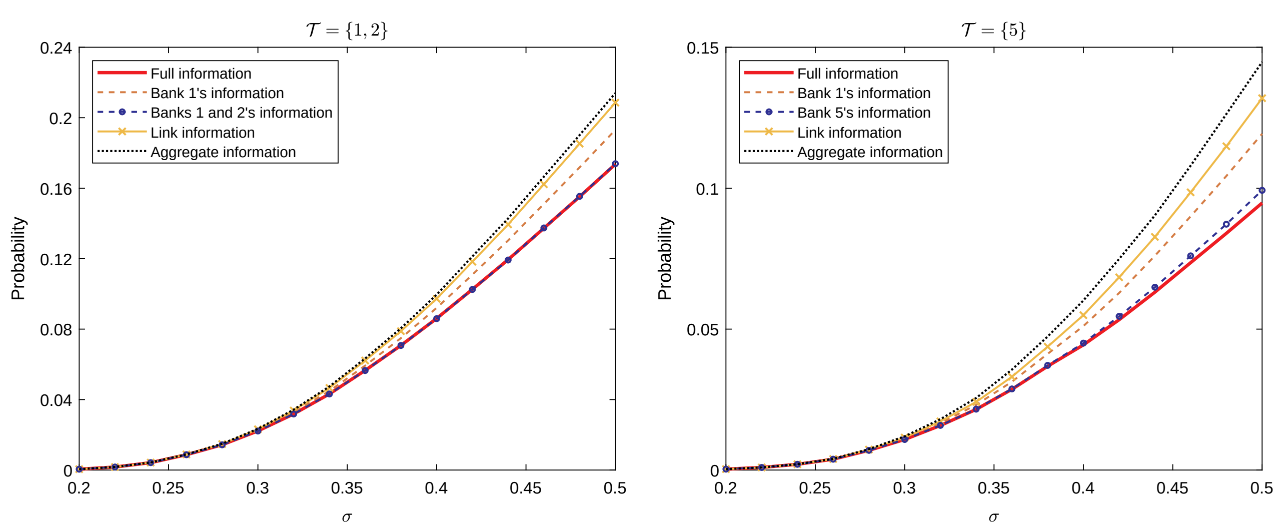
$$\mathsf{P}(\mathbf{X} \in \mathsf{D}_{\mathcal{A}}) = \mathsf{P}\left(\max_{i\in\mathcal{T}}\{x_i - w_i + (1+\eta)\boldsymbol{\beta}_{\mathcal{T}^c}^\top(\mathbf{x}_{\mathcal{T}^c} - \mathbf{w}_{\mathcal{T}^c})^+\} > 0\right). \tag{17}$$

Thus, we do not solve an optimization problem for estimating this probability, which helps us reduce the computation time greatly. Note that (17) is analogous to the result of Glasserman and Young (2015); in fact, both coincide given a single shock and a single target bank. However, we consider a multivariate shock vector $\mathbf{X}$ and the target event $\cup_{i\in\mathcal{T}} \mathsf{D}_i$ in (17), whereas their result is based on a single shock to a specific bank and the target event $\cap_{i\in\mathcal{T}} \mathsf{D}_i$.

4.2.6. Comparison of Network Information Effects.

We revisit the underlying network in Example 1 to illustrate how the worst-case default probability (4) changes according to the network information. In Figure 4, we present the estimated values of (4) with the target sets {1, 2}

Figure 4. (Color online) The Worst-Case Default Probabilities Under Different Types of Network Information



Notes. The figure exhibits the worst-case default probabilities (4) for $\mathcal{T} = \{1, 2\}$ (left panel) and $\mathcal{T} = \{5\}$ (right panel) with different σ under different types of network information. The Monte Carlo method with 10^5 replications is used for the estimation of the probabilities.

and {5} under five different types of network information: full network information, bank 1's information, target banks' information, link information, and aggregate information. The same shock distribution in Example 1 is assumed. For the link information, we arbitrarily set $\epsilon = 0.05$. Note that all interbank liabilities in this example are large exposures, equal to 80% of banks' capital, meaning that the large exposures' information is exactly the same as the full information.

In the left panel of Figure 4, we observe that the estimate of (4) given bank 1's and bank 2's information is closer to the true value (3) than the estimate of (4) given bank 1's information. This corresponds to our intuition that more information would lead to a better estimate. Also, in both panels, the information of target banks in $\mathcal{T}$ gives the best approximation, implying that the firsthand shock propagation is likely to dominate the whole shock propagation in financial networks. We shall see in Section 6 that these observations are consistent even in the case of real-world financial networks.

The magnitude of the differences between the true and worst-case default probabilities (up to 50% in this example) greatly depends on several inputs, including partial network information, shock distributions, bankruptcy costs, the size of interbank liabilities, and the number of target banks. However, no matter how large or small the gap is, our methodology allows us to compare the effects of different types of network information on the proximity between the true and worst-case default probabilities, which eventually gives us practical insights into information collection.

5. Worst-Case Risk Capital

The higher loss absorbency requirements in the Basel III framework ask global systemically important banks (G-SIBs) to hold additional capital to reduce their default probabilities. The task of gauging this risk capital poses a challenge when full network information is not available. In this section, we apply the main results in Section 4 so as to estimate the minimum additional capital amount, which we call *worst-case risk capital*, required to keep all SIFIs solvent with probability α.

Let $\mathcal{T}$ be the set of SIFIs, which is assumed to be given, and fix $\alpha \in [0,1)$. Then, using the probability (4), our problem can be mathematically formulated as follows:

$$\begin{aligned} \min \quad & \sum_{i\in\mathcal{T}} \nu_i \\ \text{s.t.} \quad & \mathsf{P}\Big(\max_{i\in\mathcal{T}}\{\overline{\Phi}_i(\mathbf{X}) - w_i - \nu_i\} > 0\Big) \le 1-\alpha, \\ & \nu_i \ge 0,\ i\in\mathcal{T}. \end{aligned} \tag{18}$$

Note that the left-hand side of the first constraint of (18) is the probability (4) with $w_i + \nu_i$ instead of w_i for $i \in \mathcal{T}$. The optimal solution to (18), say $\boldsymbol{\nu}^\alpha$, limits the probability (4) to $1-\alpha$, and the total amount $\nu_\circ^\alpha := \sum_{i\in\mathcal{T}} \nu_i^\alpha$ is the worst-case risk capital. However, (18) is a joint chance-constrained optimization problem that is known to be difficult to solve; because the probability does not have a closed form in general, it is hard to check the feasibility of the probabilistic constraint, and even if it is feasible, the feasible region might not be convex.

To tackle this issue, in the following theorem, we propose an SAA of Problem (18) assuming that the random shock vector $\mathbf{X}$ can be sampled.

Theorem 4. *Assume that $\mathbf{X}$ is a continuous random shock vector and that $\mathbf{x}^1, \dots, \mathbf{x}^N$ are its independent and identically distributed samples. Let V be the set of optimal solutions of (18) and M be a sufficiently large constant such that $\overline{\Phi}_i(\mathbf{X}) < M$ almost surely for all $i \in \mathcal{T}$. If $(\boldsymbol{\nu}^N, \mathbf{z}^N)$ solves the following MILP*:

$$\begin{aligned} \min \quad & \sum_{i\in\mathcal{T}} \nu_i \\ \text{s.t.} \quad & \sum_{j=1}^{N} z_j \le N(1-\alpha), \\ & w_i + \nu_i + Mz_j \ge \overline{\Phi}_i(\mathbf{x}^j), \quad i\in\mathcal{T},\ j=1,\dots,N, \\ & 0 \le \nu_i \le M, \quad i \in \mathcal{T}, \\ & z_j \in \{0,1\}, \quad j=1,\dots,N, \end{aligned} \tag{19}$$

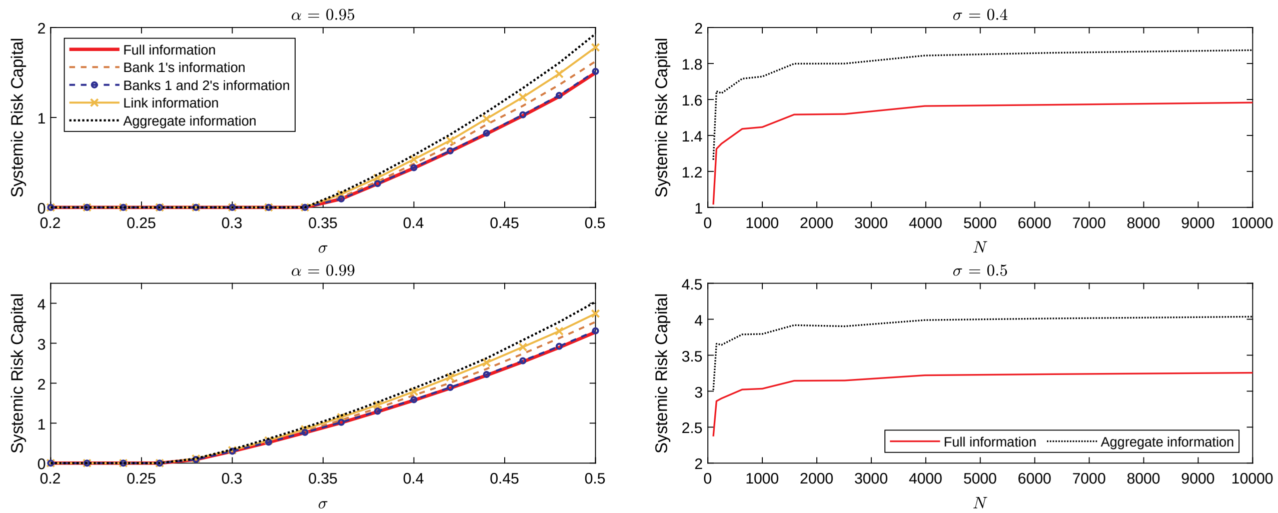
then $\nu_\circ^N := \sum_{i\in\mathcal{T}} \nu_i^N \to \nu_\circ^\alpha$ and $\inf_{\boldsymbol{\nu}^\alpha\in\mathsf{V}} \|\boldsymbol{\nu}^N - \boldsymbol{\nu}^\alpha\| \to 0$ with probability 1 as $N \to \infty$.

Theorem 4 provides significant computational benefits because the new Problem (19) is much easier to solve than the original Problem (18), and the approximation quality improves as the sample size increases. Furthermore, the random vector $\mathbf{X}$ can be a general continuous probability distribution, and hence, shocks to banks are possibly correlated.

Demange (2018) and Ahn and Kim (2019) study tractable mathematical programs for the optimal capital injection under the Eisenberg–Noe framework. However, those approaches essentially differ from ours. First, their policies are scenario-based solutions and do not adopt any probabilistic approach (i.e., each deterministic shock scenario may lead to a different capital allocation). Second, they do not take incomplete network information and SIFIs' default into special consideration.

In Figure 5, using Example 1 again, we describe the worst-case risk capital under different types of network information, different σ, and different N for the set $\mathcal{T} = \{1,2\}$. In the left panel of the figure, we observe the impact of network information on estimating the risk capital, which is similar to the results in Figure 4. The right panel, on the other hand, numerically shows that the risk capital is underestimated when the sample size is small but converges rapidly as the sample size increases. In general, the small sample size would lead to the underestimation of the (worst-case) default probability and risk capital because default events are rare.

Figure 5. (Color online) The Worst-Case Risk Capital Under Different Types of Network Information



Notes. The left panel exhibits the worst-case risk capital $\nu_\circ^N$ for $\mathcal{T} = \{1,2\}$ with $n = 10{,}000$; $\alpha = 0.95, 0.99$; and different σ under five types of network information in Figure 4. The right panel shows the worst-case risk capital $\nu_\circ^N$ for $\mathcal{T} = \{1,2\}$ with $\sigma = 0.4, 0.5$, $\alpha = 0.99$, and different N under full information and aggregate information.

Remark 4. One may also consider convex approximation (Hong et al. 2011), robust approximation (Yuan et al. 2017), and scenario approaches (Calafiore and Campi 2005, 2006) to overcome the intractability of Problem (18). However, those methods either require more distributional assumptions or lead to conservative bounds only. As long as we can sample the random vector, it is relatively easier to apply the SAA than the other methods. See Luedtke and Ahmed (2008) and Pagnoncelli et al. (2009) for more details on the SAA for joint chance-constrained optimization.

Before we end this section, we briefly discuss the applicability of our SAA approach to the measurement of external creditors' risk exposure to the financial system. If we regard the set of external creditors as node 0 and denote the associated relative liabilities of bank i by a_{i0} for each i, then a slight modification to Lemma 1 suffices to show that the creditors' worst-case losses are equal to $\overline{\Phi}_0(\mathbf{x}) := \max_{\mathbf{A}\in\mathcal{A}} \Phi_0(\mathbf{x})$, where $\Phi_0(\mathbf{x}) := \max\{\boldsymbol{\zeta}^\top(\mathbf{x}-\mathbf{w}) \,|\, (\mathbf{I}-(1+\eta)\mathbf{A})\boldsymbol{\zeta} \le (1+\eta)\mathbf{a}^0,\ \boldsymbol{\zeta} \in \mathbb{R}_+^n\}$ and $\mathbf{a}^0 = (a_{10}, a_{20}, \ldots, a_{n0})^\top$, which can be computed using an MILP similar to the one in Theorem 1. Thus, the associated value-at-risk, $\mathrm{VaR}_\alpha(\overline{\Phi}_0(\mathbf{X})) = \min\{y \in \mathbb{R}_+ \,|\, \mathsf{P}(\overline{\Phi}_0(\mathbf{X}) > y) \le 1-\alpha\}$, can be estimated via Theorem 4. According to Rockafellar and Uryasev (2000), given the samples $\overline{\Phi}_0(\mathbf{x}^1), \ldots, \overline{\Phi}_0(\mathbf{x}^N)$, the conditional value-at-risk of the worst-case losses can be approximated as $\mathrm{CVaR}_\alpha(\overline{\Phi}_0(\mathbf{X})) \approx \min\{y \in \mathbb{R}_+ \,|\, \overline{\Phi}_0(\mathbf{x}^j) + \{N(1-\alpha)\}^{-1} \sum_{j=1}^N z_j \le z_j + y,\ z_j \ge 0,\ j = 1, \ldots, N\}$.

6. Numerical Experiments

To demonstrate how our theoretical results in Sections 4 and 5 can be applied in practice, we use data from the 2011 EU-wide stress test conducted by the European Banking Authority (EBA). For illustration, we restrict our focus to 11 German banks that participated in the test as in Chen et al. (2016) and Gandy and Veraart (2017). Our numerical results based on the full data set can be found in Section EC.4 in the e-companion. Note that both data sets lead to the same conclusions.

The data set of the German banks is provided in Table 2. We use the numbers in the first column of Table 2 for the subscripts of variables (e.g., DE017's net worth is denoted by $w_1 = 30{,}361$). Because this data set does not report each bank's bilateral interbank exposures, we adopt the three types of reconstructed networks used in Chen et al. (2016): complete, ring-like, and core-periphery networks (see Figure 6). This reconstruction is based on an entropy-minimization method in Upper and Worms (2004), assuming that each bank's interbank assets and liabilities are equal to its exposure at default (EAD).[10] We refer the reader to Chen et al. (2016) for more details. For the core-periphery network, DE017, DE018, DE019, and DE020 are selected as the core banks according to the size of the total assets. The matrices of the relative liabilities for these networks are presented in Section EC.4.3 in the e-companion.

We assume that the random shocks $X_1, \ldots, X_{11}$ follow Pareto distributions, taking into account that heavy-tailed shocks could lead to large shock propagation. Note that if only a little shock propagation occurs, it is obvious that the worst-case default probability and risk capital under partial information are almost identical to the true quantities. For each i, the probability density f_i of X_i is assumed to depend on the amount of external assets as follows: $f_i(x) = \theta_i^{-1}(1+\lambda x/\theta_i)^{-(1/\lambda+1)}$, where $\theta_i = c_i/c_1$ and $\lambda \in [1,6]$. It is also assumed that the shocks are independent and constrained to be in $[\mathbf{0}, \mathbf{c}]$ to satisfy the modeling assumption in Section 2.

Table 2. Data of German Banks from the 2011 EBA EU-Wide Stress Test

No.	Code	Bank name	Total assets	EAD	Equity	External assets
1	DE017	Deutsche Bank AG	1,905,630	47,102	30,361	1,858,528
2	DE018	Commerzbank AG	771,201	49,871	26,728	721,330
3	DE019	Landesbank B-W	374,413	91,201	9,838	283,212
4	DE020	DZ Bank AG	323,578	100,099	7,299	223,479
5	DE021	Bayerische Landesbank	316,354	66,535	11,501	249,819
6	DE022	Norddeutsche Landesbank	228,586	54,921	3,974	173,665
7	DE023	Hypo Real Estate Holding AG	328,119	7,956	5,539	320,163
8	DE024	WestLB AG Dusseldorf	191,523	24,007	4,218	167,516
9	DE025	HSH Nordbank AG Hamburg	150,930	4,645	4,434	146,285
10	DE027	Landesbank Berlin AG	133,861	27,707	5,162	106,154
11	DE028	DekaBank Deutsche Girozentrale	130,304	30,937	3,359	99,367

Note. All quantities are exhibited in millions of euros.

We set Deutsche Bank (DE017) and Commerzbank (DE018) as target banks (i.e., $\mathcal{T} = \{\text{DE017}, \text{DE018}\}$) because they were the two largest banks in terms of asset size and were identified as G-SIBs by the Financial Stability Board in 2011. Further, we consider seven types of network information: full network information, DE017's information, DE018's information, DE017 and DE018's information, large exposures' information, link information, and aggregate information. Recall that the worst-case quantities under full information correspond to the true quantities and that those under partial information should be larger than the true values.

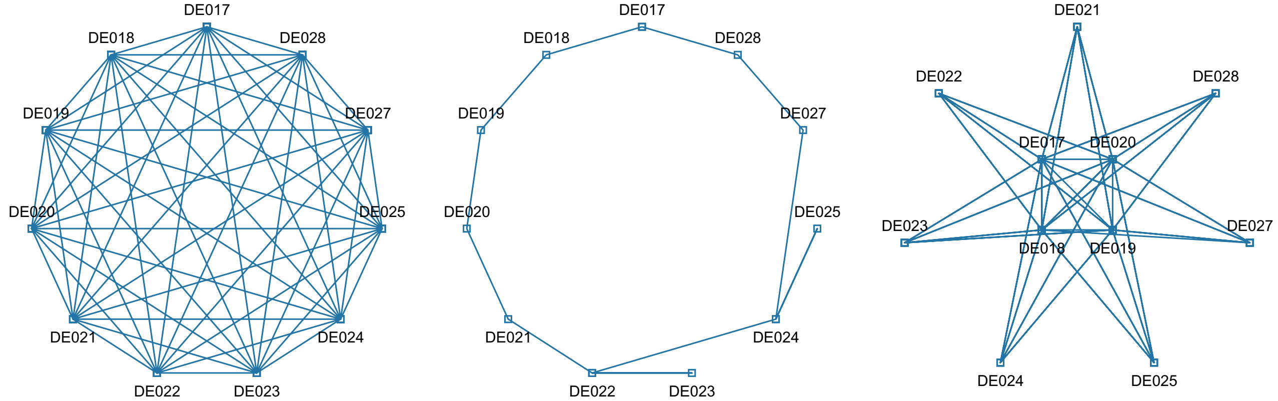
In Figures 7 and 8, we observe the impact of network information on the worst-case default probabilities (4) and risk capitals (18) as in Figures 4 and 5, respectively. We arbitrarily set $\epsilon = 0.002$ for the link information and $\eta = 0$ for all cases. We find that most interbank exposures are large exposures because of the small network size, and hence, the worst-case analysis under large exposures' information is hardly different from the quantities under full information. Although this observation does not extend to the case of larger networks, the experiments with the full EBA data set show that large exposures' information remains useful in larger networks; see Section EC.4.1 in the e-companion. This result is associated with contribution (3b) in Section 1.

Notably, the impact of link information on approximation quality is highly sensitive to the network structure. Although it is most effective under the ring network, the performance hardly improves under the complete network. This confirms that the value of the link information is greater for a sparse network than for a dense network, as previously inferred from the corresponding uncertainty set (15). This corresponds to contribution (3c) in Section 1.

More importantly, it is consistently observed that the target banks' information results in the best approximations, whereas the aggregate information and the information of a single target bank do not help much, which pertains to contributions (3a) and (3d) in Section 1. The figures nevertheless show that the information of DE018 is more useful than that of DE017 regardless of the network structure, which is because DE018 has greater financial connectivity than DE017 (see Table 2).

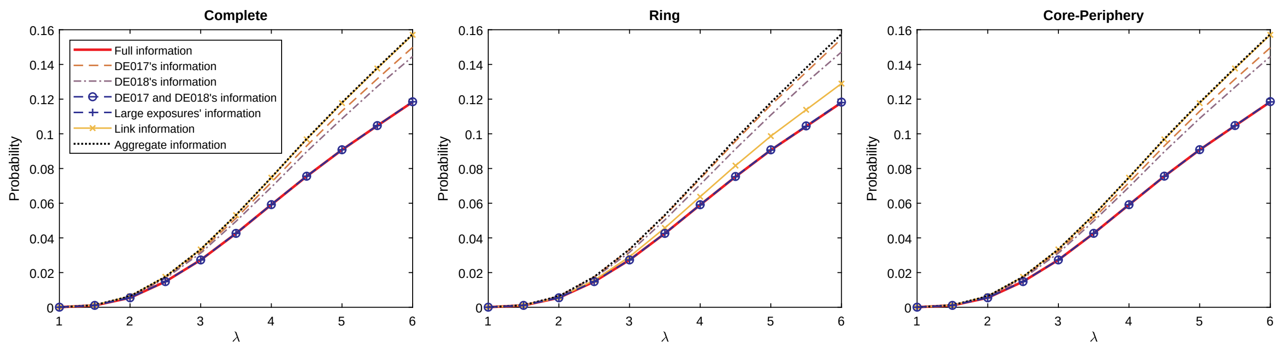
In contrast to the result in Figure 5, Figure 8 shows that the worst-case risk capital is close to the true risk capital

Figure 6. (Color online) The Recovered Interbank Network Structures from the EBA Stress Test Data



Notes. The three structures represent complete, ring-like, and core-periphery networks from left to right, respectively. For each structure, nodes represent individual banks, and edges stand for their interbank exposures.

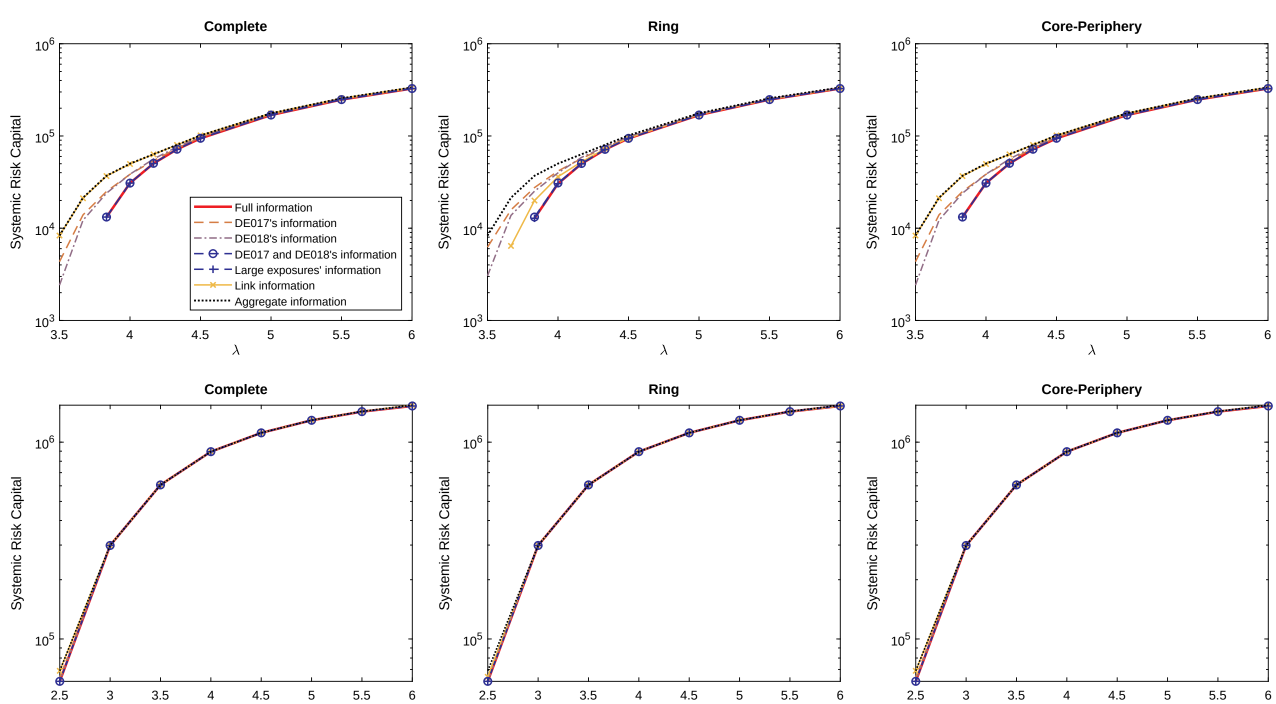
Figure 7. (Color online) The Worst-Case Default Probabilities for $\mathcal{T} = \{\text{DE017}, \text{DE018}\}$ Under Different Network Information and Different Network Structures



Notes. The Monte Carlo method with 10^5 replications is used for the estimation of the probabilities. The legend in the left panel applies to all of the panels.

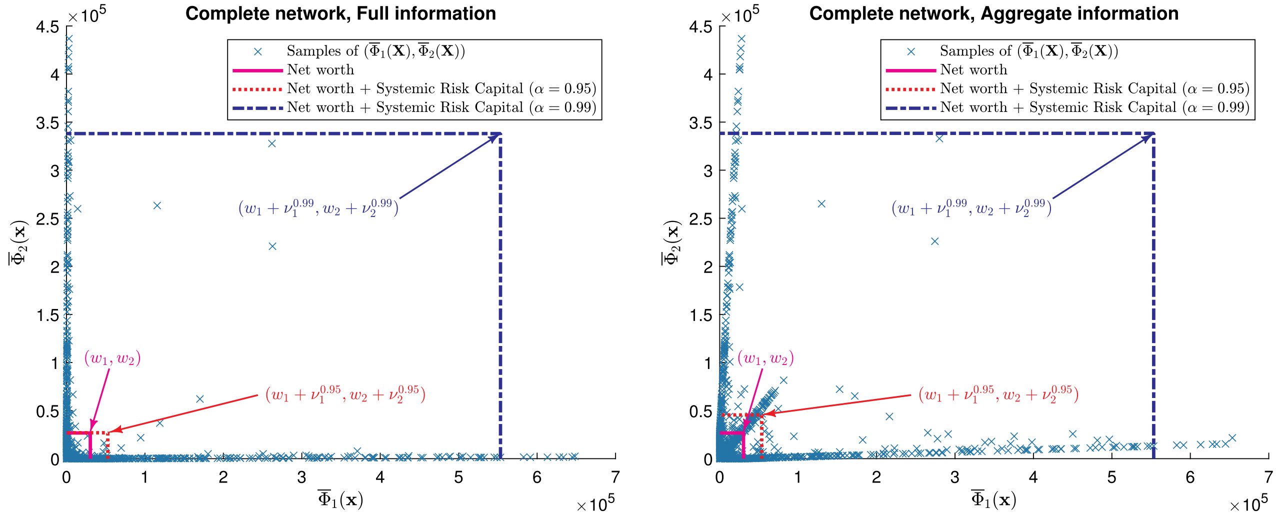
in most cases and that the degree of proximity increases as α increases. This seemingly conflicting phenomenon stems from the nature of factors that affect shock propagation and that of the data we use. Figure 9 provides two scatterplots of the samples of $(\overline{\Phi}_1(\mathbf{X}), \overline{\Phi}_2(\mathbf{X}))$ under full information and aggregate information, respectively. The figure tells us that large $\overline{\Phi}_i$'s are driven by direct shocks but that small $\overline{\Phi}_i$'s are driven by indirect shocks. Hence, the shock propagation has more effects on computing the risk capital for smaller α. Note that in the figure, the default probability estimate counts the number of samples outside the region $[0, w_1] \times [0, w_2]$, whereas for risk capital, $(\nu_1^\alpha, \nu_2^\alpha)$ is determined to limit the number of samples outside the region $[0, w_1 + \nu_1^\alpha] \times [0, w_2 + \nu_2^\alpha]$ to at most $N(1-\alpha)$. In addition, compared with the case in Figure 5, the financial connectivities of the target banks are relatively small (see Table 2), which strengthens the impact of direct shocks.

Figure 8. (Color online) The Worst-Case Risk Capitals for $\mathcal{T} = \{\text{DE017}, \text{DE018}\}$ Under Different Network Information and Different Network Structures



Notes. The upper panels are when $\alpha = 0.95$, and the lower panels are when $\alpha = 0.99$. The number of samples (N) is 10,000. All panels share the same legend in the upper left panel.

Figure 9. (Color online) The Distributions of $(\overline{\Phi}_1(\mathbf{X}), \overline{\Phi}_2(\mathbf{X}))$ with Full Information (Left Panel) and Aggregate Information (Right Panel)



Notes. The complete network is used for the illustration. We set $\lambda = 4$. Each rectangle represents the region in which the total shocks to DE017 and DE018 are completely covered by their net worths and the corresponding risk capital allocated to each of them.

7. Conclusion

In this paper, we addressed robust risk quantification under the Eisenberg–Noe model with incomplete network information. Particularly, we provided an MILP problem to identify worst-case shock propagation to a specific group of banks from other banks, based on which worst-case default probabilities are quantified. In response to recent changes in financial regulations, we also expanded our approach to the problem of computing risk capital, which secures SIFIs against the worst-case shock propagation. We formulated the problem using chance-constrained optimization and suggested a sample average approximation scheme for computational tractability. Our numerical observations revealed the impact of partial information on the worst-case default probability and risk capital. They were found to be potentially useful in estimating the true quantities in the presence of certain network data, such as target banks' information, large exposures' information, or link information.

This work opens up several interesting directions for further investigation. First, analyzing the difference between true and worst-case default probabilities and its sensitivity to information availability could be insightful. Second, relaxing specific modeling features, such as extending to the Eisenberg–Noe model with fire sales, would be interesting. Further, based on the mapping from the shock vector $\mathbf{x}$ to the loss $\Phi_i(\mathbf{x})$, one may explore the problem of selecting the most likely shock scenario given the loss outcome, which is called reverse stress testing. Lastly, assuming that unknown interbank liabilities are random, one might tackle the issue of incomplete network information differently by finding a confidence interval of the total loss given a shock realization.

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Appendix. Comparison with Different Approaches

In Table A.1, we compare the performance of our approach with that of the two alternative methods for the robust quantification of (3) in Remark 2 denoted by robust LP and bilevel NLP, respectively. We consider two different networks: the core-periphery network constructed by the data of 11 German banks in Section 6 and that constructed by the full EBA data with 80 European banks in Section EC.4.1 in the e-companion. We use the same target banks as in those sections. We assume Pareto distributions for random shocks, where $\lambda = 4$ for the small network case and $\lambda = 2$ for the large network case. Based on 1,000 presampled shock realizations, we apply different approaches to estimate the worst-case default probabilities under SIFI information and aggregate information, and thus, the computation times in Table A.1 do not include sampling times. We use the Dell PowerEdge R630 server with a Dual Intel Xeon E5-2697 2.6-GHz CPU and 128 GB of RAM.

Our approach shows outstanding performance compared with the two alternative methods. In particular, robust LP is computationally fast, but it can be easily seen in Table A.1 that the estimates are too conservative to be used in practice.

Table A.1. Estimates and Estimation Times Under Different Approaches for the Robust Quantification of (3) in Remark 2

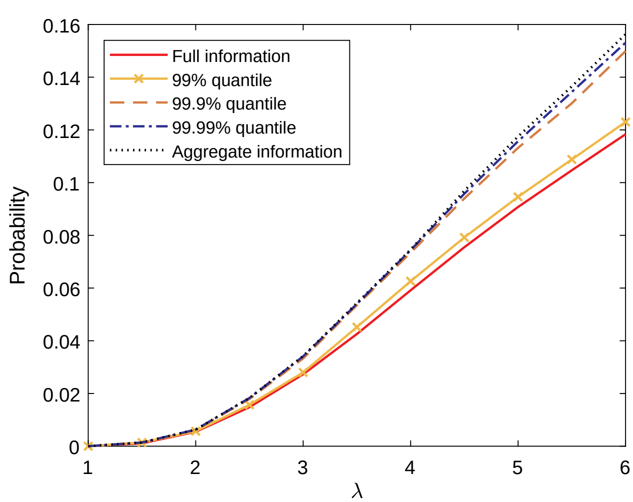
	SIFI information		Aggregate information	
Methods	Estimate	Time (seconds)	Estimate	Time (seconds)
Small network				
Our approach	0.053	4.902	0.071	0.003
Robust LP	0.062	3.498	1	0.346
Bilevel NLP	0.053	3.038×10^2	0.053	4.634×10^2
Large network				
Our approach	0.083	31.139	0.097	0.009
Robust LP	0.120	4.203	1	0.415
Bilevel NLP	0.083	1.985×10^4	0.083	3.004×10^4

Note. Robust LP means the method of using the robust counterpart of (2), and bilevel NLP represents the method of using a bilevel nonlinear program to find the worst possible network configuration that yields the greatest default probability.

Bilevel NLP turns out to be much slower than our method. More importantly, we find that this method may often produce unreliable estimates despite such high computational costs and may fail to distinguish SIFI and aggregate information as seen in their identical estimates. Note that in the case of aggregate information, our approach is fast and accurate because it does not require solving an MILP; see (17).

Furthermore, we numerically examine the practical validity of our worst-case default probabilities. We revisit the example of 11 German banks in Section 6 and assume that only aggregate information is available. We first generate 10^6 network structures that satisfy the condition in the set $\mathcal{A}$ with $\mathcal{K} = \{(1,1),(2,2)\ldots,(n,n)\}$ and $\tilde{\mathcal{K}} = \emptyset$ based on a homogeneous random graph model, in which every possible edge occurs independently with probability 1/2. Given 10^5 shock realizations, we estimate the default probabilities for each network structure and obtain their 99%, 99.9%, and 99.99% quantiles. In other words, we have an empirical distribution of the target default probability based on a million different network structures. Figure A.1 compares our worst-case default probabilities with those quantiles when $\mathcal{T} = \{\text{DE017}, \text{DE018}\}$. We use the core-periphery network in Section 6 for the full information benchmark. We observe that the 99.9% and 99.99% quantiles are close to our worst-case default probabilities with aggregate information. This implies that our worst-case default probabilities are not overly conservative. We also note that the more information that is available (e.g., SIFI information), the smaller the gap is between our worst-case default probabilities and the quantiles, which further highlights the value of information on interbank liabilities.

Figure A.1. The Worst-Case Default Probabilities and Three Different Quantiles of the Default Probabilities



Note. The Monte Carlo method with 10^5 replications is used for the estimation of the probabilities.

Endnotes

[1] Capponi and Larsson (2015) and Capponi and Weber (2023) discuss fire-sale spillover effects between banks with applications to capital constraints and portfolio diversification, respectively.

[2] Although the concept of the risk capital we consider is analogous to systemic risk measures that address capital injection to each bank (Feinstein et al. 2017, Biagini et al. 2019), our approach differs in that we consider the worst-case capital requirement against incomplete network information.

[3] Because interbank networks typically have a core-periphery structure (in 't Veld et al. 2020), the entire interbank network is generally sparse, but the network of core banks could be relatively dense in practice. In this case, link information may not be useful for approximating the true quantities.

[4] Based on the empirical data, Ahn (2020) points out that conditions 3 and 4 are not restrictive from the practical point of view.

[5] See Figure EC.9 in the e-companion for the five-bank financial network in Example 1.

[6] This lemma also allows us to visualize the default region and to understand an explicit form of the total shock; see Sections EC.3 and EC.4.2, respectively, in the e-companion. Such analyses are hardly possible if the set D_i is represented by $\mathbf{p}(\mathbf{x})$, in which $\mathbf{x}$ is implicitly entangled.

[7] The vector $\boldsymbol{\zeta}$ can be viewed as a dual of the weighted Bonacich centrality in network analysis (see, e.g., Candogan et al. 2012). This centrality is used as a measure of how influential each single node is in a network, whereas $\boldsymbol{\zeta}$ captures the influence of other nodes on a particular node.

[8] Based on empirical observations, it is widely accepted that financial shocks have heavy tails in practice; see, for example, Bradley and Taqqu (2003) and McNeil et al. (2015).

[9] See Section EC.2 in the e-companion for its mathematical definition.

[10] The EAD quantifies a bank's total claims on all other banks, and hence, it is considered as the size of its interbank assets. As

interbank liabilities are not reported in the EBA data, their size is roughly assumed to be equal to the EAD or its perturbed value in the literature (Glasserman and Young 2015, Chen et al. 2016, Gandy and Veraart 2017, Veraart 2020, Amini and Feinstein 2023).

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