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Occupation Times of Jump-Diffusion Processes with Double Exponential Jumps and the Pricing of Options

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In this paper, we provide Laplace transform-based analytical solutions to pricing problems of various occupation-time-related derivatives such as step options, corridor options, and quantile options under Kou's double exponential jump diffusion model. These transforms can be inverted numerically via the Euler Laplace inversion algorithm, and the numerical results illustrate that our pricing methods are accurate and efficient. The analytical solutions can be obtained primarily because we derive the closed-form Laplace transform of the joint distribution of the occupation time and the terminal value of the double exponential jump diffusion process. Beyond financial applications, the mathematical results about occupation times of a jump diffusion process are of more general interest in applied probability.

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1. Introduction. Occupation-time-related derivatives are recently introduced products that have been attracting much attention from investors and researchers. A defining characteristic of these contracts is an exercise payoff that depends on the time spent by the underlying asset in a predetermined region. Typically, the specification of the occupation regions involves flat barriers. In that sense, these contracts can be viewed as a generalized type of barrier option.

The payoffs of barrier options are activated or extinguished as soon as the underlying asset prices cross barriers. This discontinuity at the barriers poses an obstacle to the risk management of both option writers and buyers. Take the knock-out barrier option as an illustration. Even if the buyer has a correct view on the overall market trend, an accidental price jump across the barrier can easily wipe out his or her entire investment in the options. Furthermore, as Chesney et al. [6] and Linetsky [21] argued, market manipulators also like to take advantage of the fact that the payoffs are associated with barrier crossing, driving the underlying price to trigger a crossing and profiting from the massive losses of the other party to the transaction.

Several scholars have proposed a series of occupation-time-related options to alleviate the risk management difficulties inherent in barrier options caused by the discontinuity around the barriers. The payoffs now depend not only on the barrier crossing, but also on how long the underlying price spends above or below the barrier. Thus, option buyers can receive or lose value more gradually. One of the most popular examples is the step option suggested by Linetsky [21, 22]. This derivative's payoff is discounted at a rate defined by the occupation time. Under the geometric Brownian motion (GBM) model, Linetsky [22] derived closed-form pricing formulae for various single-barrier step options, and Davydov and Linetsky [11] investigated the pricing of double-barrier step options via Laplace inversion. A second example is the corridor option traded in the foreign exchange and interest rate markets. This option pays an amount at maturity that is associated with the time spent by a reference index, usually an exchange or interest rate, below a given level or inside a band. Fusai [13] priced this derivative under the GBM model by studying the distribution of the time spent by a Brownian motion with drift inside a band. Another important type of occupation-time-related option is the quantile option, which Miura [24] suggested as an alternative to the standard barrier option. A quantile is the minimum barrier to ensure that the fraction of the occupation time during the lifetime of the option exceeds a given level. Dassios [9] provided a formula for the quantile distribution of a Brownian motion with drift, as did Embrechts et al. [12] and Yor [29]. Akahori [2] and Dassios [9] calculated the prices of α-quantile options for the GBM model. Kwok and Lau [19] developed a pricing algorithm for quantile options based on the forward shooting grid method under the GBM model. Leung and Kwok [20] derived the distribution functions of occupation times under the constant elasticity of variance (CEV) process. Using an identity on quantiles of the processes with stationary and independent increments developed by Dassios [10], Cai [4] priced both the fixed- and floating-strike quantile options numerically by applying Laplace inversion twice under a hyper-exponential jump diffusion model.

In reality, many occupation-time-related options are based on a discrete time monitoring. In other words, such derivatives specify a series of reference dates. The occupation time is defined through the portion of monitoring dates in which the underlying price is below or above some level or between two levels. Some research is devoted to the study of such kind of options. However, the common feature of such research is that the underlying asset price is assumed to follow a GBM model. For instance, Atkinson and Fusai [3] studied discrete quantile options using the Spitzer identity of Brownian motions; Fusai and Tagliani [14] applied some numerical methods of PDEs to price discrete corridor options; and Davydov and Linetsky [11] considered step options under the discrete monitoring scheme.

In this article, we investigate the pricing and hedging problems of occupation-time-related options under Kou's double exponential jump diffusion model (Kou [16]). The model assumes the underlying asset return follows a jump diffusion process with Poisson jump intensity and double exponentially distributed jump sizes. It is appealing in two respects. The associated asset returns have heavier tails than normal distributions and hence the model is capable of generating the asymmetric leptokurtic feature for asset returns and volatility smiles for equity options, matching the empirical data better than the GBM model. The model also yields analytical solutions to many pricing problems, including both European and path-dependent derivatives, in terms of Laplace transforms. By applying numerical inversion algorithms we can easily obtain the prices.

The main result of this article is to derive the Laplace transform of the distribution of occupation times regarding one barrier under Kou's model, which enables us to calculate the prices of various related options such as step options, corridor options, and quantile options. It turns out that the Laplace transform solves a partial integro-differential equation (PIDE). We manage to reduce the equation to an ordinary integro-differential equation (OIDE) using an integral transform. Note that derivatives of exponential functions are still exponential. Then we can transform the OIDE into an ordinary differential equation (ODE) and rigorously show the existence and uniqueness of the solution to the OIDE. This article contributes to the literature of occupation-time-related options by generalizing the formulae for the GBM model to a model with discontinuous sample paths. It is simple to recover all of the classical results obtained with the GBM model from ours by letting the jump intensity be zero. The closed-form expressions of the Laplace transforms of the option prices also facilitate the calculation of price sensitivities in relation with market variables and model parameters. As shown in §4, not much extra effort is needed to obtain deltas, the price sensitivity with respect to the change of the underlying price. Such sensitivities play a vital role in risk management of derivatives, and traders can use it to rebalance the portfolio accordingly to achieve a desired exposure. In addition, our PIDE-OIDE approach can easily be extended to derive a closed-form solution for the Laplace transform of the distribution of occupation times spent within two barriers (a corridor).

Beyond financial settings, we should emphasize that the mathematical results about occupation times of a jump diffusion process may find potential applications in other branches of applied probability. One candidate case we can think of is in queuing theory. When service times or interarrival times have heavy-tailed distributions, heavy-traffic limits for the queue-length process usually are given by jump diffusions (see Whitt [28], Chapter 6). The results presented in this paper may be of interest to those who want to study the occupation time above or below single level or between two levels for a heavy-traffic queue. The literature accumulates some progress in this direction. For instance, Cohen and Hooghiemstra [8] discussed occupation times of Brownian excursions, a special kind of diffusions, and their link with the M/M/1 queue. We hope that our results may stimulate further investigation in jump-diffusion settings.

The organization of this article is as follows. Section 2 introduces Kou's model and some of its elementary properties. Section 3 demonstrates how to solve the PIDE to obtain the Laplace transform of the distribution of the occupation times. Section 4 applies the results of §3 to pricing various derivatives including step options, corridor options, and quantile options. Numerical results are given in §5. Appendices A–C are included to deal with some technical issues that arise in the body of the text, and Appendix D discusses the extension of our approach to the occupation times in a corridor.

2. Kou's model and its basic properties. Consider a market consisting of three securities only: a risk-free bond, a stock, and an occupation-time-related option contingent upon the stock. The bond offers investors risk free interest rate r. In Kou's double exponential jump diffusion model (DEM), the stock price under the physical probability measure is governed by the following dynamic,

$$\frac{dS_t}{S_{t-}} = \mu dt + \sigma dW_t + d\left(\sum_{i=1}^{N_t}(V_i - 1)\right),$$

where μ and σ are constants, $\{W_t: t \geq 0\}$ is a standard Brownian motion, $\{N_t: t \geq 0\}$ is a Poisson process with arrival rate λ, and $\{V_i: i = 1, 2, \ldots\}$ is a sequence of independent identically distributed (i.i.d.) random variables. According to the model, the instantaneous asset return rate is subject to the effects of three factors: a deterministic trend μ, small fluctuations described by the Brownian motion, and large market shocks captured by the Poisson-arrival jump part. To make the model more mathematically tractable, we further assume that $Y_i := \log(V_i)$ follows a double exponential distribution, the probability density function (pdf) of which is

$$f_Y(y) = p\eta e^{-\eta y}\mathbf{1}_{\{y \geq 0\}} + q\theta e^{\theta y}\mathbf{1}_{\{y<0\}},$$

where $\eta > 1$, $\theta > 0$, $p \geq 0$, $q \geq 0$, and $p + q = 1$. In other words, there are two types of jumps in the process: upward jumps with occurrence probability p and average jump size $1/\eta$, and downward jumps with occurrence probability q and average jump size $1/\theta$. Both types of jumps are exponentially distributed. We also assume that $\{W_t: t \geq 0\}$, $\{N_t: t \geq 0\}$, and $\{Y_i: i = 1, 2, \ldots\}$ are independent. This model, proposed by Kou [16] and Kou and Wang [17, 18], is known as the double exponential jump diffusion model in the financial engineering literature.

We need to work on a risk-neutral probability measure to calculate the option price. However, that measure is not unique because of the jump diffusion assumption. Following Lucas [23], Naik and Lee [25], and Kou [16] showed that there is a particular probability measure P^* so that the equilibrium price of an option is given by the expectation under this measure of the discounted option payoff if we consider a representative agent economy with a hyperbolic absolute risk aversion (HARA)-type utility function. We point out that our argument will work under any equivalent martingale measure that preserves the model structure, particularly the exponential type of the jumps. Under this risk-neutral probability measure P^*, S_t follows another double exponential jump diffusion model. More specifically, S_t obeys

$$\frac{dS_t}{S_{t-}} = r\,dt + \sigma^* dW_t^* + d\left(\sum_{i=1}^{N_t^*}(V_i^* - 1)\right).$$

Under P^*, $\{W_t^*: t \geq 0\}$ is a standard Brownian motion, $\{N_t^*: t \geq 0\}$ is a Poisson process with arrival rate λ^*, and $\{Y_i^* := \log(V_i^*): i = 1, 2, \ldots\}$ is also a sequence of i.i.d. double exponentially distributed random variables, but with different parameters. The distribution of Y_i^* is given by

$$f_{Y^*}(y) = p^*\eta^* e^{-\eta^* y}\mathbf{1}_{\{y \geq 0\}} + q^*\theta^* e^{\theta^* y}\mathbf{1}_{\{y<0\}},$$

where the new set of parameters satisfy $\eta^* > 1$, $\theta^* > 0$, $p^* \geq 0$, $q^* \geq 0$, and $p^* + q^* = 1$. Moreover, $\{W_t: t \geq 0\}$, $\{N_t^*: t \geq 0\}$, and $\{Y_i^*: i = 1, 2, \ldots\}$ are also independent under P^*. As we are only interested in option pricing, the difference between the physical and risk-neutral probability measures plays no role. From now on we drop the superscript *, with the understanding that all of the processes and parameters in the subsequent discussions are under P^*.

Let X_t be the log-return of the asset, i.e., $X_t := \log(S_t/S_0)$. By Itô's formula (cf. Protter [27], Theorem II. 32, p. 78), one can easily obtain

$$X_t = X_0 + \left(r - \frac{1}{2}\sigma^2 - \lambda\zeta\right)t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \quad X_0 = 0, \tag{1}$$

where ζ is the mean percentage jump size

$$\zeta = E[e^Y - 1] = \frac{p\eta}{\eta - 1} + \frac{q\theta}{\theta + 1} - 1.$$

An additional requirement $\eta > 1$ is needed to ensure that $E[V_1] = E[e^{Y_1}] < \infty$ and $E[e^{X_t}] < \infty$; this essentially means that the average upward jump cannot exceed 100%, which is quite reasonable in the reality of stock markets. For notational simplicity, denote $\bar{\mu} := r - \frac{1}{2}\sigma^2 - \lambda\zeta$.

Mathematically, the double exponential jump diffusion process (1) is a special Lévy processes because it has stationary and independent increments. Its Lévy exponent is defined as

$$G(x) := \frac{1}{t}\log E[\exp(xX_t) \mid X_0 = 0] = \frac{\sigma^2}{2}x^2 + \bar{\mu}x + \lambda\left(\frac{p\eta}{\eta - x} + \frac{q\theta}{\theta + x} - 1\right). \tag{2}$$

Consider an algebraic equation $G(x) = r + a$ for any given $a > -r$. It is easy to show that all four roots of the equation are real numbers (cf. Lemma 2.1, Kou and Wang [17]). Denote them by $\beta_{1,a}$, $\beta_{2,a}$, $-\gamma_{1,a}$, $-\gamma_{2,a}$. These roots satisfy

$$0 < \beta_{1,a} < \eta < \beta_{2,a} < \infty, \qquad 0 < \gamma_{1,a} < \theta < \gamma_{2,a} < \infty.$$

We will use these roots frequently when we derive the distributions of occupation times of (1) in §3. Explicit formulae for the four roots are also presented in Appendix A for reference.

Another important tool to establish the key results of the article is the infinitesimal generator of X_t. Note that X_t is a Markovian process and its infinitesimal generator is given by

$$\begin{aligned}(\mathscr{L}u)(x) &:= \lim_{t\downarrow 0} \frac{E[u(X_t) \mid X_0 = x] - u(x)}{t} \\ &= \frac{1}{2}\sigma^2 u''(x) + \bar{\mu}u'(x) + \lambda \int_{-\infty}^{\infty} [u(x+y) - u(x)] f_Y(y)\, dy \end{aligned} \tag{3}$$

for any twice continuously differentiable function u.

3. Distribution of the occupation times. In this section, we will present the main results of the article—the Laplace transforms of the distributions of occupation times of the double exponential jump diffusion process $\{X_t\}$ given by (1). Once it is known, in principle we are able to calculate any option prices related with occupation times. Consider a constant barrier h and let τ_t denote the occupation time the log-return process $\{X_t\}$ spends below h until t, i.e.,

$$\tau_t \equiv \tau_t(h) := \int_0^t \mathbf{1}_{\{X_u \le h\}}\, du. \tag{4}$$

An occupation-time-related option with maturity T usually has a payoff associated with τ_T and X_T. Suppose it is given by $f(\tau_T, X_T)$ for a general function f. Then pricing the option is equivalent to evaluating the following discounted expected payoff

$$e^{-rT} E[f(\tau_T, X_T) \mid X_0 = x] \tag{5}$$

under the risk-neutral probability. This section is devoted to the calculation of the expectation.

Before jumping into mathematical details, we would like to motivate readers by the intuition behind the scenes first. If the joint probability distribution of (τ_t, X_t) is available explicitly for all t, the expectation in (5) is obtainable by numerically integrating

$$E[f(\tau_T, X_T) \mid X_0 = x] = \int_0^T \int_{-\infty}^{\infty} f(s, y)\, F(ds, dy; T, x),$$

where $F(ds, dy; T, x) = P[\tau_T \in ds, X_T \in dy \mid X_0 = x]$. So our pricing strategy starts from finding a closed-form expression for the distribution $F(ds, dy; T, x)$. The Laplace transform is a powerful tool in characterizing probability distributions. We can invert the transforms to recover distributions easily, either using transform tables when possible or resorting to other numerical methods. For any $\rho > 0$ and $\gamma \in \mathbb{R}$, define $V(\rho, \gamma; t, x)$ as the Laplace transform of $F(ds, dy; t, x)$ with respect to s and y, i.e.,

$$V(\rho, \gamma; t, x) = \int_0^t \int_{-\infty}^{\infty} e^{-\rho s + \gamma y} F(ds, dy; t, x) = E[e^{-\rho \tau_t + \gamma X_t} \mid X_0 = x].$$

As mentioned in the introduction section, V can be determined by the solution of a PIDE for any fixed pair of ρ and γ. A heuristic approach is now presented to obtain the equation and a much more rigorous treatment is deferred to Theorem 3.2 below. Choose a short time duration δ. Then τ_t can be decomposed into two parts, the contribution of $\mathbf{1}_{\{X_u \le h\}}$ prior to δ and the contribution after δ:

$$\tau_t = \int_0^t \mathbf{1}_{\{X_u \le h\}}\, du = \int_0^\delta \mathbf{1}_{\{X_u \le h\}}\, du + \int_\delta^t \mathbf{1}_{\{X_u \le h\}}\, du.$$

By the Markovian property and the Lévy properties of $\{X_t\}$ we have

$$E[e^{-\rho \int_\delta^t \mathbf{1}_{\{X_u \le h\}}\, du + \gamma X_t} \mid X_\delta = x] = E[e^{-\rho \int_0^{t-\delta} \mathbf{1}_{\{X_u \le h\}}\, du + \gamma X_{t-\delta}} \mid X_0 = x] = V(t - \delta, x).$$

If applying the Taylor expansion on $V(t-\delta, X_\delta)$,

$$
\begin{aligned}
V(t,x) &= E[e^{-\rho\tau_t+\gamma X_t} \mid X_0 = x] \\
&= E[e^{-\rho\int_0^\delta \mathbf{1}_{\{X_u\le h\}}\,du}\cdot V(t-\delta, X_\delta)\mid X_0=x] \\
&\approx E\left[e^{-\rho\int_0^\delta \mathbf{1}_{\{X_u\le h\}}\,du}\cdot\left(V(t,X_\delta)-\delta\frac{\partial V}{\partial t}(t,X_\delta)\right)\middle| X_0=x\right]+o(\delta).
\end{aligned}
\tag{6}
$$

Note that $e^x \approx 1+x+o(x)$. Hence, (6) can be rewritten approximately as

$$
V(t,x)-E[V(t,X_\delta)\mid X_0=x] = -\delta E\left[\frac{\partial V}{\partial t}(t,X_\delta)\mid X_0=x\right]-\rho E\left[\int_0^\delta \mathbf{1}_{\{X_u\le h\}}\,du\cdot V(t,X_\delta)\right]+o(\delta). \tag{7}
$$

Divide both sides of (7) by δ and take it to zero. The left-hand side converges to $-\mathscr{L}V(t,x)$, thanks to (3), the definition of the infinitesimal generator $\mathscr{L}$. The right-hand side converges to

$$
-\frac{\partial V}{\partial t}(t,x)-\rho\mathbf{1}_{\{x\le h\}}V(t,x).
$$

In addition, we also know one boundary condition for the function V such that $V(0,x)=e^{\gamma x}$.

In summary, V should solve the following PIDE with Cauchy boundary condition

$$
\begin{cases}
\dfrac{\partial V}{\partial t}+\rho\mathbf{1}_{\{x\le h\}}V=\mathscr{L}V, & \text{for } t\in(0,T] \text{ and } x\in\mathbb{R}; \\
V(0,x)=e^{\gamma x}, & \text{for } x\in\mathbb{R}.
\end{cases}
\tag{8}
$$

Theorem 3.1 rigorously establishes the relationship between the Laplace transform V and the solution to PIDE (8) via the martingale problem formulation.

Theorem 3.1. *Assume that V: $[0,T]\times\mathbb{R}\to\mathbb{R}$ is a solution to PIDE (8), which is of class $C^{1,1}$ on $[0,T]\times\mathbb{R}$ and $C^{1,2}$ on $[0,T]\times\mathbb{R}\backslash\{h\}$. Moreover, the left and right second derivatives at h, $\partial^2V(t,h-)/\partial x^2$ and $\partial^2V(t,h+)/\partial x^2$, exist and V is bounded by*

$$
\max_{0\le t\le T}|V(t,x)|\le C_1e^{C_2|x|},\qquad x\in\mathbb{R}, \tag{9}
$$

for constants $C_1>0$ and $0<C_2<\min\{\eta,\theta\}$. Then V admits the following stochastic representation:

$$
V(t,x)=E[e^{-\rho\int_0^t\mathbf{1}_{\{X_s\le h\}}ds}e^{\gamma X_t}\mid X_0=x],\quad 0\le t\le T,\quad x\in\mathbb{R}. \tag{10}
$$

And such a solution is unique.

Proof. Introduce $v(t,x)=V(T-t,x)$ for any $t\in[0,T]$. Following the arguments leading to the Feynman-Kac formula (cf. Theorem 4.4.2, Karatzas and Shreve [15]), we attempt to apply Itô's formula on $v(t,X_t)\exp(-\rho\int_0^t\mathbf{1}_{\{X_s\le h\}}\,ds)$ to calculate its expectation. However the irregularity of $v(t,\cdot)$ at barrier h forbids us from doing so directly. From Lemma B.1 in Appendix B we know that there exists a series of functions $\{v_n(t,x)\colon n=1,2,\ldots\}$ such that (1) $v_n(t,x)$ converges to $v(t,x)$ as $n\to\infty$ for any $(t,x)\in[0,T]\times\mathbb{R}$; (2) $v_n(t,x)$ is of class $C^{1,2}$ in $[0,T)\times\mathbb{R}$ for any n; (3) $v_n(t,x)\equiv v(t,x)$ for any $(t,x)\in[0,T]\times(-\infty,h]\cup[h+1/n,\infty)$; and (4) for any $(t,x)\in[0,T]\times(h,h+1/n)$ and any $n\in\mathbb{N}$, $\max\{|v_n(t,x)|, |\partial v_n(t,x)/\partial t|, |\partial v_n(t,x)/\partial x|, |\partial^2 v_n(t,x)/\partial x^2|\}\le M$, where M is a positive constant independent of t, x, and n.

Define

$$
e_n(t,x):=\frac{\partial v_n}{\partial t}(t,x)-\rho\mathbf{1}_{\{x\le h\}}v_n(t,x)+\mathscr{L}v_n(t,x).
$$

According to the construction of $\{v_n(t,x)\}$ and (9), some routine algebra manipulation will yield that there exist positive constants M_1 and M_2, independent of n, t, and x, such that

$$
|e_n(t,x)|\le\frac{M_1}{n}<+\infty,\quad \text{for } (t,x)\in[0,T]\times(-\infty,h]\cup\left[h+\frac{1}{n},\infty\right) \tag{11}
$$

and

$$
|e_n(t,x)|\le M_2<+\infty,\quad \text{for } (t,x)\in[0,T]\times\left(h,h+\frac{1}{n}\right). \tag{12}
$$

Now we are able to apply Itô's formula to $v_n(t+\alpha, X_\alpha)e^{-\rho\int_0^\alpha \mathbf{1}_{\{X_s\le h\}}\,ds}$ because v_n is twice differentiable on the whole real line with respect to x. Let $T_m := \inf\{t\in[0,T]\colon |X_t|\ge m\}$ for any $m\in\mathbb{N}$. Itô's formula for jump diffusions (cf. Protter [27], Theorem II. 32, p. 78) implies that

$$\mathrm{MG}^{(n,m)}(\alpha) := v_n(t+\alpha\wedge T_m, X_{\alpha\wedge T_m})e^{-\rho\int_0^{\alpha\wedge T_m}\mathbf{1}_{\{X_s\le h\}}\,ds} - \int_{0+}^{\alpha\wedge T_m} e^{-\rho\int_0^s \mathbf{1}_{\{X_\xi\le h\}}\,d\xi} e_n(t+s-, X_{s-})\,ds$$

is a local martingale for any fixed $t\in[0,T]$, m, $n\in\mathbb{N}$, and $0\le\alpha\le T-t$. In other words, there should be a nondecreasing sequence of stopping times $\{\pi_k, k=1,2,\ldots\}$ such that $\mathrm{P}(\lim_{k\to+\infty}\pi_k=+\infty)=1$ and $\{\mathrm{MG}^{(n,m)}(\alpha\wedge\pi_k)\colon \alpha\in[0,T-t]\}$ is a true martingale. It follows that for any $0\le s<\alpha\le T-t$, we have

$$\mathrm{E}[\mathrm{MG}^{(n,m)}(\alpha\wedge\pi_k)\mid \mathscr{F}_s] = \mathrm{MG}^{(n,m)}(s\wedge\pi_k). \tag{13}$$

Fix $n\in\mathbb{N}$ and a sufficiently large m such that $m>|h|+1$. We intend to show that $\{\mathrm{MG}^{(n,m)}(\alpha)\colon \alpha\in[0,T-t]\}$ is actually a true martingale. It suffices to show that $\sup_{\alpha\in[0,T-t]}|\mathrm{MG}^{(n,m)}(\alpha)|$ is integrable. Indeed, if this is true, we can apply the dominated convergence theorem on (13). Letting $k\to+\infty$ will yield

$$\mathrm{E}[\mathrm{MG}^{(n,m)}(\alpha)\mid\mathscr{F}_s] = \mathrm{MG}^{(n,m)}(s)$$

for any $0\le s<\alpha\le T-t$; i.e., $\mathrm{MG}^{(n,m)}(\alpha)$ is a martingale.

Fortunately, the integrability of $\sup_{\alpha\in[0,T-t]}|\mathrm{MG}^{(n,m)}(\alpha)|$ is implied by the observation that the two terms in the expression of $\mathrm{MG}^{(n,m)}(\alpha)$ can be bounded as follows. For the second term, we can show that

$$\begin{aligned}
&\left|\int_{0+}^{\alpha\wedge T_m-} e^{-\rho\int_0^s\mathbf{1}_{\{X_\xi\le h\}}\,d\xi} e_n(t+s-, X_{s-})\,ds\right| \\
&\quad\le \int_{0+}^{\alpha\wedge T_m-} |e_n(t+s-, X_{s-})|\, I_{\{X_{s-}\in[h,\,h+1/n]\}}\,ds + \int_{0+}^{\alpha\wedge T_m-} |e_n(t+s-, X_{s-})|\, I_{\{X_{s-}\in[-m,\,h]\cup[h+1/n,\,m]\}}\,ds \\
&\quad\le M_2\int_{0+}^{\alpha\wedge T_m-} I_{\{X_{s-}\in[h,h+1/n]\}}\,ds + \frac{M_1}{n}(\alpha\wedge T_m-) \le \left(M_2+\frac{M_1}{n}\right)T,
\end{aligned} \tag{14}$$

where the second inequality holds due to (11) and (12). For the first term, it is easy to see that $\exp(-\rho\int_0^{\alpha\wedge T_m}\mathbf{1}_{\{X_s\le h\}}\,ds)$ is always bounded by one. Thus,

$$|v_n(t+\alpha\wedge T_m, X_{\alpha\wedge T_m})e^{-\rho\int_0^{\alpha\wedge T_m}\mathbf{1}_{\{X_s\le h\}}\,ds}| \le |v_n(t+\alpha\wedge T_m, X_{\alpha\wedge T_m})|. \tag{15}$$

When $\alpha<T_m$, $v_n(t+\alpha\wedge T_m, X_{\alpha\wedge T_m}) = v_n(t+\alpha, X_\alpha)$, which is bounded by $\max_{s\in[0,T],\,x\in[-m,m]}|v_n(s,x)|$ because $|X_\alpha|\le m$ by the definition of T_m. When $\alpha>T_m$, $v_n(t+\alpha\wedge T_m, X_{\alpha\wedge T_m}) = v_n(t+T_m, X_{T_m})$. By (9), its absolute value is bounded by

$$\left|v(t+T_m, X_{T_m})\right| \le C_1 e^{C_2\max_{0\le s\le T}|X_s|} \le C_1 e^{C_2|x|+C_2|\bar{\mu}|T} e^{C_2\sigma\max_{0\le s\le T}|W_s|} e^{C_2\sum_{i=1}^{N_T}|Y_i|}.$$

Now we intend to show $E[|v(t+T_m, X_{T_m})|]<+\infty$. On the one hand, some calculation illustrates that

$$E[e^{C_2\sum_{i=1}^{N_T}|Y_i|}] = \exp\left\{\lambda T\left(\frac{p\eta}{\eta-C_2}+\frac{q\theta}{\theta-C_2}-1\right)\right\} < +\infty, \tag{16}$$

thanks to $0<C_2<\min\{\eta,\theta\}$. On the other hand, we also have

$$E[e^{C_2\sigma\max_{0\le s\le T}|W_s|}] < +\infty. \tag{17}$$

Actually, notice that $e^{C_2\sigma\max_{0\le s\le T}|W_s|}\le Z_+Z_-$, where

$$Z_+ := e^{C_2\sigma\max_{0\le s\le T}W_s} \qquad \text{and} \qquad Z_- := e^{C_2\sigma\max_{0\le s\le T}(-W_s)}.$$

Because both $\max_{0\le s\le T}W_s$ and $\max_{0\le s\le T}(-W_s)$ have the same distribution as $|W_T|$, it follows that

$$\mathrm{E}Z_+^2 = \mathrm{E}Z_-^2 = \mathrm{E}e^{2C_2\sigma|W_T|} = 2e^{2C_2^2\sigma^2T}\Phi(2C_2\sigma\sqrt{T}),$$

where $\Phi(x)$ is the cumulative normal distribution function. According to Minkowski's integral inequality, we can obtain that

$$[\mathrm{E}e^{C_2\sigma\max_{0\le s\le T}|W_s|}]^{1/2} \le [\mathrm{E}(Z_+Z_-)]^{1/2} \le \tfrac{1}{2}[\mathrm{E}((Z_+ + Z_-)^2)]^{1/2} \le \tfrac{1}{2}[\mathrm{E}(Z_+^2)]^{1/2} + \tfrac{1}{2}[\mathrm{E}(Z_-^2)]^{1/2}$$
$$= [\mathrm{E}(Z_+^2)]^{1/2} = [2e^{2C_2^2\sigma^2 T}\Phi(2C_2\sigma\sqrt{T})]^{1/2} < +\infty.$$

Then (17) is proved.

From (16) and (17), we then have $E[|v(t+T_m, X_{T_m})|] < +\infty$. Therefore, the right-hand side of (15) will be bounded by

$$|v_n(t+\alpha\wedge T_m, X_{\alpha\wedge T_m})| \le |v_n(t+\alpha, X_\alpha)\mathbf{1}_{\{\alpha<T_m\}}| + E|v_n(t+T_m, X_{T_m})\mathbf{1}_{\{\alpha\ge T_m\}}|$$
$$\le \max_{s\in[0,T],\, x\in[-m,m]} |v_n(s,x)| + |v(t+T_m, X_{T_m})|.$$

Note that the right-hand side of this inequality has nothing to do with α. It follows that $\sup_{\alpha\in[0,T-t]}|v_n(t+\alpha\wedge T_m, X_{\alpha\wedge T_m})|$ is integrable. Combining with (14), we have already shown that $\sup_{\alpha\in[0,T-t]}|\mathrm{MG}^{(n,m)}(\alpha)|$ is integrable. Consequently, $\{\mathrm{MG}^{(n,m)}(\alpha)\colon \alpha\in[0,T-t]\}$ is a true martingale.

The martingale property of $\mathrm{MG}^{(n,m)}(\alpha)$ implies that

$$E[\mathrm{MG}^{(n,m)}(\alpha) \mid X_0 = x] = E[\mathrm{MG}^{(n,m)}(0) \mid X_0 = x] = v_n(t,x).$$

In other words,

$$v_n(t,x) = E[v_n(t+\alpha\wedge T_m, X_{\alpha\wedge T_m})e^{-\rho\int_0^{\alpha\wedge T_m}\mathbf{1}_{\{X_s\le h\}}\,ds} \mid X_0 = x]$$
$$- E\left[\int_{0+}^{\alpha\wedge T_m} e^{-\rho\int_0^s \mathbf{1}_{\{X_\xi\le h\}}\,d\xi} e_n(t+s-, X_{s-})\,ds \,\middle|\, X_0 = x\right]. \tag{18}$$

Let n go to $+\infty$ in (18). The left-hand side converges to v. Meanwhile, (14) and (15) allow us to apply the dominated convergence theorem on the right-hand side. Note that the second term on the right-hand side of (18) goes to zero. After taking the limit, (18) becomes

$$v(t,x) = E[v(t+\alpha\wedge T_m, X_{\alpha\wedge T_m})e^{-\rho\int_0^{\alpha\wedge T_m}\mathbf{1}_{\{X_s\le h\}}\,ds}|X_0 = x]. \tag{19}$$

Note that the term inside the expectation of (19) is bounded by

$$|v(t+\alpha\wedge T_m, X_{\alpha\wedge T_m})e^{-\rho\int_0^{\alpha\wedge T_m}\mathbf{1}_{\{X_s\le h\}}\,ds}| \le |v(t+\alpha\wedge T_m, X_{\alpha\wedge T_m})|$$
$$\le C_1 e^{C_2|x|+C_2|\bar{\mu}|T} e^{C_2\sigma\max_{0\le s\le T}|W_s|} e^{C_2\sum_{i=1}^{N_T}|Y_i|}$$

and the right-hand side can be shown to be integrable. We may be able to apply the dominated convergence theorem again on (19) to get the limit as m goes to $+\infty$. It follows that

$$v(t,x) = E[v(t+\alpha, X_\alpha)e^{-\rho\int_0^\alpha \mathbf{1}_{\{X_s\le h\}}\,ds}].$$

Let $\alpha = T-t$ in the last equation and recall the definition of v. We have

$$V(T-t,x) = v(t,x) = E[v(T, X_{T-t})e^{-\rho\int_0^{T-t}\mathbf{1}_{\{X_s\le h\}}\,ds}] = E[V(0, X_{T-t})e^{-\rho\int_0^{T-t}\mathbf{1}_{\{X_s\le h\}}\,ds}].$$

The right-hand side is equal to $E[e^{\gamma X_{T-t}-\rho\int_0^{T-t}\mathbf{1}_{\{X_s\le h\}}\,ds}]$. Because t is arbitrary, the proof is completed. $\square$

Equation (8) is a PIDE with a Cauchy boundary, noting that $\mathscr{L}$ involves both differential and integral operators. We intend to use the Laplace transform once again to convert it into an OIDE, which is much easier to solve. Consider the first equation in (8). Introduce the following Laplace transform on the (discounted) value of V:

$$u(x;a) = \int_0^{+\infty} e^{-at}\cdot e^{-rt}V(t,x)\,dt$$

for a sufficiently large positive a. Routine calculation shows that u must satisfy

$$(\rho\mathbf{1}_{\{x\le h\}} + r + a)u(x;a) - e^{\gamma x}$$
$$= \mathscr{L}u(x;a) = \tfrac{1}{2}\sigma^2 u''(x;a) + \bar{\mu}u'(x;a) + \lambda\int_{-\infty}^{\infty}[u(x+y;a) - u(x;a)]f_Y(y)\,dy. \tag{20}$$

Thus, we have successfully removed the partial derivative in (8). For a general jump density f_Y, it could still be very difficult to solve (20) for a closed-form solution. However, when f_Y is a double exponential density, (20) is solvable explicitly. We summarize the solution in the following theorem.

THEOREM 3.2. *For any* $0 \le \gamma < \min\{\eta, \theta\}$, $\rho > 0$ *and*

$$a + r > |\bar{\mu}|\gamma + 2\sigma^2\gamma^2 + \lambda\left(\frac{p\eta}{\eta-\gamma} + \frac{q\theta}{\theta-\gamma} - 1\right), \tag{21}$$

the Laplace transform

$$\begin{aligned} u(x;\rho,\gamma,a,h) &= \int_0^{\infty} e^{-(a+r)t} E[e^{-\rho\tau_t+\gamma X_t} \mid X_0 = x]\,dt \\ &= \begin{cases} \omega_1 e^{\beta_{1,a+\rho}(x-h)} + \omega_2 e^{\beta_{2,a+\rho}(x-h)} - c_1 e^{\gamma(x-h)}, & x \le h; \\ -\nu_1 e^{-\gamma_{1,a}(x-h)} - \nu_2 e^{-\gamma_{2,a}(x-h)} - c_2 e^{\gamma(x-h)}, & x > h, \end{cases} \end{aligned}$$

where

$$c_1 = \frac{e^{\gamma h}}{G(\gamma) - a - r - \rho}, \qquad c_2 = \frac{e^{\gamma h}}{G(\gamma) - a - r},$$

and

$$\omega_1 = \frac{(\beta_{2,a+\rho}-\gamma)(-\gamma_{1,a}-\gamma)(-\gamma_{2,a}-\gamma)(\eta-\beta_{1,a+\rho})(\theta+\beta_{1,a+\rho})}{(\beta_{2,a+\rho}-\beta_{1,a+\rho})(-\gamma_{1,a}-\beta_{1,a+\rho})(-\gamma_{2,a}-\beta_{1,a+\rho})(\eta-\gamma)(\theta+\gamma)} c_{12}, \tag{22}$$

$$\omega_2 = \frac{(\beta_{1,a+\rho}-\gamma)(-\gamma_{1,a}-\gamma)(-\gamma_{2,a}-\gamma)(\eta-\beta_{2,a+\rho})(\theta+\beta_{2,a+\rho})}{(\beta_{1,a+\rho}-\beta_{2,a+\rho})(-\gamma_{1,a}-\beta_{2,a+\rho})(-\gamma_{2,a}-\beta_{2,a+\rho})(\eta-\gamma)(\theta+\gamma)} c_{12}, \tag{23}$$

$$\nu_1 = \frac{(\beta_{1,a+\rho}-\gamma)(\beta_{2,a+\rho}-\gamma)(-\gamma_{2,a}-\gamma)(\eta+\gamma_{1,a})(\theta-\gamma_{1,a})}{(\beta_{1,a+\rho}+\gamma_{1,a})(\beta_{2,a+\rho}+\gamma_{1,a})(-\gamma_{2,a}+\gamma_{1,a})(\eta-\gamma)(\theta+\gamma)} c_{12}, \tag{24}$$

$$\nu_2 = \frac{(\beta_{1,a+\rho}-\gamma)(\beta_{2,a+\rho}-\gamma)(-\gamma_{1,a}-\gamma)(\eta+\gamma_{2,a})(\theta-\gamma_{2,a})}{(\beta_{1,a+\rho}+\gamma_{2,a})(\beta_{2,a+\rho}+\gamma_{2,a})(-\gamma_{1,a}+\gamma_{2,a})(\eta-\gamma)(\theta+\gamma)} c_{12}. \tag{25}$$

with

$$c_{12} = \frac{\rho e^{\gamma h}}{(G(\gamma)-a-r-\rho)(G(\gamma)-a-r)}. \tag{26}$$

PROOF. Fix constants ρ, γ, a, and h. Define

$$u(x) = \begin{cases} u_1(x), & x < h; \\ u_2(x), & x > h. \end{cases} \tag{27}$$

Then the nonhomogenous OIDE (20) can be rewritten as two separate equations in the regions $(-\infty, h)$ and $(h, +\infty)$. For $x < h$,

$$\begin{aligned} &\frac{\sigma^2}{2} u_1''(x) + \bar{\mu} u_1'(x) - (\lambda + a + r + \rho) u_1(x) \\ &\quad + \lambda\left[\int_{-\infty}^{0} u_1(x+y) q\theta e^{\theta y}\,dy + \int_0^{h-x} u_1(x+y) p\eta e^{-\eta y}\,dy + \int_{h-x}^{+\infty} u_2(x+y) p\eta e^{-\eta y}\,dy\right] = -e^{\gamma x}. \end{aligned} \tag{28}$$

and for $x > h$,

$$\begin{aligned} &\frac{\sigma^2}{2} u_2''(x) + \bar{\mu} u_2'(x) - (\lambda + a + r) u_2(x) \\ &\quad + \lambda\left[\int_{-\infty}^{h-x} u_1(x+y) q\theta e^{\theta y}\,dy + \int_{h-x}^{0} u_2(x+y) q\theta e^{\theta y}\,dy + \int_0^{+\infty} u_2(x+y) p\eta e^{-\eta y}\,dy\right] = -e^{\gamma x}. \end{aligned} \tag{29}$$

We claim that the solution $u_1(x)$ and $u_2(x)$ must be of the following form

$$\begin{cases} u_1(x) = \omega_1 e^{\beta_{1,a+\rho}(x-h)} + \omega_2 e^{\beta_{2,a+\rho}(x-h)} - c_1 e^{\gamma(x-h)}, & x \le h; \\ u_2(x) = -\nu_1 e^{-\gamma_{1,a}(x-h)} - \nu_2 e^{-\gamma_{2,a}(x-h)} - c_2 e^{\gamma(x-h)}, & x > h, \end{cases} \tag{30}$$

where ω_1, ω_2, ν_1, ν_2, c_1, and c_2 are constants to be determined. Indeed, for Equation (29), under a change of variable $z = x + y$, it is transformed further to

$$\frac{\sigma^2}{2}u_2''(x) = -\bar{\mu}u_2'(x) + (\lambda + a + r)u_2(x) - \lambda e^{-\theta x}\int_{-\infty}^{h} u_1(z)q\theta e^{\theta z}\,dz \\ - \lambda e^{-\theta x}\int_{h}^{x} u_2(z)q\theta e^{\theta z}\,dz - \lambda e^{\eta x}\int_{x}^{+\infty} u_2(z)p\eta e^{-\eta z}\,dz + e^{\gamma x}. \tag{31}$$

Our purpose is to remove the three integrals in (31), one by one, to reduce the OIDE to an ODE in order to make use of the theory of ODEs to solve the equation completely and to show the uniqueness of the solution at the same time. First, any solution to (31) must have the third-order derivative. This point is easily seen from the right-hand side of (31) because all terms are differentiable and so is $u''(x)$. Multiplying both sides of (31) by $e^{\theta x}$,

$$\frac{\sigma^2}{2}e^{\theta x}u_2''(x) = -\bar{\mu}e^{\theta x}u_2'(x) + (\lambda + a + r)e^{\theta x}u_2(x) - \lambda\int_{-\infty}^{h} u_1(z)q\theta e^{\theta z}\,dz \\ - \lambda\int_{h}^{x} u_2(z)q\theta e^{\theta z}\,dz - \lambda e^{(\theta+\eta)x}\int_{x}^{+\infty} u_2(z)p\eta e^{-\eta z}\,dz + e^{(\theta+\gamma)x}.$$

Take differentiation on both sides of this equation to remove the first integral. Dividing the resulting OIDE by $e^{-\theta x}$ yields

$$\frac{\sigma^2}{2}u_2'''(x) = -\left(\frac{\sigma^2}{2}\theta + \bar{\mu}\right)u_2''(x) - (\bar{\mu}\theta - \lambda - a - r)u_2'(x) + [(\lambda + a)\theta - \lambda q\theta + \lambda p\eta]u_2(x) \\ + \lambda(\eta + \theta)e^{\eta x}\int_{x}^{+\infty} u_2(z)p\eta e^{-\eta z}\,dz + (\theta + \gamma)e^{\gamma x}. \tag{32}$$

From (32), u should also be fourth-order differentiable. Hence, we can take a similar step to remove the integral in (32) to obtain a nonhomogeneous ODE with constant coefficients as follows:

$$\frac{\sigma^2}{2}u_2^{(4)}(x) + \left[-\frac{\sigma^2}{2}(\eta - \theta) + \bar{\mu}\right]u_2'''(x) + \left[-\frac{\sigma^2}{2}\eta\theta - \bar{\mu}(\eta - \theta) - \lambda - a - r\right]u_2''(x) \\ + [(\eta - \theta)(\lambda + a + r) - \bar{\mu}\eta\theta + \lambda q\theta - \lambda p\eta]u_2'(x) + a\eta\theta u_2(x) = (\eta - \gamma)(\theta + \gamma)e^{\gamma x}. \tag{33}$$

On one hand, it is easy to see that $c_2e^{\gamma x}$ is a particular solution to the ODE (33) for a constant c_2. On the other hand, the characteristic equation of the corresponding homogeneous ODE turns to be

$$(G(y) - a - r)(y + \theta)(y - \eta) = 0,$$

which has four real roots as mentioned in §2. Therefore, any solution to (33) can be expressed as

$$u_2(x) = \bar{\nu}_1 e^{\beta_{1,a}(x-h)} + \bar{\nu}_2 e^{\beta_{2,a}(x-h)} - \nu_1 e^{-\gamma_{1,a}(x-h)} - \nu_2 e^{-\gamma_{2,a}(x-h)} - c_2 e^{\gamma(x-h)}, \quad \text{for any } x > h,$$

with $\bar{\nu}_1$, $\bar{\nu}_2$, ν_1, ν_2, and c_2 undetermined. Furthermore, we can argue that the first two coefficients $\bar{\nu}_1$ and $\bar{\nu}_2$ should be zero. In fact, we know that

$$\frac{u_2(x)}{e^{\gamma x}} = \int_0^{\infty} e^{-(a+r)t}E[e^{-\rho\tau_t + \gamma(X_t - x)} \mid X_0 = x]\,dt = \int_0^{\infty} e^{-(a+r)t}E[e^{-\rho\tau_t + \gamma X_t} \mid X_0 = 0]\,dt,$$

where the last equality is because of the Lévy property of X. The right-hand side of the above equality is less than

$$\int_0^{\infty} e^{-(a+r-G(\gamma))t}\,dt = \frac{1}{a + r - G(\gamma)} < +\infty,$$

because $E[\exp(-\rho\tau_t + \gamma X_t) \mid X_0 = 0] < E[\exp(\gamma X_t) \mid X_0 = 0] = \exp(G(\gamma))$. Thus, $\lim_{x\to+\infty} u_2(x)/e^{\gamma x} < +\infty$. Note $\beta_{2,a} > \beta_{1,a} > \gamma$, which implies $\bar{\nu}_1$ and $\bar{\nu}_2$ must be zero. Consequently, any solution to the OIDE in (29) can be expressed as

$$u_2(x) = -\nu_1 e^{-\gamma_{1,a}(x-h)} - \nu_2 e^{-\gamma_{2,a}(x-h)} - c_2 e^{\gamma(x-h)}, \quad \text{for } x > h,$$

with c_2, ν_1, and ν_2 to be determined. Similarly, we also can show any solution to the first OIDE in (28) is expressed as

$$u_1(x) = \omega_1 e^{\beta_{1,a+\rho}(x-h)} + \omega_2 e^{\beta_{2,a+\rho}(x-h)} - c_1 e^{\gamma(x-h)}, \quad \text{for } x < h,$$

with c_1, ω_1, and ω_2 to be determined.

Now we need six equations to determine these coefficients. Substituting $u_1(x)$, $u_2(x)$ into (28–29) yields that for any $x < h$,

$$\begin{aligned}&[c_1 e^{-\gamma h}(G(\gamma) - a - r - \rho) - 1]e^{\gamma x}\\ &\quad + \lambda p \eta \left[\frac{\omega_1}{\eta - \beta_{1,a+\rho}} + \frac{\omega_2}{\eta - \beta_{2,a+\rho}} + \frac{\nu_1}{\eta + \gamma_{1,a}} + \frac{\nu_2}{\eta + \gamma_{2,a}} - \frac{c_1 - c_2}{\eta - \gamma}\right] e^{\eta(x-h)} = 0.\end{aligned}$$

and for any $x > h$,

$$\begin{aligned}&-[c_2 e^{-\gamma h}(G(\gamma) - a - r) - 1]e^{\gamma x}\\ &\quad + \lambda q \theta \left[\frac{\omega_1}{\theta + \beta_{1,a+\rho}} + \frac{\omega_2}{\theta + \beta_{2,a+\rho}} + \frac{\nu_1}{\theta - \gamma_{1,a}} + \frac{\nu_2}{\theta - \gamma_{2,a}} - \frac{c_1 - c_2}{\theta + \gamma}\right] e^{-\theta(x-h)} = 0.\end{aligned}$$

Therefore, u is a solution if and only if the coefficients ω_1, ω_2, ν_1, ν_2, c_1, and c_2 satisfy the following four equations:

$$\begin{aligned}&c_1(G(\gamma) - a - r - \rho) = e^{\gamma h}, \qquad \frac{\omega_1}{\eta - \beta_{1,a+\rho}} + \frac{\omega_2}{\eta - \beta_{2,a+\rho}} + \frac{\nu_1}{\eta + \gamma_{1,a}} + \frac{\nu_2}{\eta + \gamma_{2,a}} = \frac{c_1 - c_2}{\eta - \gamma},\\ &c_2(G(\gamma) - a - r) = e^{\gamma h}, \qquad \frac{\omega_1}{\theta + \beta_{1,a+\rho}} + \frac{\omega_2}{\theta + \beta_{2,a+\rho}} + \frac{\nu_1}{\theta - \gamma_{1,a}} + \frac{\nu_2}{\theta - \gamma_{2,a}} = \frac{c_1 - c_2}{\theta + \gamma}.\end{aligned}$$

In addition, we can also obtain another two equations from the fact that $u(x)$ is continuously differentiable at barrier h:

$$\begin{aligned}\omega_1 + \omega_2 - c_1 &= -\nu_1 - \nu_2 - c_2,\\ \beta_{1,a+\rho}\omega_1 + \beta_{2,a+\rho}\omega_2 - c_1\gamma &= \gamma_{1,a}\nu_1 + \gamma_{2,a}\nu_2 - c_2\gamma.\end{aligned}$$

All of these equations are linear with respect to the undetermined parameters. To solve them, first we can easily obtain that

$$c_1 = \frac{e^{\gamma h}}{G(\gamma) - a - r - \rho} \qquad \text{and} \qquad c_2 = \frac{e^{\gamma h}}{G(\gamma) - a - r}.$$

Substituting these two into the above linear system will reduce it further to

$$\mathbf{A}(\rho)\mathbf{c}(\rho, \gamma) = \mathbf{J}(\rho, \gamma), \tag{34}$$

with $\mathbf{c}(\rho, \gamma) = (\omega_1, \omega_2, \nu_1, \nu_2)^T$, $\mathbf{J}(\rho, \gamma) = c_{12}(1, \gamma, 1/(\eta - \gamma), 1/(\theta + \gamma))^T$, and

$$\mathbf{A}(\rho) = \begin{bmatrix} 1 & 1 & 1 & 1\\ \beta_{1,a+\rho} & \beta_{2,a+\rho} & -\gamma_{1,a} & -\gamma_{2,a}\\ \dfrac{1}{\eta - \beta_{1,a+\rho}} & \dfrac{1}{\eta - \beta_{2,a+\rho}} & \dfrac{1}{\eta + \gamma_{1,a}} & \dfrac{1}{\eta + \gamma_{2,a}}\\ \dfrac{1}{\theta + \beta_{1,a+\rho}} & \dfrac{1}{\theta + \beta_{2,a+\rho}} & \dfrac{1}{\theta - \gamma_{1,a}} & \dfrac{1}{\theta - \gamma_{2,a}} \end{bmatrix},$$

where $c_{12} = c_1 - c_2 = \rho e^{\gamma h}/((G(\gamma) - a - r - \rho)(G(\gamma) - a - r))$. Appendix C shows that the matrix $\mathbf{A}(\rho)$ is nonsingular and the coefficients defined by (22)–(25) solve the linear Equations (34). $\square$

We also can extend the above approach to derive the distribution of occupation times the process spends within two barriers. A minor technical gap remains. All detailed discussion is included in Appendix D.

REMARK 3.1. The key step in the whole proof lies in (31). The assumption of exponential-type jump distributions in Kou's model allows us to differentiate the OIDE in order to transform the OIDE to an ODE. It seems that our method does not apply for any jump distribution other than exponential-type distributions. For instance, this transformation will not be workable for Merton's jump diffusion model.

REMARK 3.2. Cai and Kou [5] studied a similar OIDE under a more general hyper-exponential jump diffusion model as follows:

$$\begin{cases} (\mathscr{L}\bar{u})(x) - (\bar{a}+r)u(x) = 0, & x < x_0; \\ \bar{u}(x) = \bar{g}(x), & x \geq x_0, \end{cases} \tag{35}$$

where $\bar{a} > 0$ and $\bar{g}(x)$ is a known function. By transforming (35) into a homogeneous linear ODE with constant coefficients, Cai and Kou managed to show that the solution to (35) must be of the form

$$\bar{u}(x) = \bar{\omega}_1 e^{\beta_{1,\bar{a}}(x-x_0)} + \bar{\omega}_2 e^{\beta_{2,\bar{a}}(x-x_0)} + \tilde{\omega}_1 e^{-\gamma_{1,\bar{a}}(x-x_0)} + \tilde{\omega}_2 e^{-\gamma_{2,\bar{a}}(x-x_0)}.$$

Despite the similarity, (8) is much more complicated because it is nonhomogeneous and furthermore it contains two OIDEs in two disjoint regions that are intertwined together due to the integral parts. We are still able to reduce it down to a linear ODE, applying the same technique as in Cai and Kou [5] after some modification.

REMARK 3.3. Note that several structured products issued on the real financial market have a payoff written on the occupation time, but with an interest rate or a spread of swap rates with different maturities as underlying. These underlying processes are usually of mean reversion structure. However, our approach would be hard to extend to the mean reversion jump diffusion cases. The primary technical barrier lies in the fact that the corresponding OIDE, in which the coefficient of the first derivative is not a constant but a linear function of state variable, is difficult to solve explicitly.

4. Pricing occupation-time-related options. In this section, several examples of occupation-time-related options accumulated in the literature are considered, including the step options suggested by Linetsky [21], the corridor options studied by Fusai [13], and the quantile options proposed by Miura [24]. Thanks to Theorem 3.2 and the special structures of these options, we can obtain closed-form expressions for the option prices in terms of their Laplace transforms and then make it possible to suggest hedging strategies accordingly. Furthermore, we are also able to calculate the price sensitivities very easily from the Laplace transforms, which is convenient for risk management on the options. This section uses delta as an example. The calculation of other Greeks is similar and thus omitted due to the space limitation.

From now on, we assume that L is the constant barrier to define the occupation times. Define $h = \log(L/S_0)$ as the associated barrier for the log-return process $\{X_t\}$.

4.1. Pricing step options. As mentioned in the introduction, Linetsky [22] introduced the step option to overcome the hedging problem inherent in standard barrier options around the barrier. For down-and-out step call options, the payoff at maturity is defined as the payoff of a standard European call option discounted at a rate that depends on the amount of time spent by the underlying asset below a prespecified barrier. We can classify these options into proportional step options, simple step options, and delayed barrier options according to different discounting schemes used.

4.1.1. Proportional (geometric) step options. In this section, we focus on pricing a proportional step call option, which has the payoff

$$e^{-\rho\tau_T(h)}(S_0 e^{X_T} - K)^+,$$

where ρ is the nonnegative knock-out rate, S_0 is the initial underlying asset price, X_T is the log-return value of the underlying asset price at maturity T, and $\tau_T(h)$ is the occupation time as defined in (4). The pricing method also applies to proportional step put options.

In some sense the proportional step option can be regarded as an extension of the standard barrier option and the vanilla European option. With a finite positive knock-out rate ρ, it is obvious that

$$\mathbf{1}_{\{\varsigma_h > T\}}(S_0 e^{X_T} - K)^+ \leq e^{-\rho\tau_T(h)}(S_0 e^{X_T} - K)^+ \leq (S_0 e^{X_T} - K)^+, \tag{36}$$

where ς_h is defined as the first passage time of $\{X_t\}$ to the barrier h, i.e., $\varsigma_h = \inf\{t \geq 0\colon X_t \leq h\}$. The payoff of the proportional step call option is sandwiched by the payoff of the vanilla European call on the right-hand side of (36) and the payoff of the down-and-out barrier call on the left-hand side of (36). When $\rho = 0$, the payoff of the step option coincides with that of the vanilla call. As ρ approaches $+\infty$, it tends to the payoff of the down-and-out barrier call.

Additionally, (36) also reveals one advantage of the step option over the standard barrier option. The down-and-out barrier call eliminates the payoff to the investor immediately if the underlying process $\{X_t\}$ touches the barrier h at or before T, i.e., $\mathbf{1}_{\{\varsigma_h > T\}} = 0$. However, the payoff of the step option does not disappear when X crosses the boundary. Investors still receive a portion of the original payoff discounted depending upon the length of the period that $\{X_t\}$ spends below h. This mollifies the discontinuity of the barrier options around h, which eases the difficulty of risk management on barrier options to some degree. We have discussed it briefly in the introduction section and Linetsky [22] has offered more details.

Under the risk-neutral probability measure, the proportional step call option price is

$$C_1(K, T) = e^{-rT} E[e^{-\rho\tau_T(h)}(S_0 e^{X_T} - K)^+ \mid S_0].$$

Make a change of variable $\kappa = -\log K$ for the convenience of later applying Laplace transforms. Then, we have

$$C_1(\kappa, T) = e^{-rT} E[e^{-\rho\tau_T(h)}(S_0 e^{X_T} - e^{-\kappa})^+ \mid S_0].$$

Taking double Laplace transforms on the price function $C_1(\kappa, T)$ with respect to κ and T, respectively, and applying the Fubini theorem to interchange the order of the expectation and the integral with respect to κ, we obtain

$$g_1(\varphi, a) := \int_0^\infty dT \int_{-\infty}^\infty e^{-\varphi\kappa - aT} C_1(\kappa, T)\, d\kappa = \frac{S_0^{\varphi+1}}{\varphi(\varphi+1)} \int_0^\infty e^{-aT} E[e^{-\int_0^T k(X_s)\,ds + (\varphi+1)X_T}]\, dT. \tag{37}$$

Using Theorem 3.2, we can derive an explicit closed-form expression for the double Laplace transform above.

Theorem 4.1. *With the initial underlying asset price S_0 and barrier L, assuming that (21) is satisfied, then for any $a > 0$ and $0 < \varphi < \min\{\eta, \theta\} - 1$, the double Laplace transform of the proportional step call option price $C_1(\kappa, T)$ is*

$$g_1(\varphi, a) = \frac{S_0^{\varphi+1}}{\varphi(\varphi+1)} u(0; \rho, \varphi+1, a, \log(L/S_0)),$$

where $u(x; \rho, \gamma, a, h)$ is given by Theorem 3.2.

The delta of an option is defined as the derivative of the option price with respect to the current underlying price S_0. Taking differentiation under the integral (37), we can easily see that

$$\frac{\partial}{\partial S_0} g_1(\varphi, a) = \int_0^\infty dT \int_{-\infty}^\infty e^{-\varphi\kappa - aT} \frac{\partial}{\partial S_0} C_1(\kappa, T)\, d\kappa.$$

Accordingly, the transform of the delta is just the derivative of the transform of the price function with respect to S_0. Hence, the delta of the step option is also obtainable through the Laplace transform.

4.1.2. Simple (arithmetic) step options and delayed barrier options. In addition to the proportional step options, Linetsky [21] also discussed two other kinds of step options, simple (arithmetic) step options and delayed barrier options. Laplace transform techniques can also lead to analytical solutions to pricing problems of these two step options.

The simple step option uses a discounting scheme that is different from what is used for the proportional step option. The payoff of a simple step call option is defined as

$$(1 - \tau_T(h)/\vartheta)^+ \cdot (S_T - K)^+.$$

With a positive knock-out rate $1/\vartheta$, investors will lose the option payoff gradually until the occupation time accumulates up to ϑ, when they will lose all of the value. This is a major difference from the proportional step option, where investors will never lose the entire option value.

It is simple to convert the pricing problem of simple step options into that of the proportional step options we discussed in §4.1.1 via Laplace transform. Note that for any $\rho > 0$,

$$\begin{aligned}\int_0^\infty \vartheta C_2(K, T, \vartheta) e^{-\rho\vartheta}\, d\vartheta &= e^{-rT} E\left[\int_0^\infty \vartheta(1 - \tau_T(h)/\vartheta)^+ e^{-\rho\vartheta}\, d\vartheta \cdot (S_0 e^{X_T} - K)^+ \mid S_0\right] \\ &= \frac{e^{-rT}}{\rho^2} E[e^{-\rho\tau_{T(h)}}(S_0 e^{X_T} - K)^+] = \frac{1}{\rho^2} C_1(\rho; K, T).\end{aligned}$$

The right-hand side of the formula above is calculable via double Laplace inversion. Thus, we can essentially apply triple Laplace inversion to obtain C_2, The numerical experiment in §5 indicates that the computation is still very efficient.

The delayed barrier option poses an alternative discount factor $\mathbf{1}_{\{\tau_T(h)<\vartheta\}}$ on the payoff of the vanilla European call. Hence, the option value is wiped out completely if and only if $\tau_T(h) > \vartheta$. We can also convert the associated pricing problem into that of a proportional option formulation by taking a Laplace transform with respect to ϑ:

$$\int_0^\infty e^{-\rho\vartheta} C_3(K,T,\vartheta)\,d\vartheta = e^{-rT} E\left[\int_0^\infty \mathbf{1}_{\{t<\vartheta\}} e^{-\rho\vartheta}\,d\vartheta \cdot (S_0 e^{X_T} - K)^+ \mid S_0\right] = \frac{1}{\rho} C_1(\rho; K, T).$$

Hence, triple Laplace inversion can also be applied to price delayed barrier options numerically.

4.2. Pricing corridor options. The corridor option is another example of occupation-time-related options. It pays an amount at the maturity, dependent upon the time spent by a reference market variable below (or above) a given barrier or inside an interval. The former option, i.e., the corridor option with single barrier, is usually referred to as the hurdle option. In this subsection, we will concentrate on hurdle options only. Corridor options with double barriers can be priced similarly. For details, see Appendix D. It is worth mentioning that Fusai [13] studied the pricing of corridor options with double barriers under the GBM model. His approach relied on the special properties of Brownian motion.

A corridor option with single barrier has the payoff $\max\{\tau_T(h) - K, 0\}$ for a given strike $K < T$, and its price at time zero is thus given by

$$\mathrm{Cor}(K,T) = e^{-rT} E[\max\{\tau_T(h) - K, 0\}].$$

We need the expectation of $\tau_T(h)$ to proceed the price calculation. A nice property of the Laplace transform of a probability distribution is that we can obtain any order moments of the distribution through the derivatives of its Laplace transform at zero. Keeping this property in mind and using the notations in Theorem 3.2, we have

$$\int_0^\infty e^{-(a+r)T} E[\tau_T(h)]\,dT = \int_0^\infty e^{-(a+r)T} \frac{\partial}{\partial\rho}\bigg|_{\rho=0} E[e^{-\rho\tau_T(h)+\gamma X_t} \mid X_0 = x]\,dT = \frac{\partial u}{\partial\rho}(x; 0, \gamma, a, h). \tag{38}$$

Then, taking a double Laplace transform of $\mathrm{Cor}(K,T)$ with respect to K and T, i.e.,

$$g_{\mathrm{cor}}(\varphi, a) = \int_0^\infty \int_0^\infty e^{-\varphi K - aT}\,\mathrm{Cor}(K,T)\,dK\,dT,$$

we can obtain Theorem 4.2 as follows:

THEOREM 4.2. *For any φ and $a > 0$, we have*

$$g_{cor}(\varphi, a) = -\frac{1}{\varphi}\frac{\partial u}{\partial\rho}(0; 0, 0, a, \log(L/S_0)) + \frac{1}{\varphi^2} u(0; \varphi, 0, a, \log(L/S_0)) - \frac{1}{(a+r)\varphi^2}. \tag{39}$$

PROOF. Applying the Fubini theorem to interchange the order of expectation and integrals in g_{cor}, we have

$$\begin{aligned} g_{\mathrm{cor}}(\varphi, a) &= \int_0^\infty e^{-(a+r)T}\,dT E\left[\int_0^\infty e^{-\varphi K} \max\{\tau_T(h) - K, 0\}\,dK\right] \\ &= -\frac{1}{\varphi}\int_0^\infty e^{-(a+r)T} E[\tau_T(h)]\,dT + \frac{1}{\varphi^2}\int_0^\infty e^{-(a+r)T} E[e^{-\varphi\tau_T(h)}]\,dT - \frac{1}{(a+r)\varphi^2}. \end{aligned}$$

The integral in the second term on the right-hand side above can be represented by $u(0; \varphi, 0, a, h)$. In addition, we know from (38) that the integral in the first term is $\partial u(0; \rho, 0, a, h)/\partial\rho$. The theorem is proved. □

What is interesting here is that we can also obtain a closed-form expression for $\partial u/\partial\rho$, which is convenient when calculating g_{cor}.

PROPOSITION 4.1. *For any $a > 0$, we have*

$$\frac{\partial u}{\partial\rho}(0; 0, 0, a, \log(L/S_0)) = \begin{cases} \tilde{\omega}_1\left(\dfrac{S_0}{L}\right)^{\beta_{1,a}} + \tilde{\omega}_2\left(\dfrac{S_0}{L}\right)^{\beta_{2,a}} - \dfrac{1}{(a+r)^2}, & S_0 \le L; \\ -\tilde{\nu}_1\left(\dfrac{L}{S_0}\right)^{\gamma_{1,a}} - \tilde{\nu}_2\left(\dfrac{L}{S_0}\right)^{\gamma_{2,a}}, & S_0 > L, \end{cases} \tag{40}$$

where

$$\tilde{\omega}_1 = \frac{\beta_{2,a}\gamma_{1,a}\gamma_{2,a}}{\eta\theta(a+r)^2}\frac{(\beta_{1,a}-\eta)(\beta_{1,a}+\theta)}{(\beta_{1,a}-\beta_{2,a})(\beta_{1,a}+\gamma_{1,a})(\beta_{1,a}+\gamma_{2,a})},$$
$$\tilde{\omega}_2 = \frac{\beta_{1,a}\gamma_{1,a}\gamma_{2,a}}{\eta\theta(a+r)^2}\frac{(\beta_{2,a}-\eta)(\beta_{2,a}+\theta)}{(\beta_{2,a}-\beta_{1,a})(\beta_{2,a}+\gamma_{1,a})(\beta_{2,a}+\gamma_{2,a})},$$
$$\tilde{\nu}_1 = \frac{\beta_{1,a}\beta_{2,a}\gamma_{2,a}}{\eta\theta(a+r)^2}\frac{(\gamma_{1,a}+\eta)(\gamma_{1,a}-\theta)}{(\gamma_{1,a}+\beta_{1,a})(\gamma_{1,a}+\beta_{2,a})(\gamma_{1,a}-\gamma_{2,a})},$$
$$\tilde{\nu}_2 = \frac{\beta_{1,a}\beta_{2,a}\gamma_{1,a}}{\eta\theta(a+r)^2}\frac{(\gamma_{2,a}+\eta)(\gamma_{2,a}-\theta)}{(\gamma_{2,a}+\beta_{1,a})(\gamma_{2,a}+\beta_{2,a})(\gamma_{2,a}-\gamma_{1,a})}.$$

PROOF. According to Theorem 3.2, u is a piecewise-defined function. To emphasize their dependence on ρ and γ, we rewrite c_1, c_2, ω_1, ω_2, ν_1, and ν_2 as $c_1(\rho,\gamma)$, $c_2(\gamma)$, $\omega_1(\rho,\gamma)$, $\omega_2(\rho,\gamma)$, $\nu_1(\rho,\gamma)$, and $\nu_2(\rho,\gamma)$, respectively. When $x \le h \equiv \log(L/S_0)$, by the product rule of function derivative, we have

$$\frac{\partial u}{\partial \rho} = e^{\beta_{1,a+\rho}(x-h)}\left(\frac{\partial \omega_1(\rho,\gamma)}{\partial\rho} + \omega_1(\rho,\gamma)\frac{\partial\beta_{1,a+\rho}}{\partial\rho}\right) + e^{\beta_{2,a+\rho}(x-h)}\left(\frac{\partial \omega_2(\rho,\gamma)}{\partial\rho} + \omega_2(\rho,\gamma)\frac{\partial\beta_{2,a+\rho}}{\partial\rho}\right) - \frac{\partial c_1(\rho,\gamma)}{\partial\rho}e^{\gamma(x-h)}. \tag{41}$$

Letting $x=\rho=\gamma=0$ and noting that $\omega_1(0,\gamma)=\omega_2(0,\gamma)=0$ (cf. (22), (23), and (26)), we obtain that when $S_0 \le L$,

$$\left.\frac{\partial u}{\partial\rho}\right|_{\rho=0} = \left(\frac{S_0}{L}\right)^{\beta_{1,a}}\left.\frac{\partial\omega_1(\rho,\gamma)}{\partial\rho}\right|_{(\rho,\gamma)=(0,0)} + \left(\frac{S_0}{L}\right)^{\beta_{2,a}}\left.\frac{\partial\omega_2(\rho,\gamma)}{\partial\rho}\right|_{(\rho,\gamma)=(0,0)} - \left.\frac{\partial c_1(\rho,\gamma)}{\partial\rho}\right|_{(\rho,\gamma)=(0,0)}. \tag{42}$$

Similarly, when $S_0 > L$, we have

$$\left.\frac{\partial u}{\partial\rho}\right|_{\rho=0} = -\left(\frac{L}{S_0}\right)^{\gamma_{1,a}}\left.\frac{\partial\nu_1(\rho,\gamma)}{\partial\rho}\right|_{(\rho,\gamma)=(0,0)} - \left(\frac{L}{S_0}\right)^{\gamma_{1,a}}\left.\frac{\partial\nu_2(\rho,\gamma)}{\partial\rho}\right|_{(\rho,\gamma)=(0,0)}. \tag{43}$$

Note $\mathbf{c}(\rho,\gamma) = (\omega_1(\rho,\gamma), \omega_2(\rho,\gamma), \nu_1(\rho,\gamma), \nu_2(\rho,\gamma))$ is the solution of the linear system (34), i.e., $A(\rho)c(\rho,\gamma) = J(\rho,\gamma)$. Then

$$\left.\frac{\partial\mathbf{A}(\rho)}{\partial\rho}\right|_{\rho=0}\mathbf{c}(0,0) + \mathbf{A}(0)\left.\frac{\partial\mathbf{c}(\rho,\gamma)}{\partial\rho}\right|_{(\rho,\gamma)=(0,0)} = \left.\frac{\partial\mathbf{J}(\rho,\gamma)}{\partial\rho}\right|_{(\rho,\gamma)=(0,0)}.$$

The fact that $\mathbf{c}(0,0)=\mathbf{0}$ implies

$$\mathbf{A}(0)\left.\frac{\partial\mathbf{c}(\rho,\gamma)}{\partial\rho}\right|_{(\rho,\gamma)=(0,0)} = \left.\frac{\partial\mathbf{J}(\rho,\gamma)}{\partial\rho}\right|_{(\rho,\gamma)=(0,0)}.$$

In other words, we can obtain $\partial\mathbf{c}(\rho,\gamma)/\partial\rho|_{(\rho,\gamma)=(0,0)}$, i.e., the partial derivatives of $(\omega_1(\rho,\gamma), \omega_2(\rho,\gamma), \nu_1(\rho,\gamma), \nu_2(\rho,\gamma))$ at $(0,0)$ by solving the above equations. Then substituting the result back into (42) and (43) yields (40) immediately, which completes the proof. □

4.3. Pricing quantile options. Miura [24] introduced α-quantile options as an extension of lookback options. Its payoff depends on the α-quantile of the underlying asset price process, which is defined as

$$q(\alpha,T) = \inf\{h\colon \tau_T(h) > \alpha T\}, \quad \text{for any } \alpha\in[0,1].$$

Following Dassios [9], we will investigate the pricing of the fixed-strike α-quantile call option with payoff $(S_0e^{\gamma q(\alpha,T)} - K)^+$. It is worth mentioning that when $\alpha=0$ and $\gamma=1$, $q(\alpha,T)$ is the running maximum of $\{X_t\}$ over $[0,T]$ so that the quantile option is reduced to the lookback option.

For any $0 \le v \le T$, let

$$\mathrm{Qua(v,T) = e^{-rT}E[(S_0e^{\gamma q(v/T,T)} - K)^+]}$$

be the (v/T)-quantile option price. A key observation that

$$\{\tau_T(h) < v\} \equiv \{q(v/T, T) > h\} \tag{44}$$

links the quantile options with occupation times. The Laplace transform of $\tau_T(h)$ helps us again to establish a theorem as follows on the closed-form double Laplace transform of the quantile option price. Inverting the transform can then produce numerical prices.

Theorem 4.3. *Assume that $0 < \gamma < \min\{\eta, \theta\}$. For any $a > 0$ and $\rho > 0$ such that $G(\gamma) < a + \rho + r$, the double Laplace transform of $Qua(v, T)$ with respect to v and T is given by*

$$g_{Qua}(\rho, a) = \int_0^\infty \int_0^\infty e^{-\rho v} e^{-aT} Qua(v, T) \mathbf{1}_{\{v<T\}} \, dT \, dv$$
$$= \begin{cases} \dfrac{\gamma K}{\rho} \dfrac{\omega_1}{\beta_{1,a+\rho} - \gamma} (S_0/K)^{\beta_{1,a+\rho}/\gamma} + \dfrac{\gamma K}{\rho} \dfrac{\omega_2}{\beta_{2,a+\rho} - \gamma} (S_0/K)^{\beta_{2,a+\rho}/\gamma}, & \textit{if } K \geq S_0; \\ \dfrac{\gamma S_0}{\rho} \dfrac{\omega_1}{\beta_{1,a+\rho} - \gamma} + \dfrac{\gamma S_0}{\rho} \dfrac{\omega_2}{\beta_{2,a+\rho} - \gamma} - \dfrac{\gamma S_0}{\rho} \dfrac{\nu_1}{\gamma_{1,a} + \gamma} (1 - (K/S_0)^{(\gamma_{1,a}+\gamma)/\gamma}) & \\ \quad - \dfrac{\gamma S_0}{\rho} \dfrac{\nu_2}{\gamma_{2,a} + \gamma} (1 - (K/S_0)^{(\gamma_{2,a}+\gamma)/\gamma}) + \dfrac{S_0 - K}{(a+r)(a+r+\rho)}, & \textit{if } K < S_0, \end{cases}$$

where ω_1, ω_2, ν_1, and ν_2 are given by Theorem 3.2 with both γ and h replaced by zero.

Proof. With the change of variable $s = T - v$, we have

$$g_{\text{Qua}}(\rho, a) = \int_0^\infty \int_0^\infty e^{-(a+\rho)v} e^{-as} \text{Qua}(v, v+s) \, ds \, dv. \tag{45}$$

Note that for any random variable Y,

$$E[(Y-K)^+] = \int_K^{+\infty} P(Y > u) \, du.$$

In particular,

$$\text{Qua}(v, v+s) = e^{-r(v+s)} \int_K^{+\infty} P\left[q\left(\frac{v}{v+s}, v+s\right) > \frac{1}{\gamma}\log(u/S_0)\right] du.$$

Introduce another change of variable such that $h = \log(u/S_0)/\gamma$. Then,

$$\text{Qua}(v, v+s) = e^{-r(v+s)} \gamma S_0 e^{\gamma h} \int_k^\infty P\left[q\left(\frac{v}{v+s}, v+s\right) > h\right] dh,$$

where $k = \log(K/S_0)/\gamma$. The equivalence (44) implies

$$\text{Qua}(v, v+s) = e^{-r(v+s)} \gamma S_0 e^{\gamma h} \int_k^\infty P[\tau_{v+s}(h) < v] \, dh. \tag{46}$$

Substituting (46) back into (45) leads to

$$g_{\text{Qua}}(\rho, a) = \int_k^\infty \gamma S_0 e^{\gamma h} \left(\int_0^\infty \int_0^\infty e^{-(a+r)(v+s)-\rho v} P[\tau_{v+s}(h) < v] \, ds \, dv \right) dh. \tag{47}$$

The double integral (47) becomes

$$\int_0^\infty \int_0^\infty e^{-(a+r)(v+s)-\rho v} P[\tau_{v+s}(h) < v] \, ds \, dv = \int_0^\infty e^{-(a+r)t} \, dt \int_0^t e^{-\rho v} P[\tau_t(h) < v] \, dv$$
$$= \frac{1}{\rho} \int_0^\infty e^{-(a+r)t} E[e^{-\rho \tau_t(h)}] \, dt - \frac{1}{\rho(a+r+\rho)}$$

under a change of variable $t = v + s$. The integral on the right-hand side of this equality is equal to $u(0, \rho, 0, a, h)$ by Theorem 3.2. Hence,

$$\int_0^\infty \int_0^\infty e^{-(a+r)(v+s)-\rho v} P[\tau_{v+s}(h) < v] \, ds \, dv = \begin{cases} \dfrac{1}{\rho} (\omega_1 e^{-\beta_{1,a+\rho} h} + \omega_2 e^{-\beta_{2,a+\rho} h}), & h \geq 0; \\ -\dfrac{1}{\rho} (\nu_1 e^{\gamma_{1,a} h} + \nu_2 e^{\gamma_{2,a} h}) + \dfrac{1}{(a+r)(a+r+\rho)}, & h < 0. \end{cases}$$

Plugging this into (47), routine calculation will complete the proof. $\square$

TABLE 1. The double Laplace inversion (EI price) vs. Monte Carlo simulation (MC value) under the DEM.

		$\sigma = 0.2$			$\sigma = 0.3$		
S_0	K	EI price	MC value	Std. err.	EI price	MC value	Std. err.
		Prices of proportional step options under the DEM					
100	90	13.81882988	13.84674076	0.01999824	16.46304873	16.49739972	0.02805678
	100	9.42438004	9.45073300	0.02077236	12.47130080	12.50586340	0.02936357
	110	5.97929056	6.00093565	0.02087176	9.17850577	9.21094825	0.03010849
105	90	19.04025239	19.06951901	0.01933582	21.11914511	21.15570510	0.02798558
	100	13.45926395	13.48746393	0.02121837	16.30658585	16.34289184	0.03019022
	110	8.90133738	8.93024825	0.02272951	12.24916449	12.28871355	0.03195427
		Deltas of proportional step options under the DEM					
100	90	0.96243741	0.96296267	0.00149024	0.87472711	0.87493294	0.00135611
	100	0.73048507	0.73064749	0.00128122	0.71370367	0.71343802	0.00126079
	110	0.51700296	0.51785311	0.00122150	0.56523162	0.56654006	0.00123985
102	90	1.07858913	1.07768208	0.00156524	0.94680248	0.94689904	0.00139872
	100	0.82650108	0.82629754	0.00133499	0.77681513	0.77684003	0.00129342
	110	0.59299438	0.59407950	0.00126350	0.61938116	0.62083247	0.00126643

Notes. The default choices are $\lambda = 3$, $r = 0.05$, $\eta = 30$, $\theta = 20$, $p = q = 0.5$, $L = 102$, $\rho = 1$, and $t = 1$. The CPU time for the Laplace inversion method is around 3.5 seconds. MC values along with the associated standard errors (denoted by std. err.) are obtained by using 50,000 time steps and simulating 100,000 sample paths, and the CPU time is around 10 minutes. This table shows that all of the EI prices stay within the 95% confidence intervals of the associated MC values.

REMARK 4.1. Cai [4] developed a method to price both the fixed- and floating-strike quantile options numerically using Laplace inversion twice under a more general hyper-exponential jump diffusion model. Our method improves the efficiency because it requires inversion only once. His method can also be used to price floating-strike quantile options under a more general jump diffusion model.

5. Numerical results. In this section we present numerical results of the option's prices and hedging parameters. For numerical pricing and hedging of options via Laplace inversion, we use the analytical formulae in §4 and the multidimensional Euler inversion algorithm, which was introduced by Choudhury et al. [7] and was extended to the two-sided case by Petrella [26].

5.1. Proportional step options. We use the modified two-sided Euler inversion algorithm of Petrella [26] to invert the two-sided Laplace transform with respect to κ for the proportional step option. This algorithm is faster and more stable numerically than the original Euler inversion when dealing with two-sided transforms, due to the introduction of a scaling factor. The numerical results for the proportional step option prices (denoted by EI price) are given in Table 1, where we also show the Monte Carlo simulation results (denoted by MC value) as

TABLE 2. How the prices and deltas of a proportional step option change as λ goes to zero.

Numerical results when the jump intensity is small		
	Prices	Deltas
λ	P	Δ
0.1	6.802016390247875	0.616344715465678
0.01	6.782466159399428	0.616240150248286
0.001	6.780507945848759	0.616229613080276
0.0001	6.780312092511664	0.616228558551593
0.00001	6.780292506857593	0.616228453089914
0	6.780290330669454	0.616228441372041

Notes. When $\lambda \to 0$, both of the prices and deltas converge to those under the GBM model. The parameters we use are the same as the setting in Table 5.3 of Linetsky [22]: $r = 0.05$, $\sigma = 0.6$, $L = 95$, $S_0 = 100$, $K = 100$, and $t = 0.5$. The jump parameters are $\eta = 30$, $\theta = 20$, and $p = q = 0.5$. When $\lambda = 0$, our results are the same as Linetsky's.

a benchmark together with the associated 95% confidence intervals (denoted by 95% CI). The numerical prices are given at the top and the delta values are given at the bottom. We can see that all the EI prices stay within the 95% confidence intervals of the associated MC values. The pricing method based on our analytical pricing formulae as well as the Euler inversion algorithm is accurate and efficient.

As λ approaches zero, the double exponential jump diffusion model will converge to a geometric Brownian motion. Therefore, we can expect both the price and delta of occupation-time-related options under the DEM should also converge to those under the GBM. Table 2 verifies this intuition. Furthermore, it shows that our numerical method works for GBM as well because it replicates Linesky's result when we take $\lambda = 0$.

Table 3. The Laplace inversion (EI price) vs. Monte Carlo simulation (MC value).

		$\sigma = 0.2$			$\sigma = 0.3$		
S_0	K	EI price	MC value	Std. err.	EI price	MC value	Std. err.
		Prices of simple step options under the DEM with $\lambda = 3$					
100	90	9.67457995	9.70774495	0.02985213	12.01854880	12.06531905	0.03945057
	100	7.07669587	7.10395013	0.02529039	9.56487211	9.60638463	0.03568811
	110	4.75390837	4.77502124	0.02191003	7.33033262	7.36671170	0.03279602
102	90	12.16683520	12.20418981	0.03073929	14.18169934	14.22878114	0.04039522
	100	8.92866361	8.95838153	0.02642060	11.30765966	11.34847947	0.03686850
	110	6.03645208	6.05956925	0.02348342	8.69198201	8.72920403	0.03430783
		Prices of delayed barrier options under the DEM with $\lambda = 3$					
100	90	14.25719729	14.28598897	0.03006500	17.17147708	17.21579034	0.03853737
	100	10.08003700	10.10164481	0.02591103	13.30997994	13.34463574	0.03570174
	110	6.52095740	6.53852537	0.02366194	9.92977812	9.96000715	0.03416747
102	90	16.39440581	16.43625061	0.02796657	19.05103096	19.09098213	0.03691433
	100	11.63440011	11.66789910	0.02483299	14.80000641	14.83260573	0.03481742
	110	7.59164287	7.61545252	0.02366583	11.08553402	11.11870624	0.03402495

Prices of corridor options with single barrier under the DEM with $\lambda = 3$

		$\sigma = 0.2$			$\sigma = 0.3$		
K	S_0	EI price	MC value	Std. err.	EI price	MC value	Std. err.
0.2	95	0.46627793	0.46580529	0.00060334	0.44628615	0.44596639	0.00062321
	100	0.34654861	0.34620580	0.00064820	0.35821596	0.35802303	0.00065590
	105	0.22446654	0.22460171	0.00061260	0.26863327	0.26885828	0.00064341
0.4	95	0.31194613	0.31159386	0.00050566	0.29695209	0.29670152	0.00052092
	100	0.22032156	0.22018846	0.00052382	0.22911706	0.22904177	0.00053315
	105	0.13161829	0.13177739	0.00046635	0.16241928	0.16287320	0.00050042

Prices of quantile options under the DEM with $\lambda = 3$

		$\sigma = 0.2$			$\sigma = 0.3$		
α	K	EI price	MC value	Std. err.	EI price	MC value	Std. err.
0.2	90	6.98491715	7.00339911	0.01605925	6.72911720	6.75874909	0.02063748
	100	2.08465538	2.09972912	0.01122946	2.69357957	2.71698281	0.01554383
	110	0.37724012	0.38423388	0.00552578	0.86545323	0.88162865	0.00988966
0.5	90	12.59539246	12.61168267	0.02098495	13.77086937	13.79326547	0.02941576
	100	5.90331831	5.92048348	0.01876866	7.84321530	7.87143782	0.02685232
	110	2.29109044	2.30873387	0.01459738	4.15347044	4.17954592	0.02304078

Notes. For the simple step and delayed barrier options, the default parameter choices are $\lambda = 3$, $r = 0.05$, $\eta = 30$, $\theta = 20$, $p = q = 0.5$, $L = 102$, $\vartheta = 0.5$, and $t = 1$. For the corridor options with single barrier, the default parameter choices are $\lambda = 3$, $r = 0.05$, $\eta = 30$, $\theta = 20$, $p = q = 0.5$, $L = 102$, and $t = 1$. For the quantile options, the default parameter choices are $\lambda = 3$, $r = 0.05$, $\eta = 34$, $\theta = 34$, $p = 0.6$, $q = 0.4$, $S_0 = 100$, $\gamma = 1$, and $t = 1$. All Monte Carlo values (denoted by MC value) along with the associated standard errors (denoted by std. err.) are obtained using 50,000 time steps and simulating 100,000 sample paths. The CPU time of our numerical methods for generating one price of simple step or delayed barrier options, corridor options, and quantile options is around 2 minutes, 3 seconds and 3 seconds, respectively. The CPU time for Monte Carlo simulation is around 22 minutes for the quantile options and around 10 minutes for the other three type of options. The table indicates that all the EI prices stay within the 95% confidence intervals of the associated MC values.

5.2. Simple step, delayed barrier, corridor, and quantile options. The numerical prices and delta values of other occupation-time-related options, including simple step, delayed barrier, corridor, and quantile options, are given in Tables 3 and 4.

For the pricing and hedging of the simple step and the delayed barrier options, we need to do triple Laplace inversions. First, we use a two-dimensional Euler inversion formula for the complex-valued function (Formula (2.7) in Choudhury et al. [7] with $l_1 = l_2 = 1$) and then we do an extra one-dimensional Euler inversion (Formula (4.6) in Abate and Whitt [1]). Our results show that the average time spent by one triple Laplace inversion is

TABLE 4. The Laplace inversion (EI value) vs. Monte Carlo simulation (MC value).

		$\sigma = 0.2$			$\sigma = 0.3$		
S_0	K	EI price	MC value	Std. err.	EI price	MC value	Std. err.
		Delta of simple step options under the DEM with $\lambda = 3$					
100	90	1.13763436	1.13898389	0.00377164	1.01630712	1.02072949	0.00332720
	100	0.84343613	0.84598812	0.00282432	0.81774321	0.82201992	0.00274491
	110	0.58164514	0.58407656	0.00219618	0.63765092	0.64174368	0.00233056
102	90	1.35962149	1.35725685	0.00391806	1.14862657	1.14872893	0.00336411
	100	1.01249653	1.01289927	0.00291876	0.92657250	0.92805388	0.00276581
	110	0.70396190	0.70573895	0.00226845	0.72529944	0.72715609	0.00234920
		Delta of delayed barrier options under the DEM with $\lambda = 3$					
100	90	1.04040458	1.00570473	0.02406215	0.92115385	0.90869264	0.02433650
	100	0.75353143	0.73751457	0.01574238	0.72850514	0.72237409	0.01771596
	110	0.51485647	0.50789855	0.00969742	0.56297401	0.55980380	0.01246820
102	90	1.09523707	1.07269986	0.02342268	0.95773171	0.93864846	0.02459732
	100	0.79990286	0.78274507	0.01498936	0.76107256	0.75647237	0.01839117
	110	0.55545444	0.54777543	0.00905981	0.59252471	0.59962847	0.01345097

Delta of corridor options with single barrier under the DEM with $\lambda = 3$

		$\sigma = 0.2$			$\sigma = 0.3$		
K	S_0	EI price	MC value	Std. err.	EI price	MC value	Std. err.
0.2	100	−0.02563184	−0.02556980	0.00008900	−0.01844483	−0.01845705	0.00006192
	102	−0.02669485	−0.02658852	0.00008981	−0.01900456	−0.01903014	0.00006164
	104	−0.02209558	−0.02199631	0.00008277	−0.01683142	−0.01685143	0.00005857
0.4	100	−0.01912417	−0.01908412	0.00008163	−0.01398161	−0.01399076	0.00005720
	102	−0.01957286	−0.01956383	0.00008143	−0.01424219	−0.01426468	0.00005648
	104	−0.01569789	−0.01561718	0.00007313	−0.01235004	−0.01236274	0.00005286

Delta of quantile options under the DEM with $\lambda = 3$

		$\sigma = 0.2$			$\sigma = 0.3$		
α	K	EI price	MC value	Std. err.	EI price	MC value	Std. err.
0.2	90	0.06855937	0.06908873	0.00077633	0.11127492	0.11060554	0.00093565
	100	0.33498655	0.33507900	0.00118391	0.30939518	0.30898836	0.00117698
	110	0.62926410	0.62827708	0.00108407	0.51724312	0.51748005	0.00115069
0.5	90	0.25471549	0.25497213	0.00115890	0.32410267	0.32348591	0.00124658
	100	0.57434958	0.57383915	0.00121216	0.55614658	0.55658769	0.00125713
	110	0.82522134	0.82466107	0.00096239	0.75014823	0.75064412	0.00111455

Notes. For the simple step and delayed barrier options, the default parameter choices are $\lambda = 3$, $r = 0.05$, $\eta = 30$, $\theta = 20$, $p = q = 0.5$, $L = 102$, $\vartheta = 0.5$, $\Delta S_0 = 0.1$, and $t = 1$. For the corridor options with single barrier, the default parameter choices are $\lambda = 3$, $r = 0.05$, $\eta = 30$, $\theta = 20$, $p = q = 0.5$, $L = 102$, $\Delta S_0 = 0.1$, and $t = 1$. For the quantile options, the default parameter choices are $\lambda = 3$, $r = 0.05$, $\eta = 34$, $\theta = 34$, $p = 0.6$, $q = 0.4$, $S_0 = 100$, $\gamma = 1$, $\Delta S_0 = 0.1$, and $t = 1$. Monte Carlo values for simple step and delayed barrier options (for corridor and quantile options, respectively) along with the associated standard errors (denoted by std. err.) are obtained by using 100,000 time steps (20,000 time steps, respectively) and simulating 100,000 sample paths. The CPU time of our numerical methods for generating one price of simple step or delayed barrier options, corridor options and quantile options is around 100 seconds and 3 seconds, respectively. The CPU time for Monte Carlo simulation is around 25, 25, 4.3, and 9 minutes for the simple step, delayed barrier, corridor, and quantile options, respectively. The table indicates that all the EI values stay within the 95% confidence intervals of the associated MC values.

TABLE 5. Comparison of continuous and discrete step option pricing.

	Monitoring frequency				
	Relative differences				
S_0	Monthly (%)	Biweekly (%)	Weekly (%)	Daily (%)	Continuous prices
95	4.698	1.974	1.011	0.922	7.22634078
100	3.489	1.583	0.998	0.783	10.35784700
105	9.458	4.049	2.027	0.828	14.37387610
110	8.940	3.918	2.003	0.720	18.50956926
115	8.866	3.947	1.982	0.616	22.75134627
120	8.841	3.957	1.975	0.534	27.12165429

Note. The relative difference is defined as (discrete price − continuous price)/continuous price. The default parameters of the underlying process are $r = 0.05$, $\sigma = 0.2$, $\lambda = 3$, $\eta = \theta = 15$, and $p = q = 0.5$. Consider a proportional step option with the parameters $L = 102$, $K = 100$, $\rho = 1$, and $t = 1$. The occupation time refers to the time the underlying price spends under $L = 102$. And we use 100,000 sample paths to simulate the discrete prices.

around two minutes, which is still very efficient compared to the Monte Carlo simulation. For the numerical results of corridor and quantile option prices, it suffices to use a two-dimensional Euler inversion algorithm. Our method is more efficient than Cai's method (Cai [4]).

5.3. Discretization frequency effect. Our EI price is given under an assumption that the underlying price is continuously monitored. However, in reality a sizable portion of contracts specify fixed reference times for monitoring and the occupation time is defined according to the number of the monitoring dates in which the underlying price is above or below some level or within a band. This may introduce substantial differences between the two monitoring schemes. Some scholars have already studied the effect of discretization frequency on the pricing of occupation-time-related options under GBM models. The main literature includes Atkinson and Fusai [3], Davydov and Linetsky [11], and Fusai and Tagliani [14].

In this subsection, we aim to investigate how the discretization frequency will affect the pricing results under the double exponential jump diffusions. Table 5 and Figure 1 compare our continuous-time outcomes in one proportional step option example with the prices under discrete time monitoring, which are obtained through Monte Carlo simulation, for various initial underlying prices. The monitoring frequencies we use are monthly, biweekly, weekly and daily. That is, the time horizon, one year, is divided into 12, 26, 52, and 252 subintervals, respectively. For discrete monitoring contracts, define the occupation time as follows:

$$\tau_L = \sum_{i=1}^{N} (t_i - t_{i-1}) \mathbf{1}_{\{S_{t_i} < L\}},$$

where $0 = t_0 < \cdots < t_N = T$ are the reference dates.

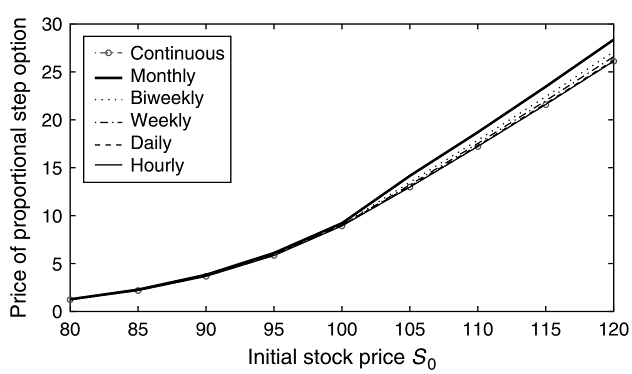

FIGURE 1. Comparison of continuous and discrete monitoring results under the DEM model.
Notes. As the discretization becomes finer, the discrete-time monitoring option prices converge to the continuous-time option prices under all initial stock prices. The default parameters of the underlying process are $r = 0.05$, $\sigma = 0.2$, $\lambda = 3$, $\eta = \theta = 15$, and $p = q = 0.5$. Consider a proportional step option with the parameters $L = 102$, $K = 100$, $\rho = 1$, and $t = 1$. The occupation time refers to the time the underlying price spends under $L = 102$. And we use 100,000 sample paths to simulate the discrete prices.

TABLE 6. Comparison of the deltas of the continuous and discrete step options.

	Monitoring frequency				
	Relative differences				
S_0	Monthly (%)	Biweekly (%)	Weekly (%)	Daily (%)	Continuous delta
95	3.990	2.366	1.368	0.361	0.53553845
100	−3.411	−1.907	−0.643	0.275	0.72754990
105	4.342	2.670	1.752	0.466	0.82098985
110	8.048	4.428	1.915	0.256	0.83620619
115	8.850	3.936	1.824	0.195	0.86111472
120	8.935	3.986	2.108	0.138	0.88675445

Note. The relative difference is defined as (discrete delta − continuous delta)/continuous delta. The default parameters of the underlying process are $r = 0.05$, $\sigma = 0.2$, $\lambda = 3$, $\eta = \theta = 15$, and $p = q = 0.5$. Consider a proportional step option with the parameters $L = 102$, $K = 100$, $\rho = 1$, and $t = 1$. The occupation time refers to the time the underlying price spends under $L = 102$. And we use 100,000 sample paths to simulate the discrete deltas.

It is clear to see that the relative differences between the two schemes reduce significantly when the discretization becomes more frequent. Therefore, the continuous results should be a good approximation to those contracts under frequent monitoring (say, daily or weekly). However, we should admit that significant differences exist (e.g., more than 9% for $S_0 = 105$ in the case of monthly monitoring) between the continuous-time scheme and the less-frequent discrete monitoring. It will then be important to distinguish these two under this scenario.

A similar convergence can be observed for the delta too. As the discretization becomes finer and finer, the deltas under discrete monitoring will converge to the delta under continuous monitoring. Table 6 and Figure 2 demonstrate the related numerical experiments.

5.4. Robustness of our pricing algorithm. We point out that our Laplace inversion-based pricing algorithm is robust. As illustrated in Figure 3, our pricing algorithm retains its accuracy when some model parameters vary within realistic ranges. More precisely, when η (θ and p, respectively) changes in [15, 100] ([15, 100] and [0, 1], respectively), the relative errors between our numerical prices and MC prices are all less than 0.3%. These ranges cover most cases in reality. For example, $\eta \in [15, 100]$ and $\theta \in [15, 100]$ mean that the expected upward and downward jump sizes of return are between 1% and 6.67%. Note that the minimum and maximum daily returns of S&P 500 from Aug. 1, 2007 to Oct. 26, 2009 (during the ongoing financial crisis) are −4.76% and 4.11%, respectively. Absolute values of them are both smaller than 6.67%. Consequently, we draw the conclusion that our pricing algorithm is robust and thus reliable.

6. Conclusion. In this paper, we investigate pricing and hedging problems of occupation-time-related options such as step options, corridor options, and quantile options under Kou's double exponential jump diffusion model. By studying the occupation-time distribution, we derive the Laplace transform-based analytical

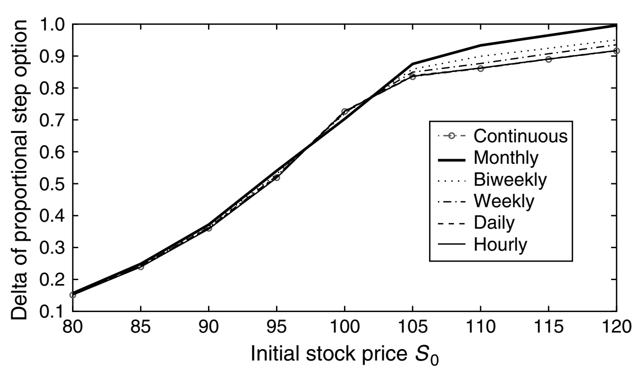

FIGURE 2. Comparison of continuous and discrete monitoring deltas under the DEM model.
Notes. As the discretization becomes finer, the deltas of discrete monitoring converge to those of continuous monitoring under all initial stock prices. The default parameters of the underlying process are $r = 0.05$, $\sigma = 0.2$, $\lambda = 3$, $\eta = \theta = 15$ and $p = q = 0.5$. Consider a proportional step option with the parameters $L = 102$, $K = 100$, $\rho = 1$, and $t = 1$. The occupation time refers to the time the underlying price spends under $L = 102$. And we use 100,000 sample paths to simulate the discrete deltas.

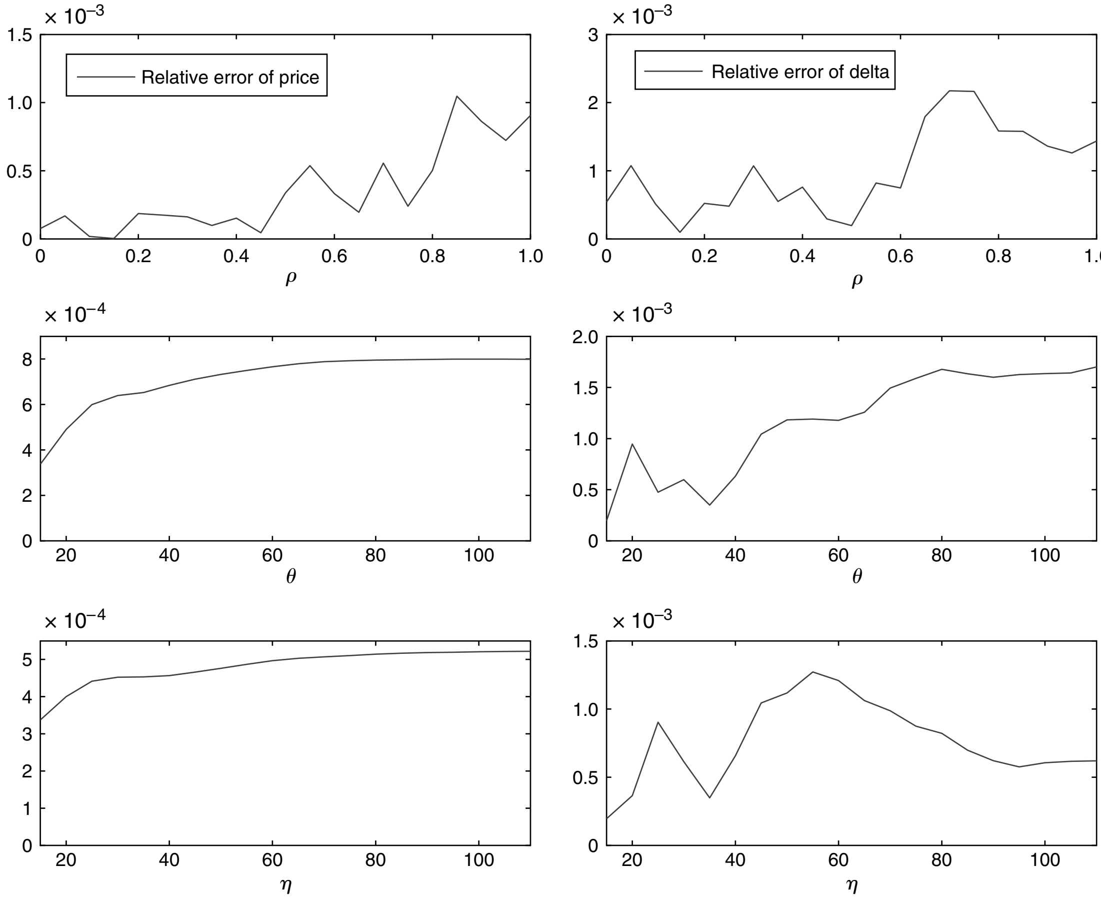

FIGURE 3. The relative errors between the Euler inversion and MC simulation for varying p, θ, and η.

Notes. We test the robustness of our method using the proportional step option. The default parameters of the jump diffusion processes are $r = 0.05$, $\sigma = 0.2$, $\lambda = 1$, $\eta = \theta = 15$, and $p = q = 0.5$. The current underlying asset price is $S_0 = 105$. The option contract parameters are $\rho = 1$, $K = 100$ and $L = 90$. The occupation time is accumulated when the underlying price is less than 90.

solutions to these pricing problems, which can be inverted numerically via the Euler Laplace inversion algorithm. The numerical results indicate that our pricing formulae are both accurate and efficient.

Appendix A. Roots of the equation $G(x) = r + a$**.** The equation $G(x) = r + a$, with $G(x)$ defined as (2), can be reduced down to

$$a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0,$$

where

$$a_4 = \sigma^2, \qquad a_3 = 2\mu - \sigma^2(\eta - \theta), \qquad a_2 = -\sigma^2\eta\theta - 2\mu(\eta - \theta) - 2\lambda - 2(r + a),$$
$$a_1 = -2\mu\eta\theta - 2\lambda p(\eta + \theta) + 2\lambda\eta + 2(r + a)(\eta - \theta), \qquad a_0 = 2(r + a)\eta\theta.$$

It has four roots given by

$$\beta_{1,a} = -\frac{a_3}{4a_4} + \frac{p_1 - p_3}{2}, \qquad \beta_{2,a} = -\frac{a_3}{4a_4} + \frac{p_1 + p_3}{2},$$
$$\gamma_{1,a} = \frac{a_3}{4a_4} + \frac{p_1 - p_2}{2}, \qquad \gamma_{2,a} = \frac{a_3}{4a_4} + \frac{p_1 + p_2}{2},$$

where

$$p_1 = \sqrt{B_3 + C_0 + C_1}, \qquad p_2 = \sqrt{B_4 - C_0 - C_1 - \frac{B_5}{4p_1}}, \qquad p_3 = \sqrt{B_4 - C_0 - C_1 + \frac{B_5}{4p_1}},$$
$$B_0 = a_2^2 - 3a_1 a_3 + 12 a_0 a_4, \qquad B_1 = 2a_2^3 - 9a_1 a_2 a_3 + 27 a_1^2 a_4 + 27 a_0 a_3^2 - 72 a_0 a_2 a_4,$$

$$B_2 = \sqrt{B_1^2 - 4B_0^3}, \qquad B_3 = \frac{a_3^2}{4a_4^2} - \frac{2a_2}{3a_4}, \qquad B_4 = \frac{a_3^2}{2a_4^2} - \frac{4a_2}{3a_4}, \qquad B_5 = \frac{4a_2a_3}{a_4^2} - \frac{8a_1}{a_4} - \frac{a_3^3}{a_4^3},$$

$$B_6 = \sqrt[3]{B_1 + B_2}, \qquad C_0 = \frac{\sqrt[3]{2}B_0}{3a_4B_6}, \qquad C_1 = \frac{B_6}{3\sqrt[3]{2}a_4}.$$

Appendix B. Lemma 1.

Lemma B.1. *Let f: $[0, T] \times \mathbb{R} \to \mathbb{R}$ be a function of class $C^{1,1}$ on $[0, T] \times \mathbb{R}$ and $C^{1,2}$ on $[0, T] \times \mathbb{R}\backslash\{h\}$. The left and right second derivatives $(\partial^2 f/\partial x^2)(t, h-)$, $(\partial^2 f/\partial x^2)(t, h+)$ exist. Then, we can find a sequence of $\{f_n\} \in C^{1,2}([0, T] \times \mathbb{R})$ and a positive constant M, independent of t, x, and n, such that (1) $f_n(t, x)$ converges to $f(t, x)$ as $n \to \infty$ for any $(t, x) \in [0, T] \times \mathbb{R}$; (2) $f_n(t, x) \equiv f(t, x)$ for any $(t, x) \in [0, T] \times (-\infty, h] \cup [h + (1/n), \infty)$; and (3) $\max\{|f_n|, |\partial f_n/\partial t|, |\partial f_n/\partial x|, |\partial^2 f_n/\partial x^2|\} \le M$ for any $(t, x) \in [0, T] \times (h, h + 1/n)$.*

Proof. Introduce a polynomial to smooth the irregular point at $x = h$ for the function f. Let $f_n(t, x) = f(t, x)$ for $(t, x) \in [0, T] \times (-\infty, h] \cup [h + 1/n, \infty)$ and $f_n(t, x) = P_n(t, n(x - h))$ for $(t, x) \in [0, T] \times (h, h + (1/n))$, where P_n is a fifth order polynomial given by

$$P_n(t, x) = \frac{a}{n^2}x^5 + \frac{b}{n^2}x^4 + \frac{c}{n^2}x^3 + \frac{(\partial^2 f/\partial x^2)(t, h-)}{2n^2}x^2 + \frac{(\partial f/\partial x)(t, h)}{n}x + V(t, h).$$

f_n must be twice differentiable at $x = h$ and $x = h + 1/n$. It is easy to check that f_n has second order derivative at $x = h$ and its differentiability at $x = h + 1/n$ is equivalent to requiring a, b, c to satisfy $P_n(t, 1) = f(t, h + 1/n)$,

$$\frac{\partial P_n(t, 1)}{\partial x} = \frac{\partial f(t, h + 1/n)}{\partial x} \qquad \text{and} \qquad \frac{\partial^2 P_n(t, 1)}{\partial x^2} = \frac{\partial^2 f(t, h + 1/n)}{\partial x^2}.$$

That is, $\{a, b, c\}$ is a set of roots of the following linear equations:

$$a + b + c = n\left(n\left(f\left(t, h + \frac{1}{n}\right) - f(t, h)\right) - \frac{\partial f(t, h)}{\partial x}\right) - \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, h-); \tag{B1}$$

$$5a + 4b + 3c = n\left(\frac{\partial f(t, h + 1/n)}{\partial x} - \frac{\partial f}{\partial x}(t, h)\right) - \frac{\partial^2 f}{\partial x^2}(t, h-); \tag{B2}$$

$$20a + 12b + 6c = \frac{\partial^2 f(t, h + 1/n)}{\partial x^2} - \frac{\partial^2 f}{\partial x^2}(t, h-). \tag{B3}$$

Note that the foregoing linear equations are solvable for any t and n. Using the conditions of f, we can show that the right-hand sides of (B1)–(B3) are in the order of $o(1)$ as $n \to +\infty$. Thus, the coefficients a, b, and c are also in the order of $o(1)$, which yields the property (3). From our construction it is also easy to see that such f_n satisfies (1) and (2). ☐

Appendix C. The property of the matrix A. By Gauss elimination of elementary column operation, we can show that the determinant of the following matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ \dfrac{1}{\eta - a_1} & \dfrac{1}{\eta - a_2} & \dfrac{1}{\eta - a_3} & \dfrac{1}{\eta - a_4} \\ \dfrac{1}{\theta + a_1} & \dfrac{1}{\theta + a_2} & \dfrac{1}{\theta + a_3} & \dfrac{1}{\theta + a_4} \end{bmatrix},$$

is given by

$$\det(\mathbf{A}) = -\frac{(\eta + \theta)\Pi_{1 \le i < j \le 4}(a_i - a_j)}{\Pi_{1 \le i \le 4, 1 \le j \le 4}(\eta - a_i)(\theta + a_j)} \neq 0.$$

$\mathbf{A}$ is thus nonsingular. Let $\mathbf{b} = (1, b, 1/(\eta - b), 1/(\theta + b))^T$, then the linear equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

have a unique solution $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*)^T$, with

$$x_i^* = \frac{\Pi_{j \neq i}(a_j - b)(\eta - a_i)(\theta + a_i)}{\Pi_{j \neq i}(a_j - a_i)(\eta - b)(\theta + b)}, \qquad i = 1, 2, 3, 4.$$

Appendix D. Occupation times with double barriers. Our Euler inversion-based approach can be extended to cover the occupation time that the underlying process spends inside two flat barriers, i.e., a corridor with double barriers. There is one minor technical difficulty remaining: we cannot show nonsingularity of an 8×8 matrix rigorously, which we believe is true. Note that numerical experiments demonstrate that the matrix should be invertible. Moreover, it turns out that this does not affect the validity of our numerical methods for pricing occupation-time-related options. In this subsection, we first present the closed-form Laplace transform of the joint distribution of the occupation time with double barriers and the log-return of the underlying at the maturity. Then this result is applied to price corridor options with double barriers, and numerical results are provided in Table D.1. To price other options related to occupation times with two barriers, readers may mimic the arguments in §4.

Consider two barriers h and H with $h<H$ and let $\tau_{(h,H)}$ denote the occupation times spent between the lower barrier h and the upper barrier H until the maturity T, that is,

$$\tau_{(h,H)} := \int_0^T \mathbf{1}_{\{h<X_t<H\}}\, dt.$$

Given any $0\le\gamma<\min\{\eta,\theta\}$ and $\rho>0$, our objective is to compute the following Laplace transform of $\tau_{(h,H)}$ and X_T:

$$V(T,x;\rho,\gamma;h,H) := e^{-rT}\cdot E[e^{-\rho\tau_{(h,H)}+\gamma X_T}\mid X_0=x]. \tag{D1}$$

Following similar derivation as in Theorem 3.1, we can show that such V uniquely solves the following PIDE system:

$$\begin{cases} \dfrac{\partial V}{\partial t}+\rho\mathbf{1}_{\{h<x<H\}}V=\mathscr{L}V, & \text{for } t\in(0,T] \text{ and } x\in\mathbb{R}\backslash\{h,H\}; \\ V(0,x)=e^{\gamma x}, & \text{for } x\in\mathbb{R}. \end{cases} \tag{D2}$$

For $a>0$ satisfying (21), consider the Laplace transform of $V(T,x;\rho,\gamma)$ with respect to the maturity T

$$\tilde{u}(x;\rho,\gamma,a;h,H) \triangleq \int_0^\infty e^{-aT}V(T,x;\rho,\gamma)\,dT.$$

TABLE D.1. Prices and deltas of corridor options with double barriers (denoted by EI value).

		$\sigma=0.2$			$\sigma=0.3$		
K	S_0	EI price	MC value	Std. Err.	EI price	MC value	Std. Err.
		Prices of corridor options with double barriers under the DEM with $\lambda=3$					
0.2	95	0.49444505	0.49360862	0.00075583	0.37046981	0.36989311	0.00076296
	100	0.45098018	0.45017582	0.00073051	0.35035352	0.34997974	0.00073555
	105	0.37305021	0.37252713	0.00070721	0.30838577	0.30811779	0.00070295
0.4	95	0.32304472	0.32235313	0.00065997	0.21695824	0.21656458	0.00062004
	100	0.28990787	0.28934707	0.00062511	0.20313338	0.20289474	0.00059457
	105	0.23235612	0.23187688	0.00058409	0.17455576	0.17424786	0.00055781
		Deltas of corridor options with double barriers under the DEM with $\lambda=3$					
0.2	100	−0.01853381	−0.01852545	0.00007402	−0.01427373	−0.01428372	0.00005625
	102	−0.02008015	−0.02014496	0.00007719	−0.01505973	−0.01507682	0.00005713
	104	−0.02149747	−0.02153651	0.00007988	−0.01577619	−0.01577376	0.00005772
0.4	100	−0.01499858	−0.01496464	0.00007158	−0.01131949	−0.01130931	0.00005388
	102	−0.01588576	−0.01590179	0.00007396	−0.01178545	−0.01179470	0.00005423
	104	−0.01664286	−0.01667591	0.00007538	−0.01219103	−0.01219225	0.00005426

Notes. The default parameter choices are $\lambda=3$, $r=0.05$, $\eta=30$, $\theta=20$, $p=q=0.5$, $l=80$ for pricing part or $l=50$ for delta part, $L=110$, and $t=1$. The Monte Carlo simulation estimates (denoted by MC value) along with the associated standard errors (denoted by std. err.) are obtained by using 50,000 time steps for pricing part or 20,000 time steps for delta part and by simulating 100,000 sample paths. The CPU time of our numerical method for generating one corridor option price or delta is around 3 seconds. The CPU times for producing one MC value of corridor option price and one MC value of delta are around 10 minutes and 4.3 minutes, respectively. The table indicates that all the EI values stay within the 95% confidence intervals of the associated MC values.

Similarly as in the case of occupation times with single barrier, we can transform the PIDE (D2) into an OIDE. Some algebra can yield the closed-form solution for $\tilde{u}$ as follows

$$\tilde{u}(x;\rho,\gamma,a;h,H)=\begin{cases}\omega_1^L e^{\beta_{1,a}(x-h)}+\omega_2^L e^{\beta_{2,a}(x-h)}-c_L e^{\gamma(x-h)}, & x\le h;\\ -\omega_1^0 e^{\beta_{1,a+\rho}(x-H)}-\omega_2^0 e^{\beta_{2,a+\rho}(x-H)}-\nu_1^0 e^{-\gamma_{1,a+\rho}(x-h)} & \\ -\nu_2^0 e^{-\gamma_{2,a+\rho}(x-h)}-c_0 e^{\gamma(x-H)}, & h<x<H;\\ \nu_1^U e^{-\gamma_{1,a}(x-H)}+\nu_2^U e^{-\gamma_{2,a}(x-H)}-c_U e^{\gamma(x-H)}, & x\ge H,\end{cases}$$

where

$$c_L=\frac{e^{\gamma h}}{G(\gamma)-a-r},\qquad c_0=\frac{e^{\gamma H}}{G(\gamma)-a-r-\rho},\qquad \text{and}\qquad c_U=\frac{e^{\gamma H}}{G(\gamma)-a-r}.$$

In other words, the solution $\tilde{u}$ is a linear combination of exponential functions. The coefficient's vector

$$\mathbf{d}=(\omega_1^L,\omega_2^L,\nu_1^0,\nu_2^0,\nu_1^U,\nu_2^U,\omega_1^0,\omega_2^0)^T$$

satisfies a linear system

$$\mathbf{Bd}=\mathbf{R}. \tag{D3}$$

Here $\mathbf{R}$ is an eight-dimensional vector

$$\mathbf{R}=(c_U-c_0)\cdot\left(\bar{x}^{\gamma},\gamma\bar{x}^{\gamma},\frac{\bar{x}^{\gamma}}{\eta-\gamma},\frac{\bar{x}^{\gamma}}{\theta+\gamma},1,\gamma,\frac{1}{\eta-\gamma},\frac{1}{\theta+\gamma}\right)^T,$$

where $\bar{x}:=e^{h-H}$. $\mathbf{B}$ is an 8×8 matrix

$$\mathbf{B}=\begin{bmatrix}\mathbf{M} & \mathbf{NZ}_\beta\\ \mathbf{MZ}_\gamma & \mathbf{N}\end{bmatrix},$$

where $\mathbf{Z}_\beta$ and $\mathbf{Z}_\gamma$ are two 4×4 diagonal matrices with the diagonal elements being $\{0,0,\bar{x}^{\beta_{1,a+\rho}},\bar{x}^{\beta_{2,a+\rho}}\}$ and $\{0,0,\bar{x}^{\gamma_{1,a+\rho}},\bar{x}^{\gamma_{2,a+\rho}}\}$, respectively, and $\mathbf{M}$ and $\mathbf{N}$ are given by

$$\mathbf{M}=\begin{bmatrix}1 & 1 & 1 & 1\\ \beta_{1,a} & \beta_{2,a} & -\gamma_{1,a+\rho} & -\gamma_{2,a+\rho}\\ \dfrac{1}{\eta-\beta_{1,a}} & \dfrac{1}{\eta-\beta_{2,a}} & \dfrac{1}{\eta+\gamma_{1,a+\rho}} & \dfrac{1}{\eta+\gamma_{2,a+\rho}}\\ \dfrac{1}{\theta+\beta_{1,a}} & \dfrac{1}{\theta+\beta_{2,a}} & \dfrac{1}{\theta-\gamma_{1,a+\rho}} & \dfrac{1}{\theta-\gamma_{2,a+\rho}}\end{bmatrix},$$

and

$$\mathbf{N}=\begin{bmatrix}1 & 1 & 1 & 1\\ -\gamma_{1,a} & -\gamma_{2,a} & \beta_{1,a+\rho} & \beta_{2,a+\rho}\\ \dfrac{1}{\eta+\gamma_{1,a}} & \dfrac{1}{\eta+\gamma_{2,a}} & \dfrac{1}{\eta-\beta_{1,a+\rho}} & \dfrac{1}{\eta-\beta_{2,a+\rho}}\\ \dfrac{1}{\theta-\gamma_{1,a}} & \dfrac{1}{\theta-\gamma_{2,a}} & \dfrac{1}{\theta+\beta_{1,a+\rho}} & \dfrac{1}{\theta+\beta_{2,a+\rho}}\end{bmatrix}.$$

It seems difficult to show the nonsingularity of $\mathbf{B}$. However, numerical experiments indicate that it should be nonsingular. Moreover, it turns out that our pricing methods for occupation-time-related options based on the Laplace transform result of the joint distribution of X_T and $\tau_{(h,H)}$ are still valid. Here we shall price corridor options with double barriers to illustrate the effectiveness of our pricing method. Because of similarities, pricing of other options related to occupation times with two barriers is omitted. Consider a corridor call option with double barriers, whose price is given by

$$\mathrm{Cor}(K,T)=e^{-rT}E[\max\{\tau_{(\log(l/S_0)),\log(L/S_0))}-K,0\}],$$

where l and L ($l < L$) are two barriers of the underlying asset price process S_t that starts from S_0. Mimicking the proofs of Theorem 4.2 and Proposition 4.1, the double Laplace transform of $\mathrm{Cor}(K, T)$ with respect to K and T

$$\tilde{g}_{\mathrm{cor}}(\varphi, a) = \int_0^\infty \int_0^\infty e^{-\varphi K - aT}\, \mathrm{Cor}(K, T)\, dK\, dT \tag{D4}$$

should be equal to

$$\tilde{g}_{\mathrm{cor}}(\varphi, a) = -\frac{1}{\varphi}\frac{\partial \tilde{u}}{\partial \rho}(0; 0, 0, a; \log(l/S_0), \log(L/S_0)) + \frac{1}{\varphi^2}\tilde{u}(0; \varphi, 0, a; \log(l/S_0), \log(L/S_0)) - \frac{1}{(a+r)\varphi^2},$$

where

$$\frac{\partial \tilde{u}}{\partial \rho}(0; 0, 0, a; \log(l/S_0), \log(L/S_0))$$
$$= \begin{cases} \tilde{\omega}_1^L \cdot (S_0/l)^{\beta_{1,a}} + \tilde{\omega}_2^L \cdot (S_0/l)^{\beta_{2,a}}, & S_0 \le l; \\ -\tilde{\omega}_1^0 \cdot (S_0/L)^{\beta_{1,a}} - \tilde{\omega}_2^0 \cdot (S_0/L)^{\beta_{2,a}} - \tilde{\nu}_1^0 \cdot (l/S_0)^{\gamma_{1,a}} - \tilde{\nu}_2^0 \cdot (l/S_0)^{\gamma_{2,a}} - \dfrac{1}{(a+r)^2}, & l < S_0 < L; \\ \tilde{\nu}_1^U (L/S_0)^{\gamma_{1,a}} + \tilde{\nu}_2^U (L/S_0)^{\gamma_{2,a}}, & S_0 \ge L; \end{cases}$$

and $\tilde{\mathbf{d}} = (\tilde{\omega}_1^L, \tilde{\omega}_2^L, \tilde{\nu}_1^0, \tilde{\nu}_2^0, \tilde{\nu}_1^U, \tilde{\nu}_2^U, \tilde{\omega}_1^0, \tilde{\omega}_2^0)^T$ satisfies the following linear system:

$$\mathbf{B}(\mathbf{0})\tilde{\mathbf{d}} = -\frac{1}{(a+r)^2} \cdot \left(1, 0, \frac{1}{\eta}, \frac{1}{\theta}, 1, 0, \frac{1}{\eta}, \frac{1}{\theta}\right)^T.$$

Here is $\mathbf{B}$ with $\rho = 0$.

Inverting the Laplace transform (D4) via the Euler inversion algorithm, we can price corridor options with double barriers numerically. Numerical results are given in Table D.1, where we can see that all the numerical prices obtained using our pricing method (denoted by EI value) stay within the 95% confidence intervals of the associated MC simulation estimates (denoted by MC value). This demonstrates that our pricing method is also accurate for pricing corridor options with double barriers. In addition, similarly as in the case of corridor options with single barriers, we can also calculate deltas for corridor options with double barriers numerically. Numerical results are also given in Table D.1, which also indicate the effectiveness of our numerical method.

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References

[1] Abate, J., W. Whitt. 1992. The Fourier-seriers method for inverting transforms of probability distributions. *Queueing Systems* **10**(1) 5–88.

[2] Akahori, J. 1995. Some formulae for a new type of path-dependent option. *Ann. Appl. Probab.* **5**(2) 383–388.

[3] Atkinson, C., G. Fusai. 2007. Discrete extrema of the Brownian motion and pricing of lookback options. *J. Computational Finance* **10**(3) 1–43.

[4] Cai, N. 2009. Pricing quantile options in a flexible jump diffusion model. Techincal report, Department of IELM, HKUST, Hong Kong.

[5] Cai, N., S. G. Kou. 2008. Option pricing under a hyper-exponential jump diffusion model. Technical report, HKUST and Columbia University, Hong Kong and New York.

[6] Chesney, M., M. Jeanblanc-Picqué, M. Yor. 1997. Brownian excursions and Parisian barrier options. *Adv. Appl. Probab.* **29**(1) 165–184.

[7] Choudhury, G. L., D. M. Lucantoni, W. Whitt. 1994. Multidimensional transform inversion with applications to the transient m/g/1 queue. *Ann. Appl. Probab*. **4**(3) 719–740.

[8] Cohen, J. W., G. Hooghiemstra. 1981. Brownian excursion, the m/m/1 queue and their occupation times. *Math. Oper. Res.* **6**(4) 608–629.

[9] Dassios, A. 1995. The distribution of the quantile of a Brownian motion with drift and the pricing of related path-dependent options. *Ann. Appl. Probab.* **5**(2) 389–398.

[10] Dassios, A. 1996. Sample quantiles of stochastic processes with stationary and independent increments. *Ann. Appl. Probab.* **6**(3) 1041–1043.

[11] Davydov, A., V. Linetsky. 2002. Structuring, pricing and hedging double-barrier step options. *J. Computational Finance* **5**(2) 55–87.

[12] Embrechts, P., L. C. G. Rogers, M. Yor. 1995. A proof of Dassios' representation of the α-quantile of Brownian motion with drift. *Ann. Appl. Probab.* **5**(3) 757–767.

[13] Fusai, G. 2000. Corridor options and arc-sine law. *Ann. Appl. Probab*. **10**(2) 634–663.

[14] Fusai, G., A. Tagliani. 2001. Pricing of occupation time derivatives: Continuous and discrete monitoring. *J. Computational Finance* **5**(1) 1–37.

[15] Karatzas, I., S. Shreve. 1991. *Brownian Motion and Stochastic Calculus*. Springer-Verlag, New York.

[16] Kou, S. G. 2002. A jump-diffusion model for option pricing. *Management Sci.* **48**(8) 1086–1101.

[17] Kou, S. G., H. Wang. 2003. First passage times of a jump diffusion processes. *Adv. Appl. Probab.* **35**(2) 504–531.

[18] Kou, S. G., H. Wang. 2004. Option pricing under a double exponential jump diffusion model. *Management Sci.* **50** 1178–1192.

[19] Kwok, Y. K., K. W. Lau. 2001. Pricing algorithms for options with exotic path-dependence. *J. Derivatives* **9**(1) 23–38.

[20] Leung, K. S., Y. K. Kwok. 2007. Distribution of occupation times for CEV diffusions and pricing of α-quantile options. *Quant. Finance* **7**(1) 87–94.

[21] Linetsky, V. 1998. Steps to the barrier. *RISK* (**April**) 62–65.

[22] Linetsky, V. 1999. Step options. *Math. Finance* **9**(1) 55–96.

[23] Lucas, R. E. 1978. Asset prices in an exchange economy. *Econometrica* **46**(6) 1429–1445.

[24] Miura, R. 1992. A note on look-back options based on order statistics. *Hitotsubashi J. Commerce Management* **27**(1) 15–28.

[25] Naik, V., M. Lee. 1990. General equilibrium pricing of options on the market portfolio with discontinuous returns. *Rev. Financial Stud.* **3**(4) 493–521.

[26] Petrella, G. 2004. An extension of the Euler Laplace transform inversion algorithm with applications in option pricing. *Oper. Res. Lett.* **32**(4) 380–389.

[27] Protter, P. 2005. *Stochastic Integration and Differntial Equations. A New Approach*. Springer, Berlin.

[28] Whitt, W. 2002. *Stochastic-Process Limits: An Introduction to Stochastic-Process Limits and Their Applications to Queues*. Springer-Verlag, New York.

[29] Yor, M. 1995. The distribution of Brownian quantiles. *J. Appl. Probab.* **32**(2) 405–416.