Inventory Routing Problem under Uncertainty

Zheng CUI, Daniel Zhuoyu LONG
Department of Systems Engineering and Engineering Management, Chinese University of Hong Kong, New Territories, Hong Kong, zcui@se.cuhk.edu.hk, zylong@se.cuhk.edu.hk

Jin QI
Department of Industrial Engineering and Decision Analytics, Hong Kong University of Science and Technology, Kowloon, Hong Kong, jinqi@ust.hk

Lianmin ZHANG
Department of Management Science and Engineering, Nanjing University, Nanjing, zhanglm@nju.edu.cn

We study a stochastic inventory routing problem with a finite horizon. The supplier acts as a central planner who determines replenishment quantities and also the times and routes for delivery to all retailers. We allow ambiguity in the probability distribution of each retailer’s uncertain demand. Adopting a service-level viewpoint, we minimize the risk of uncertain inventory levels violating a pre-specified acceptable range. We quantify that risk using a novel decision criterion, the Service Violation Index, that accounts for how often and how severely the inventory requirement is violated. The solutions proposed here are adaptive in that they vary with the realization of uncertain demand. We provide algorithms to solve the problem exactly and then demonstrate the superiority of our solutions by comparing them with several benchmarks.

Key words: inventory routing, distributionally robust optimization, risk management, adjustable robust optimization, decision rule

1. Introduction

A key concept in supply chain management is the coordination of different stakeholders to minimize the overall cost. In that regard, the vendor-managed inventory (VMI) system, as a relatively new business model, has attracted extensive attention in both academia and industry. In the VMI system, the supplier (vendor) behaves like a central decision maker determining not only the timing and quantities of the replenishment for all retailers, but also the route on which they are visited. Under this arrangement, retailers benefit from saving efforts to control inventory. Meanwhile, the supplier can improve the service level and reduce its costs by using transportation capacity more
efficiently. For a more detailed discussion of the VMI concept’s advantages, see Waller et al. (1999), Çetinkaya and Lee (2000), Cheung and Lee (2002), and Dong and Xu (2002).

This joint management of inventory and vehicle routing gives rise to the inventory routing problem (IRP). The term IRP was first used by Golden et al. (1984) when defining a routing problem with an explicit inventory feature. Federgruen and Zipkin (1984) study a single-period IRP and describe the economic benefits of coordinating the decisions related to distribution and inventory. Since then, many variants of the IRP have been proposed over the past three decades (Coelho et al. 2014a): planning horizons can be finite or infinite; excess demand can be either lost or back-ordered; and there can be either just one or more than one vehicle. Scholars have used these formulations on many real-world applications of the IRP, which include maritime logistics (Christiansen et al. 2011, Papageorgiou et al. 2014), the transportation of gas and oil (Bell et al. 1983, Campbell and Savelsbergh 2004, Grønhaug et al. 2010), groceries (Gaur and Fisher 2004, Custódio and Oliveira 2006), perishable products (Federgruen et al. 1986), and bicycle sharing (Brinkmann et al. 2016). We refer interested readers to Andersson et al. (2010) for more on IRP applications.

Since the basic IRP incorporates a classical vehicle routing problem that is already NP-hard, it follows that the stochastic IRP must be even more complex. We next focus our review on the stochastic version of the IRP, especially in the case of uncertain demand. A classical way to formulate the stochastic IRP is using Markov decision process, which assumes that the joint probability distribution of retailers’ demands is known. Campbell et al. (1998) introduce a dynamic programming model in which the state is the current inventory level at each retailer and the Markov transition matrix is obtained from the known probability distribution of demand; in their model, the objective is to minimize the expected total discounted cost over an infinite horizon. This work is extended by Kleywegt et al. (2002, 2004), who solve the problem by constructing an approximation to the optimal value function. Adelman (2004) decomposes the optimal value function as the sum of single-customer inventory value functions, which are approximated by the optimal dual prices
derived under a linear program. Hvattum and Løkketangen (2009) and Hvattum et al. (2009) solve the same problem using heuristics based on finite scenario trees.

Another way to solve the stochastic IRP is via stochastic programming. Federgruen and Zipkin (1984) accommodate inventory and shortage cost in the vehicle routing problem and offer a heuristic for that problem in a scenario-based random demand environment. Coelho et al. (2014b) address the dynamic and stochastic inventory routing problem in the case where the distribution of demand changes over time. They give the algorithms for four solution policies whose use depends on whether demand forecasts are used and whether emergency trans-shipments are allowed. Adulyasak et al. (2015) consider the stochastic production routing problem with demand uncertainty under both two-stage and multi-stage decision processes. They minimize average total cost by enumerating all possible scenarios, and solve the problem by deriving some valid inequalities and taking a Benders decomposition approach.

The stochastic IRP literature is notable, however, for not adequately addressing two important issues. The first of these involves demand uncertainty. Both the Markov decision process and stochastic programming methods assume a known probability distribution for the uncertain demand, but full distributional information is seldom available in practice. Owing to various reasons, such as estimation error and the lack of historical data, we may have only partial information on the probability distribution. To cope with this shortcoming, classical robust optimization methodology has been applied to the IRP (see e.g., Aghezzaf 2008, Solyah et al. 2012, Bertsimas et al. 2016). These authors use a polyhedron to characterize the realization of uncertain demand and then minimize the cost for the worst-case scenario. In particular, Aghezzaf (2008) assumes normally distributed demands and minimizes the worst-case cost for possible realizations of demands within certain confidence level. Solyah et al. (2012) consider a robust IRP by using the “budget of uncertainty” approach proposed by Bertsimas and Sim (2003, 2004), and solve the problem via a branch-and-cut algorithm. Bertsimas et al. (2016) consider the robust IRP from the perspective of scalability; they propose a robust and adaptive formulation that can be solved for instances of
around 6,000 customers. However, traditional robust optimization does not capture any frequency information and focuses only on the worst-case realization — an approach that some consider to be unnecessarily conservative. We therefore propose a distributionally robust optimization framework that does exploit the frequency information. The resulting model allows us to make robust inventory replenishment and routing decisions that can immune against the effect of data uncertainty.

Furthermore, since this is a multi-period problem with uncertain demand, each period’s operational decisions should be adjusted in response to previous demand realization in order to achieve appealing performance. For such multi-stage problems, Ben-Tal et al. (2004) propose an adjustable decision rule approach that models decisions as functions of the uncertain parameters’ realizations. Yet because finding the optimal decision rule is computationally intractable, various decision rules have been proposed to solve the problem approximately by restricting the set of feasible adaptive decisions to some particular and simple functional forms (see e.g., Ben-Tal et al. 2004, Chen et al. 2008, Goh and Sim 2010, Bertsimas et al. 2011, Kuhn et al. 2011, Georghiou et al. 2015); readers are referred to the tutorial by Delage and Iancu (2015). To reduce computational complexity, we will use a linear decision rule that restricts the decisions in each period to be affine functions of previous periods’ uncertain demands. The effectiveness of this decision rule has been established both computationally (e.g., Ben-Tal et al. 2005, Bertsimas et al. 2018) and analytically (e.g., Kuhn et al. 2011, Bertsimas and Goyal 2012). To the best of our knowledge, Bertsimas et al. (2016) is the only paper that incorporates the linear decision rule in IRP and hence is able to solve large-scale instances. However, our paper differs from theirs in three ways: we account for the frequency information of demands, we incorporate an enhanced decision rule, and we propose a new decision criterion that is consistent with risk aversion.

The second issue that is largely ignored in the literature arises from the stochastic IRP’s objective function. Most research on this topic minimizes the expectation of operational cost, which is the sum of holding cost, shortage cost, and transportation cost. However, the calibration of cost parameters, especially the per-unit shortage cost, is challenging in many practical cases (Brandimarte and
Zotteri 2007). And even if cost parameters could be precisely estimated, the inventory cost is a piecewise linear function and so calculating its expectation involves the sort of multi-dimensional integration that increases the computational complexity (Ardestani-Jaafari and Delage 2016). Last but not least, expected cost is a risk-neutral criterion and thus fails to account for decision makers’ attitudes toward risk. Yet risk aversion is well documented empirically (as in the St. Petersburg paradox) and applied in inventory literatures (e.g., Chen et al. 2007). So instead of optimizing the expected cost, we present an alternative approach that resolves the issues previously described.

Our objective function is related to the concept of service level, which is a prevalent requirement in supply chain management (Miranda and Garrido 2004, Bernstein and Federgruen 2007, Bertsimas et al. 2016). From a practical standpoint, it is extremely important for retailers to maintain inventory levels within a certain range. On the one hand, the upper limit corresponds to a retailer’s capacity, e.g., storage capacity. On the other hand, running out of inventory will damage brand image, and also impose substantial cost due to emergent replenishments. It follows that establishing a lower limit will also help control inventory levels. The rationale of such a service-level requirement also relates to the target-driven decision making. It can be traced back to Simon (1955), who suggests that the main goal for most decision making is not to maximize returns or minimize costs but rather to achieve certain targets. Since then, a rich body of descriptive literatures has confirmed that the decision making is often driven by targets (Mao 1970). Indeed, with the absence of inventory control, Jaillet et al. (2016) and Zhang et al. (2018) have investigated the target-driven decision making in the vehicle routing problem. It is therefore natural to assume that in stochastic IRP, the objective is to maintain the inventory within certain target levels defined by an upper and a lower bound.

Given such a range imposed on inventory, there are two classical ways to model the service-level requirement. The first approach is to ensure that the inventory level remains within the desired range for all possible demand realizations; however, this approach is often viewed as being too conservative. The second way of modeling the service-level requirement is to minimize the violation
probability via chance constrained optimization. Yet chance constraints are well known to be nonconvex and computationally intractable in most cases (e.g., Ben-Tal et al. 2009); that approach has also been criticized for being unable to capture a violation’s magnitude (Diecidue and van de Ven 2008). Hence, we introduce the Service Violation Index (SVI), a general decision criterion that we use to evaluate the risk of violating the service-level requirement and to address the issues discussed so far. Our SVI decision criterion generalizes the performance measure proposed by Jaillet et al. (2016) when studying the stochastic vehicle routing problem. It can be represented from classical expected utility theory, and can therefore be constructed using any utility function. In the case of single period, the SVI is associated with the satisficing measure axiomatized by Brown and Sim (2009).

We summarize our main contributions as follows.

- Instead of minimizing the expectation of inventory and transportation cost, we study robust IRP from the service-level perspective and aim to keep the inventory level within a certain range. The existence of uncertainty in demand motivates us to propose a multi-period decision criterion that allows decision makers to account for the risk that the uncertain inventory level of each retailer in each period will violate the stipulated service requirement window.

- We use a distributionally robust optimization framework to calibrate uncertain demand, by way of descriptive statistics, to derive solutions that protect against demand variation. This approach incorporates more distributional information and will yield less conservative results than does the classical robust optimization approach.

- We derive adaptive inventory replenishment solutions that can be updated with observed demand by implementing a decision rule approach. Then the robust IRP can be formulated as a large-scale mixed-integer linear optimization problem.

- We provide algorithms that can solve our model efficiently by way of the Benders decomposition approach.

This rest of our paper proceeds as follows. Section 2 defines the problem and discusses how we model and evaluate uncertainty in the multi-period case. In Section 3, we formulate adaptive
decisions using our decision rule approach, and Section 4 proposes a solution procedure under various information sets. Section 5 reports results from several computational studies. We conclude in Section 6 with a brief summary.

**Notations:** A boldface lowercase letter, such as \( \mathbf{x} \), represents a column vector; row vectors are given within parentheses and with a comma separating each element: \( \mathbf{x}' = (x_1, \ldots, x_n) \). We use \( (y, \mathbf{x}_{-i}) \) to denote the vector with all elements equal to those in \( \mathbf{x} \) except the \( i \)th element, which is equal to \( y \); that is, \( (y, \mathbf{x}_{-i}) = (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_T) \). Boldface uppercase letters, such as \( \mathbf{Y} \), represent matrices; thus \( \mathbf{Y}_{i} \) is the \( i \)th column vector of \( \mathbf{Y} \) and \( \mathbf{y}_{i} \) is its \( i \)th row vector. For two matrices \( \mathbf{X}, \mathbf{Y} \in \mathbb{R}^{m \times n} \), the standard inner product is given by \( \langle \mathbf{X}, \mathbf{Y} \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij} Y_{ij} \). We use “\( \leq \)” to represent an element-wise comparison; for example, \( \mathbf{X} \leq \mathbf{Y} \) means \( X_{ij} \leq Y_{ij} \) for all \( i = 1, \ldots, m, j = 1, \ldots, n \). We denote uncertain quantities by characters with “\( \sim \)” sign, such as \( \tilde{d} \), and the corresponding characters themselves, such as \( d \), as the realization. We denote by \( \mathcal{V} \) the space of random variables and let the probability space be \( (\Omega, \mathcal{F}, \mathbb{P}) \). We assume that full information about the probability distribution \( \mathbb{P} \) is not known. Instead, we know only that \( \mathbb{P} \) belongs to an uncertainty set \( \mathcal{P} \), which is a set of probability distributions characterized by particular descriptive statistics. An inequality between two uncertain parameters, \( \tilde{t} \geq \hat{v} \), implies statewise dominance; that is, it implies \( \tilde{t}(\omega) \geq \hat{v}(\omega) \) for all \( \omega \in \Omega \). In addition, the strict inequality \( \tilde{t} > \hat{v} \) implies that there exists an \( \epsilon > 0 \) for which \( \tilde{t} \geq \hat{v} + \epsilon \). We let \( \mathbf{e}_m \in \mathbb{R}^T \) be the \( m \)th standard basis vector. We use \([\cdot]\) to represent a set of running indices, e.g., \( [T] = \{1, \ldots, T\} \) and denote the cardinality of the set \( \mathcal{S} \) as \( |\mathcal{S}| \).

2. **Model**

Section 2.1 defines the inventory routing problem. In Section 2.2, we introduce a new service measure to evaluate the uncertain outcome from any inventory routing decision.

2.1. **Problem statement**

We consider a basic stochastic inventory routing problem in which there is one supplier and \( N \) retailers over \( T \) periods. The problem is defined formally on a directed network \( \mathcal{G} = ([N] \cup \{0\}, \mathcal{A}) \), where
node 0 is the supplier node and $\mathcal{A}$ is the set of arcs. Given any subset of nodes $\mathcal{S} \subseteq ([N] \cup \{0\})$, we define $\mathcal{A}(\mathcal{S})$ as the set of all arcs that connect two nodes in $\mathcal{S}$. That is, $\mathcal{A}(\mathcal{S}) = \{(i,j) \in \mathcal{A} : i,j \in \mathcal{S}\}$.

We let $\tilde{D} \in \mathcal{V}^{N \times t}$ denote the matrix of uncertain demand, where $\tilde{D}_{nt}$ is retailer $n$’s uncertain demand in period $t$. Also, $\tilde{D}' = (\tilde{D}_1, \ldots, \tilde{D}_t) \in \mathcal{V}^{N \times t}$ is the matrix of uncertain demand from period 1 to period $t$; hence, we have $\tilde{D}'^T = \tilde{D}$. We assume that $\tilde{D}$ has an upper and a lower bound, which are denoted (respectively) $\bar{D}$ and $\underline{D}$. The support of $\tilde{D}$ we denote by $\mathcal{W} = \{D | \underline{D} \leq D \leq \bar{D}\}$.

Any excess demand is assumed to be backlogged. For the sake of simplicity, we assume that the transportation cost from node $i$ to node $j$ is deterministic and denoted by $c_{ij}$ for $(i,j) \in \mathcal{A}$.

We assume that the supplier uses only a single vehicle, but one of unconstrained capacity, to deliver products. In each period, the supplier observes the inventory levels of all retailers and then makes three decisions jointly: which subset of retailers to visit, which route to use, and what quantities of product to replenish.

The decisions made in period $t \in [T]$ are affected by the demand realization in previous periods; we therefore represent those decisions as functions of previous uncertain demands (i.e., $\tilde{D}^{t-1}$). The routing decisions are denoted by $y^t_n, z^t_{ij} \in \mathcal{B}_t$, where $\mathcal{B}_t$ is the space of all measurable functions that map from $\mathbb{R}^{N \times (t-1)}$ to $\{0,1\}$. More specifically, in period $t$, we have $y^t_n (\tilde{D}^{t-1}) = 1$ if retailer $n$ is visited and $y^t_n (\tilde{D}^{t-1}) = 1$ if the supplier’s vehicle is used to replenish a subset of retailers.

Moreover, for $(i,j) \in \mathcal{A}$, we have $z^t_{ij} (\tilde{D}^{t-1}) = 1$ if that vehicle travels directly from node $i$ to node $j$. Retailer $n$’s replenishment decision is denoted by $q^t_n (\tilde{D}^{t-1}) \in \mathcal{R}_t$, where $\mathcal{R}_t$ is the space of measurable functions that map from $\mathbb{R}^{N \times (t-1)}$ to $\mathbb{R}$. For retailer $n$ in period $t$, we use $x^t_n (\tilde{D}^{t}) \in \mathcal{R}_{t+1}$ to represent the ending inventory. Without loss of generality (WLOG), we assume that the system has no inventory at the start of the planning horizon: $x^0_n = 0$. Furthermore, we put $\tilde{D}^0 = \emptyset$ so that any function on $\tilde{D}^0$ is essentially a constant. The inventory dynamics can therefore be calculated as

$$x^t_n (\tilde{D}^{t}) = x^{t-1}_n (\tilde{D}^{t-1}) + q^t_n (\tilde{D}^{t-1}) - \tilde{D}_{nt} = \sum_{m=1}^{i} (q^m_n (\tilde{D}^{m-1}) - \tilde{D}_{nm}), \quad \forall t \in [T].$$

We summarize the decision variables and parameters in Table 1.
Z. Cui et al.: Inventory Routing Problem under Uncertainty
Article submitted to ; manuscript no. (Please, provide the manuscript number!)

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{ij}, (i, j) \in \mathcal{A}$</td>
<td>Transportation cost from node $i$ to node $j$;</td>
</tr>
<tr>
<td>$\tilde{D}_{nt}, n \in [N], t \in [T]$</td>
<td>Uncertain demand of retailer $n$ in period $t$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Decision variables</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_0^t \in \mathcal{B}_t, t \in [T]$</td>
<td>$y_0^t(\tilde{D}^{t-1}) = 1$ if the vehicle is used in period $t$ (= 0 otherwise)</td>
</tr>
<tr>
<td>$y_n^t \in \mathcal{B}_t, n \in [N], t \in [T]$</td>
<td>$y_n^t(\tilde{D}^{t-1}) = 1$ if retailer $n$ is replenished in period $t$ (= 0 otherwise)</td>
</tr>
<tr>
<td>$z_{ij}^t \in \mathcal{B}_t, (i, j) \in \mathcal{A}$</td>
<td>$z_{ij}^t(\tilde{D}^{t-1}) = 1$ if the arc $(i, j)$ is traversed in period $t$ (= 0 otherwise)</td>
</tr>
<tr>
<td>$q_n^t \in \mathcal{R}_t, n \in [N], t \in [T]$</td>
<td>$q_n^t(\tilde{D}^{t-1})$ is the replenishment quantity of retailer $n$ in period $t$;</td>
</tr>
<tr>
<td>$x_n^t \in \mathcal{R}_{t+1}, n \in [N], t \in [T]$</td>
<td>$x_n^t(\tilde{D}^{t})$ is the inventory level of retailer $n$ at the end of period $t$.</td>
</tr>
</tbody>
</table>

Table 1 Parameters and decisions

We now present the constraints on the routing and replenishment decisions with a feasible set $\mathcal{Z}$ as

$$
\mathcal{Z} = \begin{cases}
    y_0^t, y_n^t \in \mathcal{B}_t, & \forall n \in [N], t \in [T], \\
    z_{ij}^t \in \mathcal{B}_t, & \forall n \in [N], t \in [T], \\
    q_n^t \in \mathcal{R}_t, & \forall n \in [N], t \in [T], \\
    \forall n \in [N], t \in [T], (i, j) \in \mathcal{A} & \sum_{(i, j) \in \mathcal{A} \cap \mathcal{S}} z_{ij}^t(\tilde{D}^{t-1}) \leq \sum_{n \in \mathcal{S}} y_n^t(\tilde{D}^{t-1}) - y_k^t(\tilde{D}^{t-1}), \\
    \forall \mathcal{S} \subseteq [N], |\mathcal{S}| \geq 2, k \in \mathcal{S}, t \in [T] & (e)
\end{cases}
$$

(1)

With $M$ being a large number, constraint (1a) uses the Big-M method to infer that the ordering decision can be made only when retailer $n$ is served. Constraint (1b) ensures that, if any retailer $n$ is served in period $t$, then the route in that period must “visit” the supplier at node 0. The constraints (1c) and (1d) guarantee that each visited node $n$ has only one arc entering $n$ and one arc exiting $n$. Constraint (1e) uses a two-index vehicle flow formulation, which has been shown to be computationally attractive, to eliminate subtours (see e.g., Solyali et al., 2012).
We require the period-$t$ inventory level $x^t_n \left( D^t \right)$ to be within a pre-specified interval known as the requirement window: $[\tau^t_n, \bar{\tau}^t_n]$. The requirement window can be determined by practical considerations. For example, $\tau^t_n = 0$ indicates a strong preference for avoiding stockouts, and an upper bound $\bar{\tau}^t_n$ can be specified in accordance with the storage capacity which, if exceeded, would incur a high holding cost. Similarly, we set a budget $B^t$ for the transportation cost in each period $t \in [T]$.

Therefore, our major concern now is to evaluate the risk that the uncertain inventory level fails to satisfy the requirement, or that the transportation cost violates the budget constraint. We propose a new performance measure to evaluate this risk and then present a comprehensive model.

### 2.2. Service Violation Index

We now introduce a service quality measure to evaluate the risk of service violation in terms of inventory level and transportation cost. Inspired by Chen et al. (2015) and Jaillet et al. (2016), we propose a new and more general index to evaluate this risk. Our construction of this index is based on classical expected utility theory and risk measures.

In particular, given an $I$-dimensional random vector $\tilde{x} \in \mathbb{V}^I$, we must evaluate the risk that $\tilde{x}$ realizes to be outside of the required window $[\tau, \bar{\tau}] \subseteq \mathbb{R}^I$. Toward that end, we propose the following notion of Service Violation Index. We first define a function $v_{\tau, \bar{\tau}}(\cdot)$ as $v_{\tau, \bar{\tau}}(x) = \max \{ x - \bar{\tau}, \tau - x \} = (\max \{ x_i - \bar{\tau}_i, \tau_i - x_i \})_{i \in [I]}$, which is the violation of $x$ with regard to the requirement window $[\tau, \bar{\tau}]$. Specifically, a negative value of violation, e.g., $v_{0,3}(2) = -1$, implies that a change can still be made, in any arbitrary direction, without violating the requirement window.

**Definition 1** A class of functions $\rho_{\tau, \bar{\tau}}(\cdot) : \mathbb{V}^I \to [0, \infty]$ is a Service Violation Index (SVI) if, for all $\tilde{x}, \tilde{y} \in \mathbb{V}^I$, it satisfies the following properties.

1. **Monotonicity:** $\rho_{\tau, \bar{\tau}}(\tilde{x}) \geq \rho_{\tau, \bar{\tau}}(\tilde{y})$ if $v_{\tau, \bar{\tau}}(\tilde{x}) \geq v_{\tau, \bar{\tau}}(\tilde{y})$.

2. **Satisficing:**
   
   (a) **Attainment content:** $\rho_{\tau, \bar{\tau}}(\tilde{x}) = 0$ if $v_{\tau, \bar{\tau}}(\tilde{x}) \leq 0$;

   (b) **Starvation aversion:** $\rho_{\tau, \bar{\tau}}(\tilde{x}) = \infty$ if there exists an $i \in [I]$ such that $v_{\tau, \bar{\tau}}(\tilde{x}_i) > 0$. 

3. Convexity: $\rho_{\mathbf{x}, \mathbf{y}}(\lambda \bar{x} + (1 - \lambda) \bar{y}) \leq \lambda \rho_{\mathbf{x}, \mathbf{y}}(\bar{x}) + (1 - \lambda) \rho_{\mathbf{x}, \mathbf{y}}(\bar{y})$ for any $\lambda \in [0, 1]$.

4. Positive homogeneity: $\rho_{\lambda \mathbf{x}, \lambda \mathbf{y}}(\lambda \bar{x}) = \lambda \rho_{\mathbf{x}, \mathbf{y}}(\bar{x})$ for any $\lambda \geq 0$.

5. Dimension-wise additivity: $\rho_{\mathbf{x}, \mathbf{y}}(\bar{x}) = \sum_{i=1}^{I} \rho_{\mathbf{x}, \mathbf{y}}((\bar{x}_i, \mathbf{w}_{-i}))$ for any $\mathbf{w} \in [\mathbf{r}, \mathbf{t}]$.

6. Order invariance: $\rho_{\mathbf{x}, \mathbf{y}}(\bar{x}) = \rho_{P \mathbf{x}, P \mathbf{y}}(P \bar{x})$ for any permutation matrix $P$.

7. Left continuity: $\lim_{a \downarrow 0} \rho_{-\infty \mathbf{0}, 0}(\bar{x} - a \mathbf{1}) = \rho_{-\infty \mathbf{0}, 0}(\bar{x})$, where $(-\infty)$ is the vector all of whose components are $-\infty$.

We define the SVI such that a low value indicates a low risk of violating the requirement window. Monotonicity states that if a given violation is always greater than another, then the former will be associated with higher risk and hence will be less preferred. Satisficing specifies the riskiness in two extreme cases. Under attainment content, if any realization of the uncertain attributes lies within the requirement window, then there is no risk of violation. In contrast, starvation aversion implies that risk is highest when there is at least one attribute that always violates the requirement window. Our convexity property mimics risk management’s preference for diversification. Positive homogeneity enables the cardinal nature, the motivation of which can be found in Artzner et al. (1999). Dimension-wise additivity and order invariance imply that overall risk is the aggregation of individual risk and the insensitivity to the sequence of dimensions, respectively. Both properties are justified in a multi-stage setting by Chen et al. (2015). Finally, for solutions to be tractable, we need left continuity; together with other properties, it also allows us to represent the SVI in terms of a convex risk measure as follows.

**Theorem 1** A function $\rho_{\mathbf{x}, \mathbf{y}}(\cdot) : \mathcal{V} \to [0, \infty]$ is an SVI if and only if it has the representation as

$$
\rho_{\mathbf{x}, \mathbf{y}}(\bar{x}) = \inf \left\{ \sum_{i=1}^{I} \alpha_i \left| \mu \left( \frac{v_{\mathbf{x}, \mathbf{y}}(\bar{x}_i)}{\alpha_i} \right) \right| \leq 0, \alpha_i > 0, \ \forall i \in [I] \right\},
$$

where we define $\inf \emptyset = \infty$ by convention and $\mu(\cdot) : \mathcal{V} \to \mathbb{R}$ is a normalized convex risk measure, i.e., $\mu(\cdot)$ satisfies the following properties for all $\bar{x}, \bar{y} \in \mathcal{V}$.

1. **Monotonicity**: $\mu(\bar{x}) \geq \mu(\bar{y})$ if $\bar{x} \succeq \bar{y}$.

2. **Cash invariance**: $\mu(\bar{x} + w) = \mu(\bar{x}) + w$ for any $w \in \mathbb{R}$. 
3. Convexity: \( \mu(\lambda \hat{x} + (1 - \lambda)\hat{y}) \leq \lambda \mu(\hat{x}) + (1 - \lambda)\mu(\hat{y}) \) for any \( \lambda \in [0, 1] \).

4. Normalization: \( \mu(0) = 0 \).

Conversely, given an SVI \( \rho_{\Sigma} \varphi(\cdot) \), the underlying convex risk measure is given by

\[
\mu(\hat{x}) = \min \{ a | \rho_{-\infty, 0}((\hat{x} - a)e_1) \leq 1 \}.
\]

(3)

For the sake of brevity, we present all proofs in the appendix.

To calculate the SVI value efficiently and to illustrate some insights into its connection with the classical expected utility criterion, we focus on the SVI whose underlying convex risk measure is the shortfall risk measure defined by Föllmer and Schied (2002). In order to incorporate distributional ambiguity, we redefine it as follows.

**Definition 2** A shortfall risk measure is any function \( \mu_u(\cdot) : \mathcal{V} \to \mathbb{R} \) defined by

\[
\mu_u(\hat{x}) = \inf_\eta \left\{ \eta \mid \sup_{\varphi \in \mathcal{P}} \mathbb{E}_\varphi [u(\hat{x} - \eta)] \leq 0 \right\},
\]

(4)

where \( u \) is an increasing convex normalized utility function such that \( u(0) = 0 \).

The operation \( \sup_{\varphi \in \mathcal{P}} \) allows us to take the robustness into account. Hence the shortfall risk can be interpreted as the minimum amount that needs to be subtracted from the uncertain attribute so that this attribute’s expected utility is less than 0, i.e., the risk becomes acceptable. We now use the dual representation described in Theorem 1 to show that, when shortfall risk measure is the underlying risk measure, the corresponding SVI has the following representation.

**Proposition 1** The shortfall risk measure is a normalized convex risk measure. Its corresponding SVI, which we refer to as the utility-based SVI, can be represented as follows:

\[
\rho_{\Sigma} \varphi(\hat{x}) = \inf \left\{ \sum_{i \in [I]} \alpha_i \left| \sup_{\varphi \in \mathcal{P}} \mathbb{E}_\varphi \left( u \left( \frac{v_{\Sigma_{\tau_i}}(\hat{x}_i)}{\alpha_i} \right) \right) \right| \leq 0, \alpha_i > 0, \forall i \in [I] \right\}.
\]

(5)

According to the equation (5), the utility-based SVI is the aggregation of the smallest positive scalar factors \( \alpha_i \) over all indexes \( i \in [I] \) such that \( v_{\Sigma_{\tau_i}}(\hat{x}_i)/\alpha_i \) is within the acceptable set, or
equivalently, $v_{x_i} \tau_i (\tilde{x}_i) / \alpha_i$ is with nonpositive expected utility. This particular index allows the utility function to take different forms. For instance, if the utility function is exponential as $u(x) = \exp(x) - 1$, then the index can be reformulated as

$$\rho_{\mathcal{T}, \mathcal{P}}(\tilde{x}) = \inf \left\{ \sum_{i \in [I]} \alpha_i \left| C_{\alpha_i}(\tilde{x}_i) \leq 0, \alpha_i \geq 0, \forall i \in [I] \right\} ,$$

where

$$C_{\alpha_i}(\tilde{x}_i) = \alpha_i \ln \left( \sup_{p \in \mathcal{P}} \mathbb{E}_p \left( \exp \left( \frac{v_{x_i} \tau_i (\tilde{x}_i)}{\alpha_i} \right) \right) \right).$$

Function $C_{\alpha_i}(\tilde{x}_i)$ is the worst-case certainty equivalent of the uncertain violation of $\tilde{x}_i$ with respect to the requirement window $[\tau_i, \bar{\tau}_i]$ when the parameter of absolute risk aversion is $1 / \alpha_i$ (Gilboa and Schmeidler, 1989). More details can be found in Hall et al. (2015) and Jaillet et al. (2016).

For the sake of computational tractability, we focus on the SVI whose construction is based on the piecewise linear utility function $u$. In particular, we let $u(x) = \max_{k \in [K]} \{a_k x + b_k\}$ with $a_k \geq 0$ for all $k \in [K]$. Adopting this piecewise linear utility $u$ preserves the optimization problem’s linear structure, which eases the computational burden even while accounting for risk. We remark that, practically speaking, a piecewise linear utility can be used to approximate any type of increasing convex utility. Because $\alpha_i > 0$ for the piecewise linear function considered here, we multiply both the left-hand side (LHS) and the right-hand side (RHS) of the constraint by $\alpha_i$. Then the SVI can be simplified as

$$\rho_{\mathcal{T}, \mathcal{P}}(\tilde{x}) = \inf \left\{ \sum_{i \in [I]} \alpha_i \left| \sup_{p \in \mathcal{P}} \mathbb{E}_p \left( \max_{k \in [K]} \left\{a_k v_{x_i} \tau_i (\tilde{x}_i) + b_k \alpha_i \right\} \right) \leq 0, \right. \right. \left. \left. \alpha_i > 0, \forall i \in [I] \right\} .$$

Having specified the objective function, we can now present our stochastic IRP model as follows.
\[
\inf \sum_{t \in [T]} \sum_{n \in [N]} \alpha_n^t + \sum_{t \in [T]} \beta_n^t,
\]
\[
\text{s.t. } \sup_{P \in \mathcal{P}} \mathbb{E}_P \left( \max_{k \in [K]} \left\{ a_k \max \left\{ t_n^t - x_n^t \left( \tilde{D}^t \right) + b_k \alpha_n^t \right\} + b_k \alpha_n^t \right\} \leq 0, \forall n \in [N], t \in [T] \right)
\]
\[
\sup_{P \in \mathcal{P}} \mathbb{E}_P \left( \max_{k \in [K]} \left\{ a_k \left( \sum_{(i,j) \in \mathcal{A}} c_{ij} z_{ij} \left( \tilde{D}^{t-1} \right) - B^t \right) + b_k \beta_n^t \right\} \right) \leq 0, \forall t \in [T]
\]
\[
x_n^t \left( \tilde{D}^t \right) = \sum_{m=1}^t (q_n^m - D^{m-1} \tilde{D}_{nm}), \forall n \in [N], t \in [T]
\]
\[
\alpha_n^t > 0, \forall n \in [N], t \in [T]
\]
\[
(q_n^t, y_n^t, z_{ij}^t, n \in [N], t \in [T], (i,j) \in \mathcal{A}) \in \mathcal{Z}.
\]

3. Adaptive Decisions

We facilitate the optimization by adopting a decision rule approach to derive adaptive decisions. In particular, each period’s decision is assumed to be a specific mapping from previous demand realizations. At each stage, the decisions involve both binary ones (i.e., routing) and continuous ones (i.e., replenishment). Although decision rules with respect to continuous decisions have been investigated by a rich body of literatures, the rules related to binary decisions have attracted much less attention and is still computationally challenging (e.g., Bertsimas and Caramanis 2007, Bertsimas and Georgiou 2014, 2015). Since this study is the first to consider such a complex stochastic IRP, we simplify the model by assuming that the binary decisions are nonadaptable. That is, we consider the problem in two steps. In the first step, the visiting decisions and routing decisions for all periods are determined at the beginning of planning horizon. Thus all routing decisions are made a priori and irrespective of demand realizations. This approach approximates the optimal solution and, indeed, is a reasonable one in practice. Visiting plans must be communicated to retailers in advance so that they can adequately prepare; besides, scheduling the vehicle and informing the staff on short notice is not always practically feasible (Bertsimas et al. 2016). In the second step, the supplier makes replenishment decisions at the start of each period after demand realizations in previous periods. In this sense, then, inventory decisions are made adaptively. Figure 1 summarizes the timeline applicable to these decisions.
For the sake of tractability, we now introduce a special case of decision rules — namely, a linear decision rule — whereby each decision is assumed to be an affine function of uncertainty realizations. Following this linear decision rule, we can write the order quantity as

$$q_n^t \left( \check{D}^{t-1} \right) = q_{n0}^t + \langle Q_n^t, \check{D} \rangle, \quad n \in [N], t \in [T],$$

where $q_{n0}^t \in \mathbb{R}$ and $Q_n^t \in \mathbb{R}^{N \times T}$ are decision variables. In particular, we let the $l$th column of $Q_n^t$, denoted by $(Q_n^t)_l$, be the zero vector $\mathbf{0}$ when $l \geq t$. If $t = 1$, then $Q_n^1 = \mathbf{0}$. That captures the “non-anticipative” property, since there is no way for the order quantity to depend on future demand realization.

So at the end of each period, the inventory level is also an affine function of demand realization. This follows because, for all $t \in [T]$,

$$x_n^t \left( \check{D} \right) = \sum_{m=1}^t \left( q_{n0}^m \left( \check{D}^{m-1} \right) - \check{D}_{nm} \right) = \sum_{m=1}^t \left( q_{n0}^m + \langle Q_n^m, \check{D} \rangle - \check{D}_{nm} \right) = x_{n0}^t + \langle X_n^t, \check{D} \rangle, \quad (7)$$

here $x_n^t = \sum_{m=1}^t q_{n0}^m$ and $X_n^t = \sum_{m=1}^t (Q_n^m - e_n e_n')$.

Since now the routing decisions $z_{ij}^t$ for $(i, j) \in \mathcal{A}$ and $t \in [T]$ are nonadaptable, the transportation cost $\sum_{(i,j) \in \mathcal{A}} c_{ij} z_{ij}^t$ is independent of the demand realizations. Hence the constraint in model (6), $\sup_{p \in \mathcal{P}} \mathbb{E}_p \left( u \left( \left( \sum_{(i,j) \in \mathcal{A}} c_{ij} z_{ij}^t - B^t \right) / \beta_t \right) \right) \leq 0$ reduces to the deterministic budget constraint $\sum_{(i,j) \in \mathcal{A}} c_{ij} z_{ij}^t \leq B^t$ because the utility function $u$ is non-decreasing and $u(0) = 0$. Under this decision rule approach, our model can be formulated as follows:
\[
\inf \sum_{t \in [T]} \sum_{n \in [N]} \alpha^t_n,
\]
\[
\text{s.t. } \sup_{\mathcal{P} \in \mathcal{P}} \mathbb{E}_{\mathcal{P}} \left( \max_{k \in [K]} \left\{ a_k \max \left\{ \sum^t_n - x^t_{n0} - \langle X^t_n, \tilde{D} \rangle, x^t_{n0} + \langle X^t_n, \tilde{D} \rangle - \pi^t_n \right\} + b_k \alpha^t_n \right\} \right) \leq 0, \forall n \in [N], t \in [T], \quad (a)
\]
\[
q^t_{n0} + \langle Q^t_n, D \rangle \geq 0, \quad \forall D \in \mathcal{W}, n \in [N], t \in [T], \quad (b)
\]
\[
q^t_{n0} + \langle Q^t_n, D \rangle \leq My^t_n, \quad \forall D \in \mathcal{W}, n \in [N], t \in [T], \quad (c)
\]
\[
x^t_{n0} = \sum_{m=1}^t q^m_{n0}, \quad \forall n \in [N], t \in [T]
\]
\[
X^t_n = \sum_{m=1}^t (Q^m_n - e_n e'_m), \quad \forall n \in [N], t \in [T]
\]
\[
(Q^t_n)_{ij} = 0, \quad \forall n \in [N], l \geq t, l, t \in [T]
\]
\[
\alpha^t_n \geq \epsilon, \quad \forall n \in [N], t \in [T],
\]
\[
(y^t_0, y^t_n, z^t_{ij}, n \in [N], t \in [T], (i, j) \in \mathcal{A}) \in \mathcal{Z}_R.
\]

Here \( \mathcal{Z}_R \) is the feasible set for static routing decisions:

\[
\mathcal{Z}_R = \left\{ \left( y^t_0, y^t_n, z^t_{ij} \in \{0, 1\}, \sum_{(i,j) \in \mathcal{A}} c_{ij} z^t_{ij} \leq B^t, \quad \forall t \in [T] \right) \right\}
\]
\[
\mathcal{Z}_R = \left\{ \left( y^t_0, y^t_n, \forall n \in [N], t \in [T], \sum_{j:(n,j) \in \mathcal{A}} z^t_{nj} = y^t_n, \forall n \in [N] \cup \{0\}, t \in [T] \right) \right\}.
\]

Instead of directly using the constraint \( \alpha^t_n > 0 \) as in the original problem (6), here we use \( \alpha^t_n \geq \epsilon \) so that the feasible set will be closed. Indeed, we can always choose a positive \( \epsilon \) small enough that optimality is not compromised (Chen et al. 2015).

4. Solution Procedure

The main problem (8) involves an inner optimization over the set of possible distributions, i.e., the operation of \( \sup_{\mathcal{P} \in \mathcal{P}} \) in the first constraint. Therefore, the optimization procedure inevitably
depends on the information available about uncertain demand. In this section, we describe different types of information sets \( \mathcal{P} \) and then give the corresponding reformulations and solution techniques.

### 4.1. Stochastic approach

We first consider the case when uncertain demand follows a known distribution, i.e., \( \mathcal{P} \) is a singleton. Thus we assume that \( \tilde{D} \) takes the value of \( D^{(m)} \) with probability \( p_m, m \in [M_s] \), where \( \sum_{m \in [M_s]} p_m = 1 \). In this case,

\[
\mathcal{P} = \left\{ \mathcal{P} \left| \mathbb{P}\left( \tilde{D} = D^{(m)} \right) = p_m, m \in [M_s], \sum_{m \in [M_s]} p_m = 1 \right. \right\}
\]

and the LHS of (8a) becomes

\[
sup_{\mathcal{P} \in \mathcal{P}} E_{\mathcal{P}} \left( \max_{k \in [K]} \left\{ a_k \max \left\{ \sum_{n \in [N]} x_n^t - x_{n0}^t - \left\langle X_n^t, \tilde{D} \right\rangle, x_{n0}^t + \left\langle X_n^t, \tilde{D} \right\rangle - \pi_n^t \right\} + b_k \alpha_n^t \right\} \right)
\]

\[
= \sum_{m \in [M_s]} p_m \left( \max_{k \in [K]} \left\{ a_k \max \left\{ \sum_{n \in [N]} x_n^t - x_{n0}^t - \left\langle X_n^t, D^{(m)} \right\rangle, x_{n0}^t + \left\langle X_n^t, D^{(m)} \right\rangle - \pi_n^t \right\} + b_k \alpha_n^t \right\} \right).
\]

Hence, we can now reformulate the overall problem as a mixed-integer linear programming (MILP).

**Proposition 2** Given \( \mathcal{P} \) as defined by the equation (10), the constraints (8a)–(8c) are equivalent to these constraints:

\[
\sum_{m \in [M_s]} p_m \nu_{nt}^m \leq 0, \quad \forall n \in [N], t \in [T],
\]

\[
\nu_{nt}^m \geq a_k \left( \sum_{n \in [N]} x_{n0}^t - \left\langle X_n^t, D^{(m)} \right\rangle \right) + b_k \alpha_n^t, \quad \forall m \in [M_s], k \in [K], n \in [N], t \in [T],
\]

\[
\nu_{nt}^m \geq a_k \left( x_{n0}^t + \left\langle X_n^t, D^{(m)} \right\rangle - \pi_n^t \right) + b_k \alpha_n^t, \quad \forall m \in [M_s], k \in [K], n \in [N], t \in [T],
\]

\[
q_{n0}^t + \left\langle Q_n^t, D^{(m)} \right\rangle \geq 0, \quad \forall m \in [M_s], n \in [N], t \in [T],
\]

\[
q_{n0}^t + \left\langle Q_n^t, D^{(m)} \right\rangle \leq M y_n^t, \quad \forall m \in [M_s], n \in [N], t \in [T].
\]

It follows that problem (8) can be reformulated as an MILP.

### 4.2. Robust approach with mean absolute deviation

When knowledge about distribution is incomplete, we can also characterize uncertain demand using a distributionally robust optimization framework (see e.g., Delage and Ye 2010, Goh and Sim 2010, Wiesemann et al. 2014). Because the deterministic version of IRP is already difficult to solve, we need to choose the ambiguity set \( \mathcal{P} \) so as not to significantly increase computational complexity.
We accommodate that consideration by introducing a $\mathcal{P}$ that allows us to capture the correlation effect among uncertain demands. Formally,

$$
\mathcal{P} = \left\{ \mathbb{P} \left( \frac{D}{\bar{D}} \leq \frac{\bar{D}}{\bar{D}} \right) = 1, \quad \mathbb{E}_p \left( \bar{D} \right) = \bar{D}, \quad \mathbb{E}_p \left( \left| \bar{D} - \bar{D} \right| \right) \leq \Sigma, \quad \mathbb{E}_p \left( \sum_{(i,\tau) \in S_h} \frac{\bar{D}_{i\tau} - \bar{D}_{\tau i}}{\Sigma_{i\tau}} \right) \leq \epsilon_{lh}, \forall l \in [L], h \in [H] \right\}.
$$

(11)

In this set $\mathcal{P}$, the equation (11a) specifies the uncertain demand’s bound support while the equation (11b) specifies its mean. The inequality (11c) specifies the parameter $\Sigma_{nl}$ as the bound of uncertain demand’s mean absolute deviation. Similar to the standard deviation, this mean absolute deviation provides a direct measure of uncertain demand’s dispersion about its mean; in addition, it can be obtained in closed form for most common distributions (see e.g., Pham-Gia and Hung, 2001). For example, if $\bar{D}_{i\tau}$ is distributed uniformly on $[0,1]$, then the mean absolute deviation takes the value $1/4$. The mean absolute deviation of a normal distribution amounts to the standard deviation multiplied by $\sqrt{2/\pi}$.

In constraint (11d), $f_i(\cdot)$ is a piecewise linear convex function that can be represented by $K_i$ pieces as $f_i(x) = \max_{k \in [K_i]} \{ a_{ik}x + b_{ik} \}$. This information set is intended to capture the correlation of uncertain demand among different retailers over several periods. In practice, the respective demand of these retailers may be correlated since retailers in general are geographically dispersed within a nearby region and serve common customers. As an illustrative example, we consider a special case of (11d) in which $f_i(x) = \max \{ x, -x \} = |x|$. If $S_1 = \{(i,\tau)_{i \in [N]}\}$, then we have $\mathbb{E}_p \left( \left| \sum_{i \in [N]} \frac{\bar{D}_{i\tau} - \bar{D}_{\tau i}}{\Sigma_{i\tau}} \right| \right) \leq \epsilon_{11}$; in essence, this is a bound on the variation of total normalized demand over all retailers in period $\tau$. Similarly, for $S_2 = \{(i,\tau)_{\tau \in [\mathcal{T}]}\}$, we have $\mathbb{E}_p \left( \left| \sum_{\tau \in [\mathcal{T}]} \frac{\bar{D}_{i\tau} - \bar{D}_{\tau i}}{\Sigma_{i\tau}} \right| \right) \leq \epsilon_{12}$, which bounds the variation of the total normalized demand over all periods at node $i$. We remark that with typical characterization of correlation effect such as the covariance matrix, the inventory management problem is computationally challenging even when there is just one retailer. Nonetheless, we shall
later show that our problem remains tractable with the joint dispersion information described by constraint (11d).

We now demonstrate that, for the distributional uncertainty set $\mathcal{P}$ defined by the equation (11), the constraints in problem (8) can be transformed into linear ones.

**Proposition 3** Let $\mathcal{P}$ be as defined in the equation (11). Then, for any $n \in [N]$ and $t \in [T]$, the constraint (8a) can be reformulated as the following set of linear constraints:

$$s^t_{n0} + \langle S^t_n, \Xi \rangle + \langle T^t_n, \Sigma \rangle + \sum_{i \in [L]} \sum_{h \in [H]} \epsilon_{ih} r^t_{nih} \leq 0,$$

$$\langle D^t_{nk}, U^t_{nk} \rangle - \langle D^t_{nk}, V^t_{nk} \rangle + \langle \Xi, G^t_{nk} - F^t_{nk} \rangle + \sum_{i \in [L]} \sum_{h \in [H]} \sum_{k' \in [K_i]} \left( p_{ik} - q_{ik} \sum_{(i, r) \in S_h} X_{ir} \right) \bar{w}^t_{nkikr} \geq a_k \big( x^t_{n0} - \tau^t_n \big) + b_k \alpha^t_n - s^t_{n0}, \quad \forall k \in [K]$$

$$\langle D^t_{nk}, U^t_{nk} \rangle - \langle D^t_{nk}, V^t_{nk} \rangle + \langle \Xi, G^t_{nk} - F^t_{nk} \rangle + \sum_{i \in [L]} \sum_{h \in [H]} \sum_{k' \in [K_i]} \left( p_{ik} - q_{ik} \sum_{(i, r) \in S_h} X_{ir} \right) \bar{w}^t_{nkikr} \geq a_k \big( \bar{r}^t_n - x^t_{n00} \big) + b_k \alpha^t_n - s^t_{n0}, \quad \forall k \in [K]$$

$$\big( U^t_{nk} - V^t_{nk} - F^t_{nk} + G^t_{nk} \big)_{i_r} - \sum_{h \in [H]} \sum_{k' \in [K_i]} \sum_{(i, r) \in S_h} \frac{q_{ik}}{\sum_{i' r} \bar{w}^t_{nkikr'}} \bar{w}^t_{ nkikr } = \left( S^t_n - a_k X^t_n \right)_{i_r}, \quad \forall i \in [N], r \in [T], k \in [K]$$

$$\big( U^t_{nk} - V^t_{nk} - F^t_{nk} + G^t_{nk} \big)_{i_r} - \sum_{h \in [H]} \sum_{k' \in [K_i]} \sum_{(i, r) \in S_h} \frac{q_{ik}}{\sum_{i' r} \bar{w}^t_{nkikr'}} \bar{w}^t_{ nkikr } = \left( S^t_n + a_k X^t_n \right)_{i_r}, \quad \forall i \in [N], r \in [T], k \in [K]$$

$$F^t_{nk} + G^t_{nk} = T^t_{nk}, \quad \forall k \in [K]$$

$$F^t_{nk} + G^t_{nk} = T^t_{nk}, \quad \forall k \in [K]$$

$$\sum_{k' \in [K_i]} \bar{w}^t_{ nkikr } = r^t_{nhr}, \quad \forall l \in [L], h \in [H], k \in [K]$$

$$\sum_{k' \in [K_i]} \bar{w}^t_{ nkikr } = r^t_{nhr}, \quad \forall l \in [L], h \in [H], k \in [K]$$

$$r^t_{nhr} \geq 0, \quad \forall l \in [L], h \in [H]$$

$$T^t_{n} \geq 0$$

$$U^t_{nk}, V^t_{nk}, F^t_{nk}, G^t_{nk}, U^t_{nk}, V^t_{nk}, F^t_{nk}, G^t_{nk} \geq 0, \quad \forall k \in [K]$$

$$\bar{w}^t_{nkikr}, \bar{w}^t_{nkikr'} \geq 0, \quad \forall l \in [L], h \in [H], k \in [K], k' \in [K_i]$$
A similar technique can be used to transform constraint (8b) as follows.

**Proposition 4** Let \( \mathcal{P} \) be defined as in the equation (11). Then, for any \( n \in [N] \) and \( t \in [T] \), the constraint (8b) can be written as

\[
\begin{align*}
-q^n_t + \langle \overline{D}, L^n_t \rangle - \langle D, H^n_t \rangle & \leq 0, \\
Q^n_t + L^n_t - H^n_t & = 0, \\
L^n_t, H^n_t & \geq 0.
\end{align*}
\]

We remark that, much as in Proposition 4, the constraint (8c) can also be represented equivalently by a set of linear constraints. Now we can achieve the following reformulation of the overall problem.

**Corollary 1** Given \( \mathcal{P} \) as defined in the equation (11), the problem (8) can be formulated as an MILP.

A full description of the reformulation is given in the Appendix F.

The solution quality can be further improved by incorporating a more sophisticated decision rule. For instance, Bertsimas et al. (2018) define the lifted ambiguity set by introducing certain auxiliary random variables and then incorporating them into an approximation of a linear decision rule. To differentiate that approach from the traditional one, we call it the enhanced linear decision rule (ELDR). We can use this ELDR in the case of the ambiguity set \( \mathcal{P} \) considered previously. For that purpose, we introduce the auxiliary random variables \( \tilde{U} \) and \( \tilde{V} \) and then define the following lifted ambiguity set \( Q \):

\[
Q = \left\{ Q \left( \left( \overline{D}, \tilde{U}, \tilde{V} \right) \in \tilde{W} \right) = 1 \right\}.
\]

\[
\begin{align*}
E_Q \left( \overline{D} \right) & = \Xi, \\
E_Q \left( \tilde{U} \right) & \leq \Sigma, \\
E_Q \left[ \tilde{v}_{lh} \right] & \leq \epsilon_{lh}, \forall l \in [L], h \in [H].
\end{align*}
\]
here $\hat{W}$ is the lifted support, defined formally as:

$$
\hat{W} = \left\{ (\hat{D}, \hat{U}, \hat{V}) \in \mathbb{R}^{N \times T} \times \mathbb{R}^{N \times T} \times \mathbb{R}^{L \times H} \middle| \begin{array}{l}
D \leq \hat{D} \leq \bar{D} \\
\hat{D} - \Xi \leq \hat{U} \\
f_i \left( \sum_{(i, \tau) \in S_h} \frac{\hat{D}_{i\tau} - \Xi_{i\tau}}{\sum_{i\tau}} \right) \leq \bar{v}_{ih}, \forall l \in [L], h \in [H]
\end{array} \right\},
$$

where $\bar{v}_{ih}$ denotes the element located at the $l$th row and $h$th column of matrix $\hat{V}$. Wiesemann et al. (2014) show that the ambiguity set $\mathcal{P}$ is equivalent to the set of marginal distributions of $\hat{D}$ under $\mathbb{Q}$ for all $\mathbb{Q} \in \mathcal{Q}$. Under ELDR, the ordering quantity $q_n^t$, $n \in [N], t \in [T]$ is restricted to depend linearly not only on the primary random variable $\hat{D}$ but also on the auxiliary random variables $\hat{U}$ and $\hat{V}$. Then the order quantity can be written as

$$
q_n^t (\hat{D}, \hat{U}, \hat{V}) = q_n^{t0} + \langle Q_{0n}^t, \hat{D} \rangle + \langle Q_{1n}^t, \hat{U} \rangle + \langle Q_{2n}^t, \hat{V} \rangle, \quad n \in [N], t \in [T],
$$

here $q_n^{t0} \in \mathbb{R}$, $Q_{0n}^t \in \mathbb{R}^{N \times T}$, $Q_{1n}^t \in \mathbb{R}^{N \times T}$, and $Q_{2n}^t \in \mathbb{R}^{L \times H}$ are decision variables. Since the adaptive decisions cannot be anticipated, both $(Q_{0n}^t)_l$ and $(Q_{1n}^t)_l$ are set to 0 for all $l \geq t$; we also put $(Q_{2n}^t)_h = 0$ when there exists some $\tau \geq t$ such that $(i, \tau) \in S_h$. Now constraint (8a) can be written as

$$
\sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_\mathbb{Q} \left( \max_{k \in [K]} \left\{ a_k \max L_{i\tau} - q_{n0}^t - (Q_{0n}^t, \hat{D}) - (Q_{1n}^t, \hat{U}) - (Q_{2n}^t, \hat{V}) \\
q_n^{t0} + (Q_{0n}^t, \hat{D}) + (Q_{1n}^t, \hat{U}) + (Q_{2n}^t, \hat{V}) - S_{i\tau} + b_k \alpha_n^t \right\} \right) \leq 0
$$

(13)

Here, problem (8) can be solved by the corresponding reformulation method. This claim is formalized in our next proposition.

**Proposition 5** Let the distribution ambiguity set $\mathbb{Q}$ be as defined in the equation (12), and let constraint (8a) be replaced by (13). Then problem (8) can be equivalently formulated as an MILP.

### 4.3. Robust approach with Wasserstein distance

Considerable interest has recently been shown in ambiguity sets whose construction is based on the notion of statistical distance. In particular, by measuring the proximity of probability distributions with statistical distance measures, the ambiguity sets are defined as the set of probability distributions which are “close” to the reference distribution. The examples of such statistical distance
are $\phi$-divergence (Ben-Tal et al. 2013, Wang et al. 2016, Jiang and Guan 2016) and Wasserstein distance (Esfahani and Kuhn 2017, Gao and Kleywegt 2016). Among others, Wasserstein distance’s increasing popularity is due to its powerful out-of-sample performance. Here we show that our model also enables use of an ambiguity set based on Wasserstein distance when the matrix norm is chosen as entrywise 1-norm.

We denote by $\mathcal{P}_0(\mathcal{W})$ the space of all probability distributions supported on $\mathcal{W}$. The reference probability distribution $\mathbb{P}^\dagger$, is given for the random variable $\tilde{D}^\dagger$ which takes values $\tilde{D}^{(1)}, \ldots, \tilde{D}^{(M_r)}$ with equal likelihood; that is,

$$\mathbb{P}^\dagger \left( \tilde{D}^\dagger = D^{(m)} \right) = \frac{1}{M_r}, \quad \forall m \in [M_r].$$

Given any distribution $\mathbb{P} \in \mathcal{P}_0(\mathcal{W})$ on $\tilde{D}$, the Wasserstein distance between $\mathbb{P}$ and $\mathbb{P}^\dagger$ is defined as

$$d_W(\mathbb{P}, \mathbb{P}^\dagger) := \inf \mathbb{E}_\mathbb{P} \left( \left\| \tilde{D} - \tilde{D}^\dagger \right\| \right)$$

s.t. $(\tilde{D}, \tilde{D}^\dagger) \sim \mathbb{P}$

$$\Pi_{\tilde{D}}^\mathbb{P} = \mathbb{P}$$

$$\Pi_{\tilde{D}^\dagger}^\mathbb{P} = \mathbb{P}^\dagger$$

$$\mathbb{P} \left( (\tilde{D}, \tilde{D}^\dagger) \in \mathcal{W} \times \mathcal{W} \right) = 1.$$

Here $\mathbb{P}$ is the joint distribution of $\tilde{D}$ and $\tilde{D}^\dagger$; the terms $\Pi_{\tilde{D}}^\mathbb{P}$ and $\Pi_{\tilde{D}^\dagger}^\mathbb{P}$ stand for the marginal distribution of $\mathbb{P}$ on $\tilde{D}$ and $\tilde{D}^\dagger$, respectively; and $\| \cdot \|$ represents an arbitrary norm. Now the ambiguity set can be constructed as the Wasserstein ball of radius $\theta > 0$ centered at the reference probability distribution $\mathbb{P}^\dagger$:

$$\mathcal{P} = \left\{ \mathbb{P} \in \mathcal{P}_0(\mathcal{W}) | d_W(\mathbb{P}, \mathbb{P}^\dagger) \leq \theta \right\}. \quad (14)$$

Esfahani and Kuhn (2017) and Gao and Kleywegt (2016) show that strong duality holds while studying the distributionally robust optimization problem with Wasserstein distance. We now incorporate that result into our problem. In this case, reformulating constraints (8b)-(8c) to a finite set of linear constraints can proceed as described in Proposition 4; hence we focus on constraints (8a).
Proposition 6 Let $\mathcal{P}$ be as defined in the equation (14), and let the norm $\| \cdot \|$ be chosen as the entrywise 1-norm. Then, for any given $n \in [N]$ and $t \in [T]$, the constraints (8a) are equivalent to:

$$r^t_n + \sum_{m \in [M_r]} s^t_{nm} \leq 0,$$

$$\langle \mathcal{D}, \mathcal{V}^t_{nmk} \rangle - \langle \mathcal{D}, \mathcal{U}^t_{nmk} \rangle + \langle \hat{\mathcal{D}}, \mathcal{R}^t_{nmk} \rangle \leq M_r s^t_{nm} - a_k (x^t_n - x^t_{n0}) - b_k \alpha^t_n, \forall m \in [M_r], k \in [K],$$

$$\langle \mathcal{D}, \mathcal{V}^t_{nmk} \rangle - \langle \mathcal{D}, \mathcal{U}^t_{nmk} \rangle + \langle \hat{\mathcal{D}}, \mathcal{R}^t_{nmk} \rangle \leq M_r s^t_{nm} - a_k (x^t_{n0} - \tau^t_n) - b_k \alpha^t_n, \forall m \in [M_r], k \in [K],$$

$$-U^t_{nmk} + V^t_{nmk} = -a_k X^t_n - \mathcal{R}^t_{nmk}, \quad \forall m \in [M_r], k \in [K],$$

$$-U^t_{nmk} + V^t_{nmk} = a_k X^t_n - \mathcal{R}^t_{nmk}, \quad \forall m \in [M_r], k \in [K],$$

$$(\mathcal{R}^t_{nmk})_{ir} \leq r^t_n, \quad \forall i \in [N], \tau \in [T], m \in [M_r], k \in [K],$$

$$(\mathcal{R}^t_{nmk})_{ir} \geq -r^t_n, \quad \forall i \in [N], \tau \in [T], m \in [M_r], k \in [K],$$

$$(\mathcal{R}^t_{nmk})_{ir} \leq r^t_n, \quad \forall i \in [N], \tau \in [T], m \in [M_r], k \in [K],$$

$$(\mathcal{R}^t_{nmk})_{ir} \geq -r^t_n, \quad \forall i \in [N], \tau \in [T], m \in [M_r], k \in [K],$$

$$U^t_{nmk}, V^t_{nmk}, \mathcal{R}^t_{nmk}, \mathcal{V}^t_{nmk} \geq 0, \quad \forall m \in [M_r], k \in [K],$$

$$r^t_n \geq 0, s^t_{nm} \in \mathbb{R}^{M_r}.$$ 

Therefore, problem (8) can be formulated as an MILP.

Remark 1 After accounting for the risk described in Section 2.2 and for the uncertainty sets introduced in Sections 4.1-4.3, we find that our problem’s final formulation is still an MILP. Compared with the deterministic inventory routing formulation, only a set of linear constraints and continuous random variables are added.

Note that the computation can be expedited by applying, in our stochastic framework, methods designed for the deterministic IRP. Most valid cuts developed for deterministic problem to strengthen the routing decisions remain applicable in this context; example include $z^t_{ij} \leq y^t_j, z^t_{ij} \leq y^t_j$ and $\sum_{n \in [N]} D^t_{in} \leq y^t_0$ (Archetti et al. 2007). We can also facilitate computation by limiting the feasible routes to a pre-determined subset such that instances of much larger scale can be solved (Bertsimas et al. 2016).

Finally, there are many possible directions in which this framework can be extended. From the operational perspective, one could forgo specifying the budget for each period and instead stipulate
a total budget that controls the transportation cost within a given planning horizon. Furthermore, it would be straightforward to consider the case of multiple vehicles with capacity constraints. The model we propose also allows for incorporating the production decisions (Adulyasak et al. 2013, 2015).

4.4. Benders decomposition

In Sections 4.1–4.3 we showed that our IRP under uncertainty can be formulated as an MILP. Although the size of the resulting MILP is large, we can solve it using the Benders (1962) decomposition approach (for additional details, see Rahamanian et al. 2016). For completeness, we also describe how our own model can incorporate the Benders decomposition and hence be solved more efficiently.

Our preceding reformulations of the inventory routing problem as an MILP differ since they use different information set to characterize uncertainty. Here we express all those formulations in the following general form:

$$
\begin{align*}
\min & \quad g' \phi \\
\text{s.t.} & \quad a'_i \phi + b'_i y_0 + \langle C^y_i, Y \rangle + \langle C^z_i, Z \rangle \leq s_i, \quad \forall i \in [I] \\
& \quad (y_0, Y, Z) \in Z_R,
\end{align*}
$$

(15)

where $\phi$ is used to represent all continuous decision variables; the terms $y_0 \in \{0,1\}^T$, $Y \in \{0,1\}^{N \times T}$, and $Z \in \{0,1\}^{|A| \times T}$ are the routing decisions, all of which are binary. So if we now define a function $f : Z_R \to \mathbb{R} \cup \{\infty\}$ as

$$
f(y_0, Y, Z) = \min g' \phi
$$

(16)

s.t. $a'_i \phi \leq s_i - b'_i y_0 - \langle C^y_i, Y \rangle - \langle C^z_i, Z \rangle, \quad \forall i \in [I],$

then the main problem (15) is equivalent to $\min_{(y_0,Y,Z) \in Z_R} f(y_0, Y, Z)$. In defining $f(y_0, Y, Z)$, we consider the minimization problem to be a primal problem; its dual problem can be easily obtained as

$$
f^d(y_0, Y, Z) = \max \sum_{i \in [I]} (s_i - b'_i y_0 - \langle C^y_i, Y \rangle - \langle C^z_i, Z \rangle) \psi_i
$$

(17)

s.t. $\sum_{i \in [I]} a_i \psi_i = g$.
We note that problem (15) is bounded from below by $NT^e$. Hence the primal problem (16) must be either finite or infeasible, from which it follows that the dual problem (17) must be either finite or unbounded. In the former case, by strong duality we have $f(y_0, Y, Z) = f^d(y_0, Y, Z)$. The latter case occurs if and only if $\sum_{i \in [I]} (s_i - b'_i y_0 - (C^y_i, Y) - (C^z_i, Z)) \psi_i > 0$ for some $\psi$ with $\sum_{i \in [I]} a_i \psi_i = 0$. Define an optimization problem as follows:

$$\begin{align*}
\min & \quad \eta \\
\text{s.t.} & \quad \sum_{i \in [I]} (s_i - b'_i y_0 - (C^y_i, Y) - (C^z_i, Z)) \psi_i \leq \eta \quad \forall \psi \in G^1 \\
& \quad \sum_{i \in [I]} (s_i - b'_i y_0 - (C^y_i, Y) - (C^z_i, Z)) \psi_i \leq 0 \quad \forall \psi \in G^2 \\
& \quad (y_0, Y, Z) \in Z_R.
\end{align*}$$

If $G^1 = \{ \psi | \sum_{i \in [I]} a_i \psi_i = g \}$ and $G^2 = \{ \psi | \sum_{i \in [I]} a_i \psi_i = 0 \}$, then the main problem (15) is equivalent to (18).

We incorporate the Benders decomposition as described in the following algorithm.

**Algorithm Benders Decomposition**

1. Initialize $G^1$ as a singleton containing an arbitrary extreme point of the polyhedron $\{ \psi | \sum_{i \in [I]} a_i \psi_i = g \}$ and initialize $G^2 = \emptyset$.
2. Solve the problem (18) and denote the optimal solution by $(\eta^*, y^*_0, Y^*, Z^*)$.
3. Solve problem (17) with $(y_0, Y, Z) = (y^*_0, Y^*, Z^*)$. If $f^d(y^*_0, Y^*, Z^*)$ is finite, then let $\psi^*$ denote an optimal extreme-point solution and update $G^1 = G^1 \cup \{ \psi^* \}$. Otherwise: let $\psi^*$ signify the direction, with $\sum_{i \in [I]} (s_i - b'_i y_0 - (C^y_i, Y) - (C^z_i, Z)) \psi_i > 0$ and $\sum_{i \in [I]} a_i \psi_i = 0$, and update $G^2 = G^2 \cup \{ \psi^* \}$.
4. If $\eta^* = f^d(y^*_0, Y^*, Z^*)$, then terminate the algorithm and return $(x^*, y^*_0, Y^*, Z^*)$ as the optimal solution; otherwise, return to Step 2.

It is easy to show that, in theory, the algorithm can be terminated within a finite number of iterations, since in Step 3, there is only a finite number of extreme points/ rays. On a practical level, it has been well recognized that this approach can help solve MILP with greater efficiency (Rahmanian et al. 2016).
5. Computational Study

In this section, we carry out computational studies to show that our proposed model is practically solvable and can yield an appealing solution under uncertainty. The program is coded in C++ and run on an Intel Core i5 PC with a 2.70-GHz CPU and 8 GB of RAM using CPLEX 12.5. Unless otherwise stated, the maximum CPU time is set to two hours.

To assess the solvability of our model and the quality of its solutions, we test instances with $N = 8$ retailers on a randomly generated graph with $T = 4$ planning periods. The transportation cost between any two nodes is proportional to the distance between them.

Taking the network as given, we calculate the minimum cost to visit all nodes based on the traditional traveling salesman problem. We denote this minimum cost as $TSP$. Since the initial inventory level of each retailer is set as zero, we let the budget for the first period be the $TSP$ such that all retailers can be served. For subsequent periods, we let the budget be $\kappa \times TSP$; in this setup, $\kappa < 1$ indicates that the supplier’s vehicle cannot visit all retailers in a single period. Our experiment considers two cases: $\kappa = 0.7$ and $\kappa = 0.8$. We put $\bar{D}_{nt} = 10 + 30\bar{\sigma}_{nt}$, where $\bar{\sigma}_{nt}$ follows the distribution of Beta(2,4) for all $n \in [N]$ and $t \in [T]$. In all periods, all retailers share a common inventory level requirement window $[0, \bar{\tau}]$; we vary $\bar{\tau}$ to capture the effects of different requirement windows. For simplicity, we choose $u(x) = \max\{-1, x\}$ as the utility function. We also conduct these computational studies for other distributions. The results are quite consistent across distributions, so we only report the case with a Beta distribution.

5.1. Comparison between our approaches and benchmark approach

As a means of further evaluating and understanding the benefit of our approach, we compare the quality of solutions from three approaches. Two of them are variants of our SVI method, and one is a benchmark approach based on minimizing the probability of violation.

5.1.1. Robust approach with mean absolute deviation (SVI-R) The first approach, which we dub the SVI-R model, is from our robust optimization using information of mean absolute
deviation (Section 4.2). When solving for the optimal solution, we use the information characterized as follows:

$$\mathcal{P} = \left\{ \begin{array}{l}
\mathbb{P}\left(d \leq \tilde{D}_{nt} \leq \bar{d}\right) = 1 \\
\mathbb{E}_{\mathbb{P}}\left(\tilde{D}_{nt}\right) = \mu \\
\mathbb{E}_{\mathbb{P}}\left(\left|\tilde{D}_{nt} - \mu\right|\right) \leq \sigma \\
\mathbb{E}_{\mathbb{P}}\left(\sum_{t \in [T]} \frac{\tilde{D}_{nt} - \mu}{\sigma}\right) \leq e^o, \forall n \in [N] \\
\mathbb{E}_{\mathbb{P}}\left(\sum_{u \in [N]} \frac{\tilde{D}_{nt} - \mu}{\sigma}\right) \leq e^*, \forall t \in [T]
\end{array} \right\}.$$ 

Here \(d = 10 + 30 \times 0 = 10, \bar{d} = 10 + 30 \times 1 = 40\), and the mean value is \(\mu = 10 + 30 \times \frac{2}{2+4} = 20\). The values of \(\sigma, e^o, e^*\) can similarly be obtained from the underlying distribution of \(\tilde{D}_{nt}\). Observe that the penultimate constraint is to incorporate the correlation effect among uncertain demand of all periods for each retailer; the last constraint is for the correlation among uncertain demand of all retailers for each period. We lift the ambiguity set \(\mathcal{P}\) to \(\mathcal{Q}\) as described in the equation (12), after which we solve the problem using ELDR as described in Proposition 5.

### 5.1.2. Stochastic approach (SVI-S)

In the second approach, called the SVI-S model, we have exact information on the distribution of \(\hat{D}\) (Section 4.1). Following the underlying distribution on \(\hat{D}\), we generate \(M_s\) samples of \(\hat{D}\) and use the sampling distribution as \(\mathbb{P}\) in Section 4.1. As expected, performance and computation time depend strongly on the number \(M_s\) of training samples. A large number of samples will describe the distribution better and lead to a good solution, but computation time will increase significantly because there are more constraints involved. We test the performance of our SVI-S model in two cases: \(M_s = 100\) and \(M_s = 500\).

Note that as the feature of distributional robustness has been reflected by the SVI-R model, here we do not incorporate the model based on Wasserstein distance (Section 4.3) to avoid the specification of \(\theta\), which might be abstract in the computational study. Indeed, the SVI-S is its special case with \(\theta = 0\).
5.1.3. Minimizing violation probability (MVP) Since we focus on the service level, a natural benchmark is to minimize the probability of an inventory requirement violation, that is, to minimize \( \frac{1}{NT} \sum_{n \in [N]} \sum_{t \in [T]} \mathbb{P}(v_{0,t}(\hat{x}_{nt}) > 0) \). Because this so-called MVP model is highly intractable, we can use the sampling average approximation to derive only static decisions, not adjustable ones. More specifically, we generate \( M_e \) sample paths for uncertain demands as \( D^{(m)} \), \( m \in [M_e] \). Then the problem of MVP can be approximated as

\[
\min \frac{1}{M_eNT} \sum_{m \in [M_e]} \sum_{t \in [T]} \sum_{n \in [N]} \mathbb{f}_{nt}^{(m)} \\
\text{s.t.} \quad \mathbb{f}_{nt}^{(m)} \geq \frac{x_{nt}^{(m)} - \bar{r}}{M}, \quad \forall m \in [M_e], n \in [N], t \in [T], \\
\mathbb{f}_{nt}^{(m)} \geq -\frac{x_{nt}^{(m)}}{M}, \quad \forall m \in [M_e], n \in [N], t \in [T], \\
g_n^{t} \leq M g_n^{t}, \quad \forall n \in [N], t \in [T], \\
g_n^{t} \geq 0, \quad \forall n \in [N], t \in [T], \\
\mathbb{f}_{nt}^{(m)} \in \{0, 1\}, \quad \forall m \in [M_e], n \in [N], t \in [T], \\
(y_n^{t}, y_n^{t}, z_{ij}^{t}, n \in [N], t \in [T], (i, j) \in A) \in Z_R.
\]

Here \( Z_R \) is as defined in the equation (9). \( M \) is a sufficiently large scalar for the use of big-M method. If the sample size \( M_e \) is large enough, then this formulation closely approximates the MVP model; however, a large sample size will also result in long computation times. In this experiment, we set \( M_e = 100 \).

5.1.4. Comparison For each approach, we first derive the optimal solution and then generate \( M_o = 10,000 \) sample paths of demand \( \hat{D} \) to test the quality of solutions. We simplify the performance analysis by aggregating the uncertain violation at all nodes and all periods into a single uncertain violation \( \bar{v} \). In particular, \( \bar{v} \) is the violation at node \( n \) in period \( t \) with equal likelihood for all \( n \in [N] \) and \( t \in [T] \); formally, \( \mathbb{P}(\bar{v} = \max\{\hat{x}_{nt} - \bar{r}, -\hat{x}_{nt}, 0\}) = 1/NT \). So for each solution, we have \( M_o \times N \times T \) instances of \( \bar{v} \). Using this sample set, we report each model’s performance as measured by five criteria: a probability of there being a violation, \( \mathbb{P}(\bar{v} > 0) \); the violation’s expected value, \( \mathbb{E}(\bar{v}) \), and standard deviation, \( \text{STD}(\bar{v}) \); and the Value at Risk (VaR) at the 95th and 99th
percentiles, denoted \( \text{VaR}_{0.95}(\hat{v}) \) and \( \text{VaR}_{0.99}(\hat{v}) \). For each criterion, a smaller value corresponds to better performance. We also report how long, in seconds, it takes to obtain the solution as “CPU time”.

The performance of our SVI-R model is presented in Table 2. As the transportation budget increases (higher \( \kappa \)), the supplier has more flexibility in choosing routes, which reduces the risk of violation. If the transportation budget is held constant then, when \( \tau \) increases, the ending inventory requirement is less restrictive; once again, violations are thus less likely. The SVI-R model entails reasonable computation time: fewer than 100 seconds are needed to solve for an eight-node network over four periods.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
\( \kappa \) & \( \tau \) & \( \mathbb{P}(\hat{v} > 0) \) & \( \mathbb{E}(\hat{v}) \) & \( \text{STD}(\hat{v}) \) & \( \text{VaR}_{0.95}(\hat{v}) \) & \( \text{VaR}_{0.99}(\hat{v}) \) & \text{CPU time} \\
\hline
0.7 & 27 & 13.13\% & 0.53 & 1.92 & 3.89 & 10.44 & 56 \\
    & 30 & 9.11\% & 0.42 & 1.85 & 2.65 & 10.44 & 92 \\
    & 35 & 5.53\% & 0.27 & 1.48 & 0.64 & 8.49 & 86 \\
0.8 & 27 & 11.23\% & 0.46 & 1.80 & 3.38 & 9.90 & 65 \\
    & 30 & 7.78\% & 0.36 & 1.71 & 1.90 & 9.75 & 62 \\
    & 35 & 4.72\% & 0.23 & 1.36 & 0.00 & 7.79 & 62 \\
\hline
\end{tabular}
\caption{Performance of the SVI-R model.}
\end{table}

We summarize the performance of other models in Table 3. For purposes of comparison, we normalize the performance of these models by that of our SVI-R model. Hence, a value greater than 1 in Table 3 indicates that the risk is higher than under the SVI-R model. We observe that, for \( M_s = 100 \), the computation times for solving the SVI-S model and SVI-R models are similar. Yet performance is poor, as reflected by most values exceeding unity. When the sample size \( M_s \) increases to 500, computation time increases significantly, but performance is only slightly better than under SVI-S with \( M_s = 100 \) and is still worse than the SVI-R model’s performance. Recall that performance of the MVP model is also affected by the training sample size. But even for 100 samples, this approach cannot deliver an optimal solution within the two-hour time limit. The MVP performance is the worst among all these models, in terms not only of the violation probability but also of the violation magnitude. We therefore conclude that our proposed SVI is a good measure for capturing the risk of a requirement violation, and we know that it can be computed efficiently.
For the case $\kappa = 0.8$ and $\bar{\tau} = 35$, since $\text{VaR}_{0.95}(\bar{v}) = 0$, we can only report “n/a” (i.e., not applicable) in the table. Indeed, all of the $\text{VaR}_{0.95}(\bar{v})$ values for models SVI-S and MVP are positive.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\kappa$</th>
<th>$\bar{\tau}$</th>
<th>$\mathbb{P}(\bar{v} &gt; 0)$</th>
<th>$\mathbb{E}(\bar{v})$</th>
<th>$\text{STD}(\bar{v})$</th>
<th>$\text{VaR}_{0.95}(\bar{v})$</th>
<th>$\text{VaR}_{0.90}(\bar{v})$</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>SVI-S</td>
<td>0.7</td>
<td>27</td>
<td>1.32</td>
<td>1.30</td>
<td>1.10</td>
<td>1.29</td>
<td>1.02</td>
<td>1.59</td>
</tr>
<tr>
<td></td>
<td></td>
<td>30</td>
<td>1.72</td>
<td>1.45</td>
<td>1.08</td>
<td>1.67</td>
<td>0.99</td>
<td>0.91</td>
</tr>
<tr>
<td></td>
<td></td>
<td>35</td>
<td>1.72</td>
<td>1.22</td>
<td>1.00</td>
<td>3.56</td>
<td>0.93</td>
<td>1.05</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>27</td>
<td>1.39</td>
<td>1.35</td>
<td>1.12</td>
<td>1.36</td>
<td>1.04</td>
<td>1.65</td>
</tr>
<tr>
<td></td>
<td></td>
<td>30</td>
<td>1.85</td>
<td>1.58</td>
<td>1.15</td>
<td>2.23</td>
<td>1.06</td>
<td>1.45</td>
</tr>
<tr>
<td></td>
<td></td>
<td>35</td>
<td>2.01</td>
<td>1.43</td>
<td>1.09</td>
<td>n/a</td>
<td>1.02</td>
<td>1.55</td>
</tr>
<tr>
<td>(M_s = 100)</td>
<td>0.7</td>
<td>27</td>
<td>1.13</td>
<td>1.06</td>
<td>0.98</td>
<td>1.07</td>
<td>0.93</td>
<td>22.30</td>
</tr>
<tr>
<td></td>
<td></td>
<td>30</td>
<td>1.27</td>
<td>0.90</td>
<td>0.81</td>
<td>1.09</td>
<td>0.76</td>
<td>12.43</td>
</tr>
<tr>
<td></td>
<td></td>
<td>35</td>
<td>1.24</td>
<td>0.70</td>
<td>0.68</td>
<td>1.45</td>
<td>0.62</td>
<td>11.28</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>27</td>
<td>1.14</td>
<td>1.07</td>
<td>0.99</td>
<td>1.10</td>
<td>0.96</td>
<td>7.52</td>
</tr>
<tr>
<td></td>
<td></td>
<td>30</td>
<td>1.27</td>
<td>0.92</td>
<td>0.84</td>
<td>1.29</td>
<td>0.78</td>
<td>5.37</td>
</tr>
<tr>
<td></td>
<td></td>
<td>35</td>
<td>1.25</td>
<td>0.74</td>
<td>0.71</td>
<td>n/a</td>
<td>0.64</td>
<td>3.95</td>
</tr>
<tr>
<td>SVI-S</td>
<td>0.7</td>
<td>27</td>
<td>1.58</td>
<td>2.23</td>
<td>1.70</td>
<td>2.14</td>
<td>1.55</td>
<td>128.57</td>
</tr>
<tr>
<td>(M_s = 500)</td>
<td>30</td>
<td>1.81</td>
<td>2.14</td>
<td>1.57</td>
<td>2.53</td>
<td>1.43</td>
<td>78.26</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>35</td>
<td>1.95</td>
<td>2.00</td>
<td>1.51</td>
<td>6.11</td>
<td>1.41</td>
<td>83.72</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>27</td>
<td>1.70</td>
<td>2.28</td>
<td>1.71</td>
<td>2.27</td>
<td>1.55</td>
<td>110.77</td>
</tr>
<tr>
<td></td>
<td></td>
<td>30</td>
<td>1.96</td>
<td>2.25</td>
<td>1.60</td>
<td>3.24</td>
<td>1.45</td>
<td>116.13</td>
</tr>
<tr>
<td></td>
<td></td>
<td>35</td>
<td>2.15</td>
<td>2.17</td>
<td>1.58</td>
<td>n/a</td>
<td>1.48</td>
<td>116.13</td>
</tr>
</tbody>
</table>

Table 3  Comparison of performance for different benchmarks

5.2. Robustness

We shall now explain the need to incorporate distributional ambiguity. From a practical standpoint, given a set of historical data, we can extract certain information and then derive the corresponding optimal solution. For the SVI-R model, we use information of mean absolute deviation; for both the SVI-S and MVP models, we use the sampling distribution. Nevertheless, the samples may deviate from the true distribution. We test the performance of different models under this scenario.

In particular, we assume that the true distribution of demand is as given in Section 5.1: $\tilde{D}_{nt} = 10 + 30\tilde{\sigma}_{nt}$, where $\tilde{\sigma}_{nt}$ follows the Beta(2,4) distribution. For this test, however, the samples are drawn from the “wrong” distribution: $\tilde{D}_{nt}^w = 10 + 30\tilde{\sigma}_{nt}^w$, where $\tilde{\sigma}_{nt}^w$ follows the Beta(4,8) distribution. Note that the true and wrong distributions do not deviate too much in the sense that $\tilde{D}_{nt}$ and $\tilde{D}_{nt}^w$ have the same mean and bound. We derive the optimal solution for all models based on the samples drawn from the wrong distribution, $\tilde{D}_{nt}^w$, and then test the solution using samples drawn from the true distribution, $\tilde{D}_{nt}$. Because the mean absolute deviation information we extract from
both $\bar{D}_{nt}^w$ and $\bar{D}_{nt}$ is the same, the SVI-R model’s solution here is as the same as in Section 5.1. And since the testing samples are both from $\bar{D}_{nt}$, it follows that the SVI-R model’s performance is identical to that reported in Table 2. In Table 4 we summarize the other models’ performances, which are again normalized by that of the SVI-R model.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\kappa$</th>
<th>$\tau$</th>
<th>$\mathbb{P}(\bar{v} &gt; 0)$</th>
<th>$\mathbb{E}(\bar{v})$</th>
<th>$\text{STD}(\bar{v})$</th>
<th>VaR$_{0.95}(\bar{v})$</th>
<th>VaR$_{0.99}(\bar{v})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SVI-S ($M_s = 100$)</td>
<td>0.7</td>
<td>27</td>
<td>1.75</td>
<td>2.13</td>
<td>1.58</td>
<td>1.86</td>
<td>1.43</td>
</tr>
<tr>
<td>&amp;</td>
<td>30</td>
<td>1.99</td>
<td>1.86</td>
<td>1.30</td>
<td>2.04</td>
<td>1.17</td>
<td></td>
</tr>
<tr>
<td>&amp;</td>
<td>35</td>
<td>2.26</td>
<td>1.74</td>
<td>1.22</td>
<td>5.34</td>
<td>1.10</td>
<td></td>
</tr>
<tr>
<td>&amp;</td>
<td>0.8</td>
<td>27</td>
<td>1.94</td>
<td>2.20</td>
<td>1.53</td>
<td>2.01</td>
<td>1.36</td>
</tr>
<tr>
<td>&amp;</td>
<td>30</td>
<td>2.27</td>
<td>2.14</td>
<td>1.42</td>
<td>2.83</td>
<td>1.26</td>
<td></td>
</tr>
<tr>
<td>&amp;</td>
<td>35</td>
<td>2.68</td>
<td>2.35</td>
<td>1.55</td>
<td>NA</td>
<td>1.41</td>
<td></td>
</tr>
<tr>
<td>SVI-S ($M_s = 500$)</td>
<td>0.7</td>
<td>27</td>
<td>1.29</td>
<td>1.21</td>
<td>1.04</td>
<td>1.19</td>
<td>0.97</td>
</tr>
<tr>
<td>&amp;</td>
<td>30</td>
<td>1.50</td>
<td>1.12</td>
<td>0.90</td>
<td>1.32</td>
<td>0.83</td>
<td></td>
</tr>
<tr>
<td>&amp;</td>
<td>35</td>
<td>1.62</td>
<td>1.00</td>
<td>0.86</td>
<td>2.73</td>
<td>0.80</td>
<td></td>
</tr>
<tr>
<td>&amp;</td>
<td>0.8</td>
<td>27</td>
<td>1.37</td>
<td>1.24</td>
<td>1.04</td>
<td>1.25</td>
<td>0.98</td>
</tr>
<tr>
<td>&amp;</td>
<td>30</td>
<td>1.61</td>
<td>1.17</td>
<td>0.93</td>
<td>1.68</td>
<td>0.85</td>
<td></td>
</tr>
<tr>
<td>&amp;</td>
<td>35</td>
<td>1.77</td>
<td>1.13</td>
<td>0.90</td>
<td>NA</td>
<td>0.85</td>
<td></td>
</tr>
<tr>
<td>MVP</td>
<td>0.7</td>
<td>27</td>
<td>1.66</td>
<td>2.23</td>
<td>1.68</td>
<td>2.09</td>
<td>1.51</td>
</tr>
<tr>
<td>&amp;</td>
<td>30</td>
<td>1.92</td>
<td>2.07</td>
<td>1.48</td>
<td>2.38</td>
<td>1.32</td>
<td></td>
</tr>
<tr>
<td>&amp;</td>
<td>35</td>
<td>2.10</td>
<td>2.00</td>
<td>1.53</td>
<td>5.58</td>
<td>1.40</td>
<td></td>
</tr>
<tr>
<td>&amp;</td>
<td>0.8</td>
<td>27</td>
<td>1.84</td>
<td>2.28</td>
<td>1.64</td>
<td>2.20</td>
<td>1.46</td>
</tr>
<tr>
<td>&amp;</td>
<td>30</td>
<td>2.07</td>
<td>2.17</td>
<td>1.49</td>
<td>3.06</td>
<td>1.35</td>
<td></td>
</tr>
<tr>
<td>&amp;</td>
<td>35</td>
<td>2.34</td>
<td>2.13</td>
<td>1.49</td>
<td>NA</td>
<td>1.39</td>
<td></td>
</tr>
</tbody>
</table>

Table 4  Performance of different models when given the wrong distribution information

Since the performance of models SVI-S and MVP are highly dependent on the training samples, the solutions these models derive from the wrong distribution can easily lead to performance far inferior to that of SVI-R. This claim is confirmed by the Table 4 values exceeding those reported in Table 3. So in comparison with the other models evaluated here, adopting our SVI-R approach will better immunize the decision maker against the effects of data ambiguity.

6. Conclusions

We take a distributionally robust optimization approach to exploring the stochastic inventory routing problem. After proposing a new decision criterion for evaluating the associated risk, we develop methods geared to solve for the optimal affine policies. Our model can easily be extended to incorporate various operational features (e.g., production), and its computational efficiency can be
improved by employing some well-developed heuristics (e.g., restricting to only a subset of routes).

We undertake computational studies to illustrate the strength of our method.
Appendix A: Proof of Theorem 1

We first prove the "if" direction. If $\mu$ is a convex risk measure, we prove that the $\rho_{\mathcal{X},T}(\tilde{x})$ defined in the equation (2) is an SVI by showing its satisfaction of the properties in Definition 1.

1. Monotonicity: If $v_{\mathcal{X},T}(\tilde{x}_1) \geq v_{\mathcal{X},T}(\tilde{y}_1), \forall i \in [I]$, then by the monotonicity of $\mu$, we have

$$\mu\left(\frac{v_{\mathcal{X},T}(\tilde{x}_i)}{\alpha_i}\right) \geq \mu\left(\frac{v_{\mathcal{X},T}(\tilde{y}_i)}{\alpha_i}\right), \quad \forall \alpha_i > 0.$$  

Hence, $\rho_{\mathcal{X},T}(\tilde{x}) \geq \rho_{\mathcal{X},T}(\tilde{y})$.

2(a). Attainment content: If $v_{\mathcal{X},T}(\tilde{x}_i) \leq 0, \forall i \in [I]$, we have for any $\alpha_i > 0$,

$$\mu\left(\frac{v_{\mathcal{X},T}(\tilde{x}_i)}{\alpha_i}\right) \leq \mu(0) = 0,$$

Hence, $\rho_{\mathcal{X},T}(\tilde{x}) = 0$.

2(b). Starvation aversion: If $\exists i \in [I]$ such that $v_{\mathcal{X},T}(\tilde{x}_i) > 0$, then for any $\alpha_i > 0$, we have that $v_{\mathcal{X},T}(\tilde{x}_i)/\alpha_i > 0$. Hence, $\exists \epsilon > 0$ such that $v_{\mathcal{X},T}(\tilde{x}_i)/\alpha_i \geq \epsilon$ and

$$\mu\left(\frac{v_{\mathcal{X},T}(\tilde{x}_i)}{\alpha_i}\right) \geq \mu(\epsilon) = \mu(0) + \epsilon = \epsilon > 0,$$

Therefore, $\rho_{\mathcal{X},T}(\tilde{x}) = \infty$.

3. Convexity: Given any $\tilde{x} \in \mathcal{X}$, we define a set of $\alpha$ as

$$S(\tilde{x}) = \left\{ \alpha \left| \mu\left(\frac{v_{\mathcal{X},T}(\tilde{x}_i)}{\alpha_i}\right) \leq 0, \alpha_i > 0, \forall i \in [I] \right. \right\}.$$

We consider any $\alpha^x \in S(\tilde{x})$, $\alpha^y \in S(\tilde{y})$, and define $\alpha^\lambda = \lambda \alpha^x + (1 - \lambda) \alpha^y$ for any $\lambda \in [0,1]$. Based on the definition of $\rho_{\mathcal{X},T}(\tilde{x}) = \max\{\tilde{x} - \mathcal{X}, \tilde{x} - \tilde{x}\}$, the function $v_{\mathcal{X},T}(\cdot)$ is convex. Hence,

$$\mu\left(\frac{v_{\mathcal{X},T}(\lambda \tilde{x}_i + (1 - \lambda) \tilde{y}_i)}{\alpha_i^\lambda}\right) \leq \mu\left(\frac{\lambda v_{\mathcal{X},T}(\tilde{x}_i) + (1 - \lambda) v_{\mathcal{X},T}(\tilde{y}_i)}{\alpha_i^\lambda}\right) = \mu\left(\frac{\lambda v_{\mathcal{X},T}(\tilde{x}_i)}{\alpha_i^\lambda} + \frac{(1 - \lambda) v_{\mathcal{X},T}(\tilde{y}_i)}{\alpha_i^\lambda}\right) \leq \mu\left(\frac{v_{\mathcal{X},T}(\tilde{x}_i)}{\alpha_i^y}\right) + (1 - \lambda) \mu\left(\frac{v_{\mathcal{X},T}(\tilde{y}_i)}{\alpha_i^y}\right) \leq 0,$$
where \( o = \lambda \alpha_i^x / \alpha_i^y \in [0, 1] \). The first inequality holds because of the convexity of \( v_{\xi, \tau_i}(\cdot) \) and the monotonicity of \( \mu(\cdot) \), the second inequality is due to the convexity of \( \mu \) and the last inequality holds based on the definition of \( S(\bar{x}) \) and \( S(\bar{y}) \). Hence, for any \( \lambda \in [0, 1], \alpha^x \in S(\bar{x}), \alpha^y \in S(\bar{y}), \alpha^t \in S(\lambda \bar{x} + (1 - \lambda) \bar{y}) \) and we get

\[
\lambda \rho_{\xi, \tau}(\bar{x}) + (1 - \lambda) \rho_{\xi, \tau}(\bar{y}) = \lambda \inf \left\{ \sum_{i=1}^{I} \frac{\lambda \alpha_i^x}{\alpha_i} + (1 - \lambda) \frac{\alpha_i^y}{\alpha_i} \right\} \\
= \inf \left\{ \sum_{i=1}^{I} \frac{\lambda \alpha_i^x}{\alpha_i} + (1 - \lambda) \frac{\alpha_i^y}{\alpha_i} \right\} \\
\geq \inf \left\{ \sum_{i=1}^{I} \alpha_i \right\} \\
= \lambda \rho_{\xi, \tau}(\lambda \bar{x} + (1 - \lambda) \bar{y})
\]

4. Positive homogeneity: For all \( \lambda > 0 \),

\[
\rho_{\xi, \tau}(\lambda \bar{x}) = \inf \left\{ \sum_{i=1}^{I} \alpha_i \right\} \\
\leq \inf \left\{ \sum_{i=1}^{I} \alpha_i \right\} \\
= \lambda \rho_{\xi, \tau}(\bar{x})
\]

5. Dimension-wise additivity: It is obvious due to representation (2).

6. Order invariance: It is obvious due to representation (2).

7. Left continuity: Denote \( \alpha^* = \rho_{-\infty, 0}(\bar{x}), \alpha_i^* = \rho_{-\infty, 0}(\bar{x}, e_i), i \in [I] \). By dimension-wise additivity, we have \( \alpha^* = \sum_{i \in [I]} \alpha_i^* \). We next prove the left continuity in two cases.

Case 1: If \( \alpha_i^* \in \mathbb{R}_+, \forall i \in [I] \), then \( \alpha^* \in \mathbb{R}_+ \). We need to show that \( \forall \epsilon > 0 \), there exists \( \delta > 0 \) such that \( \forall \delta \in (0, \delta), |\rho_{-\infty, 0}(\bar{x} - \delta 1) - \rho_{-\infty, 0}(\bar{x})| \leq \epsilon \). Since \( v_{-\infty, 0}(x - \delta 1) = x - \delta 1 \), by Monotonicity, we have \( \rho_{-\infty, 0}(\bar{x} - \delta 1) \leq \rho_{-\infty, 0}(\bar{x}) \). Hence, we only need to prove \( \rho_{-\infty, 0}(\bar{x} - \delta 1) \geq \rho_{-\infty, 0}(\bar{x}) - \epsilon = \alpha^* - \epsilon \). The case of \( \epsilon \geq \alpha^* \) is trivial. For \( \epsilon \in (0, \alpha^* ) \), we define \( \mathcal{I} = \{ i \in [I] | \alpha_i^* > 0 \} \), and \( \bar{\mathcal{I}} = [I]/\mathcal{I} = \{ i \in [I] | \alpha_i^* = 0 \} \). Then we have \( \epsilon < \alpha^* = \sum_{i \in \mathcal{I}} \alpha_i^* \). Hence, we can find a vector \( \{ \epsilon_i \}_{i \in \mathcal{I}} \) such that \( \sum_{i \in \mathcal{I}} \epsilon_i \epsilon \in (0, \alpha^* ) \), \( \forall i \in \mathcal{I} \). Choose \( \bar{\delta} = \min_{i \in \mathcal{I}} (\alpha_i^* - \epsilon_i) \mu(\frac{v_{-\infty, 0}(\bar{x})}{\alpha_i^* - \epsilon_i}) \). Since \( \alpha_i^* = \rho_{-\infty, 0}(\bar{x}, e_i) = \inf \{ \alpha > 0 | \mu(v_{-\infty, 0}(\bar{x})/\alpha) \leq 0 \}, \forall i \in \mathcal{I}, \bar{\delta} \) is strictly positive. Considering \( \forall \delta \in (0, \bar{\delta}) \) and \( i \in \mathcal{I} \), we have

\[
\mu \left( \frac{v_{-\infty, 0}(\bar{x}) - \delta}{\alpha_i^* - \epsilon_i} \right) = \mu \left( \frac{v_{-\infty, 0}(\bar{x}) - \delta}{\alpha_i^* - \epsilon_i} \right) > \mu \left( \frac{v_{-\infty, 0}(\bar{x})}{\alpha_i^* - \epsilon_i} \right)
\]
1 in Chen et al. (2015) and the representation (2), we know \( \rho_{-\infty,0}(\bar{x}_i - \delta) e_i \geq \alpha^*_i - \epsilon \). For \( i \in \mathbb{I} \), obviously we have \( \rho_{-\infty,0}(\bar{x}_i - \delta) e_i = 0 \). Therefore,

\[
\rho_{-\infty,0}(\bar{x} - \delta 1) = \sum_{i \in \mathbb{I}} \rho_{-\infty,0}(\bar{x}_i - \delta) e_i \geq \sum_{i \in \mathbb{I}} (\alpha^*_i - \epsilon_i) = \alpha^* - \epsilon.
\]

Case 2: If \( \exists i \in [I] \) such that \( \alpha^*_i = \infty \), then \( \alpha^* = \infty \). Suppose \( \lim_{a \downarrow \theta} \rho_{-\infty,0}(\bar{x} - \delta) e_i \) is not \( \infty \), i.e. \( \lim_{a \downarrow \theta} \rho_{-\infty,0}(\bar{x} - \delta) e_i = \alpha^* \in \mathbb{R}_+ \). It implies that \( \forall \delta > 0, \exists \delta > 0 \) such that \( \forall \delta \in (0, \delta) \), we have \( \rho_{-\infty,0}(\bar{x} - \delta 1) \in [\alpha^* - \epsilon, \alpha^* + \epsilon] \).

Since \( \alpha^*_i = \rho_{-\infty,0}(\bar{x}_i e_i) = \infty \), we have \( \mu\left(\frac{v_{-\infty,0}(\bar{x}_i)}{\alpha}ight) > 0, \forall \alpha > 0 \). Hence, we are able to choose

\[
\delta = \min\left\{ \frac{\delta}{2}, \frac{\mu\left(\frac{v_{-\infty,0}(\bar{x}_i)}{\alpha}ight)}{\eta} \right\} \in (0, \delta),
\]

where \( \alpha = \alpha^* + 2\epsilon \), and \( \eta \) is any real number in \( (0, \mu(v_{-\infty,0}(\bar{x}_i)/\alpha)) \). As \( \delta \in (0, \delta) \), we still have \( \rho_{-\infty,0}(\bar{x}_i - \delta) e_i \leq \rho_{-\infty,0}(\bar{x} - \delta 1) \leq \alpha^* + \epsilon \). So by Lemma 1 in Chen et al. (2015), \( \mu\left(\frac{v_{-\infty,0}(\bar{x}_i - \delta)}{\alpha} \right) \leq 0 \) since \( \alpha > \alpha^* + \epsilon \).

We can also get that

\[
\mu\left(\frac{v_{-\infty,0}(\bar{x}_i - \delta)}{\alpha} \right) = \mu\left(\frac{v_{-\infty,0}(\bar{x}_i)}{\alpha} \right) - \frac{\delta}{\alpha} \geq \mu\left(\frac{v_{-\infty,0}(\bar{x}_i)}{\alpha} \right) - \left( \mu\left(\frac{v_{-\infty,0}(\bar{x}_i)}{\alpha} \right) - \eta \right) = \eta > 0,
\]

which contradicts \( \mu\left(\frac{v_{-\infty,0}(\bar{x}_i - \delta)}{\alpha} \right) \leq 0 \). So the assumption is false, \( \lim_{a \downarrow \theta} \rho_{-\infty,0}(\bar{x} - \delta 1) = \infty \).

We next prove the “only if” direction. First, we need to show function \( \mu \) defined by the equation (3) is a convex risk measure. Notice that \( \forall x \in \mathbb{R}^l, v_{-\infty,0}(x) = \max\{ -\infty - x, x - 0 \} = x \).

1. Monotonicity: If \( \bar{x} \geq \bar{y} \), then \( v_{-\infty,0}(\bar{x} - a e_1) \geq v_{-\infty,0}(\bar{y} - a e_1) \) for any \( a \in \mathbb{R} \). Hence, based on the monotonicity property, we have \( \rho_{-\infty,0}(\bar{x} - a e_1) \geq \rho_{-\infty,0}(\bar{y} - a e_1) \). According to the definition of \( \mu(\cdot) \), we get \( \mu(\bar{x}) \geq \mu(\bar{y}) \).

2. Cash invariance: Notice that \( \forall w \in \mathbb{R} \),

\[
\mu(\bar{x} + w) = \min\{ a \mid \rho_{-\infty,0}(\bar{x} + w - a e_1) \leq 1 \} = \min\{ a + w \mid \rho_{-\infty,0}(\bar{x} - a e_1) \leq 1 \} = \mu(\bar{x}) + w
\]

3. Convexity:

\[
\rho_{-\infty,0}(\lambda \bar{x} + (1 - \lambda)\bar{y} - \lambda \mu(\bar{x}) - (1 - \lambda) \mu(\bar{y})) e_1 \leq \lambda \rho_{-\infty,0}(\bar{x} - \mu(\bar{x})) e_1 + (1 - \lambda) \rho_{-\infty,0}(\bar{y} - \mu(\bar{y})) e_1 \leq \lambda + (1 - \lambda) = 1,
\]
where the first inequality follows from the convexity of $\rho_{\Sigma,T}(\hat{x})$, and the second inequality holds based on the definition of $\mu$. Hence we have

$$\mu(\lambda \hat{x} + (1 - \lambda)\hat{y}) = \min \left\{ a \mid \rho_{-\infty,0}((\lambda \hat{x} + (1 - \lambda)\hat{y} - a) e_1) \leq 1 \right\} \leq \lambda \mu(\hat{x}) + (1 - \lambda)\mu(\hat{y}).$$

4. Normalization: $\rho_{-\infty,0}(-a e_1)$ is 0 if $a \geq 0$, and it is $\infty$ if $a < 0$. Therefore

$$\mu(0) = \min \left\{ a \mid \rho_{-\infty,0}(-a e_1) \leq 1 \right\} = 0.$$

Finally we need to show for any SVI defined as (2), the equation (3) holds. With Left continuity of SVI, the minimum in (3) is achievable. Therefore,

$$\min \left\{ a \mid \rho_{-\infty,0}((\hat{x} - a) e_1) \leq 1 \right\} = \min \left\{ a \left| \sum_{j \in [I]} \alpha_j \leq 1, \mu \left( \frac{v_{-\infty,0}(\hat{x} - a)}{\alpha_j} \right) \leq 0, j = 2, \ldots, I, \alpha_i > 0, i \in [I] \right\}$$

$$= \min \left\{ a \left| \alpha \leq 1, \mu \left( \frac{v_{-\infty,0}(\hat{x} - a)}{\alpha} \right) \leq 0, \alpha > 0 \right\}$$

$$= \min \left\{ a \mu \left( v_{-\infty,0}(\hat{x} - a) \right) \leq 0 \right\}$$

$$= \min \left\{ a \mu (\hat{x} - a) \leq 0 \right\}$$

$$= \min \left\{ a \mu (\hat{x}) \leq a \right\}$$

$$= \mu(\hat{x}),$$

where the second equality is due to $\mu(v_{-\infty,0}(0))/\alpha_j \leq 0, \forall \alpha_j > 0$, the fourth equality holds for the definition of function $v$ and the fifth equality holds for the cash invariance of $\mu$. In particular, the third equality is due to the equivalence between condition A) $\mu(\hat{w}) \leq 0$ and condition B) $\exists \alpha \in (0,1]$ such that $\mu(\hat{w}/\alpha) \leq 0$, which we prove as follows. The direction of “A$\Rightarrow$B” is trivial. To see the direction of “B$\Rightarrow$A”, we observe that for any such $\alpha \in (0,1)$,

$$\mu(\hat{w}) = \mu \left( \alpha \frac{\hat{w}}{\alpha} + (1 - \alpha)0 \right) \leq \alpha \mu \left( \frac{\hat{w}}{\alpha} \right) + (1 - \alpha)\mu(0) = \alpha \mu \left( \frac{\hat{w}}{\alpha} \right) \leq 0,$$

where inequalities are due to the convexity of $\mu$ and the condition B), respectively. Q.E.D.

**Appendix B: Proof of Proposition 1**

The shortfall risk measure without $\sup$ has been shown to be a convex risk measure (Föllmer and Schied 2002). Incorporating distributional ambiguity, it is straightforward to show our shortfall risk measure to be convex risk measure as well.
By the definition of shortfall risk measure (4), the SVI defined by (2) can be formulated as

\[
\rho_T^{\mathbb{P}}(\bar{x}) = \inf \left\{ \sum_{i=1}^{I} \alpha_i \left( \frac{u_{x_i} - (\bar{x}_i)}{\alpha_i} \right) \leq 0, \alpha_i > 0, \forall i \in [I] \right\}
\]

= \inf \left\{ \sum_{i=1}^{I} \inf_{\eta \in \mathbb{R}} \left\{ \sup_{P \in \mathbb{P}} \left[ u \left( \frac{u_{x_i} - (\bar{x}_i)}{\alpha_i} \right) - \eta \right] \leq 0, \alpha_i > 0, \forall i \in [I] \right\} \right\}

= \inf \left\{ \sum_{i=1}^{I} \sup_{P \in \mathbb{P}} \left[ u \left( \frac{u_{x_i} - (\bar{x}_i)}{\alpha_i} \right) \right] \leq 0, \alpha_i > 0, \forall i \in [I] \right\}.

The last equation holds due to the equivalence between condition A) \( \exists \eta \leq 0 \) such that

\[ \sup_{P \in \mathbb{P}} \left[ u \left( \frac{u_{x_i} - (\bar{x}_i)}{\alpha_i} \right) - \eta \right] \leq 0 \]

and condition B) \( \sup_{P \in \mathbb{P}} \left[ u \left( \frac{u_{x_i} - (\bar{x}_i)}{\alpha_i} \right) \right] \leq 0 \). “B\( \Rightarrow \)A” is trivial by setting \( \eta = 0 \).

For the direction “A\( \Rightarrow \)B”, we see that \( u \left( \frac{u_{x_i} - (\bar{x}_i)}{\alpha_i} \right) \leq u \left( \frac{u_{x_i} - (\bar{x}_i)}{\alpha_i} \right) - \eta \) for \( \eta \leq 0 \) because \( u \) is an increasing function. So \( \sup_{P \in \mathbb{P}} \left[ u \left( \frac{u_{x_i} - (\bar{x}_i)}{\alpha_i} \right) \right] \leq \sup_{P \in \mathbb{P}} \left[ u \left( \frac{u_{x_i} - (\bar{x}_i)}{\alpha_i} \right) - \eta \right] \leq 0. \)

Q.E.D.

Appendix C: Proof of Proposition 2

It is straightforward. Q.E.D.

Appendix D: Proof of Proposition 3

For any node \( n \in [N] \) and period \( t \in [T] \), consider

\[
\sup_{P \in \mathbb{P}} \left( \max_{k \in [K]} \left\{ a_k \max \left\{ x_{n0}^t - x_{n0}^t - (X_{n}^t, \bar{D}), x_{n0}^t + (X_{n}^t, \bar{D}) - \tau_{n}^t \right\} + b_k \alpha_n \right\} \right)
\]

as a primal optimization problem where the decision variable is \( P \). This is a semi-infinite linear programming problem with infinite number of decision variables and finite constraints. The strong duality holds (Isii 1962,
Wiesemann et al. 2014, Bertsimas et al. 2018) and we can write its dual form. Then

\[
\begin{align*}
\sup_{\mathbf{D}} & \quad \mathbb{E}_F \left( \max_{k \in [K]} \left\{ a_k \max \left\{ \sum_{i=1}^n \left( x_i^t - x_i^t - \langle X_i^t, D \rangle, x_i^t + \langle X_i^t, \tilde{D} \rangle - \tau_i^t + b_k \alpha_i^t \right) \right\} \right\} \\
& = \sup \mathbb{E}_F \left( \max_{k \in [K]} \left\{ a_k \max \left\{ \sum_{i=1}^n \left( x_i^t - x_i^t - \langle X_i^t, D \rangle, x_i^t + \langle X_i^t, \tilde{D} \rangle - \tau_i^t + b_k \alpha_i^t \right) \right\} \right\} \\
\text{s.t.} & \quad P \left( \mathbf{D} \leq \tilde{D} \leq \mathbf{D} \right) = 1 \\
& \quad \mathbb{E}_F \left( \tilde{D} - \mathbf{D} \right) = \Xi \\
& \quad \mathbb{E}_F \left[ f_i \left( \sum_{(i,r) \in S_h} \frac{\tilde{D}_{ir} - \Xi_{ir}}{\Sigma_{ir}} \right) \right] \leq \epsilon_{ih}, \quad \forall l \in [L], h \in [H] \\
& = \min s_{n0}^t + \langle S_n^t, \Xi \rangle + \langle T_n^t, \Sigma \rangle + \sum_{l \in [L]} \sum_{h \in [H]} \epsilon_{ih} r_{n0}^t \\
\text{s.t.} & \quad s_{n0}^t + \langle S_n^t, D \rangle + \langle T_n^t, |D - \Xi| \rangle + \sum_{l \in [L]} \sum_{h \in [H]} r_{n0}^t f_i \left( \sum_{(i,r) \in S_h} \frac{D_{ir} - \Xi_{ir}}{\Sigma_{ir}} \right) \\
& \quad \geq \max_{k \in [K]} \left\{ a_k \max \left\{ \sum_{i=1}^n \left( x_i^t - x_i^t - \langle X_i^t, D \rangle, x_i^t + \langle X_i^t, \tilde{D} \rangle - \tau_i^t + b_k \alpha_i^t \right) \right\} \right\}, \forall D \leq \mathbf{D} \\
& \quad r_{n0}^t \geq 0, \quad \forall l \in [L], h \in [H] \\
& \quad T_n^t \geq 0 \\
& = \min s_{n0}^t + \langle S_n^t, \Xi \rangle + \langle T_n^t, \Sigma \rangle + \sum_{l \in [L]} \sum_{h \in [H]} \epsilon_{ih} r_{n0}^t \\
\text{s.t.} & \quad s_{n0}^t + \langle S_n^t, D \rangle + \langle T_n^t, |D - \Xi| \rangle + \sum_{l \in [L]} \sum_{h \in [H]} r_{n0}^t f_i \left( \sum_{(i,r) \in S_h} \frac{D_{ir} - \Xi_{ir}}{\Sigma_{ir}} \right) \\
& \quad \geq \max_{k \in [K]} \left\{ a_k \max \left\{ \sum_{i=1}^n \left( x_i^t - x_i^t - \langle X_i^t, D \rangle, x_i^t + \langle X_i^t, \tilde{D} \rangle - \tau_i^t + b_k \alpha_i^t \right) \right\} \right\}, \forall D \leq \mathbf{D}, k \in [K] \quad (a) \\
& \quad s_{n0}^t + \langle S_n^t, D \rangle + \langle T_n^t, |D - \Xi| \rangle + \sum_{l \in [L]} \sum_{h \in [H]} r_{n0}^t f_i \left( \sum_{(i,r) \in S_h} \frac{D_{ir} - \Xi_{ir}}{\Sigma_{ir}} \right) \\
& \quad \geq \max_{k \in [K]} \left\{ a_k \max \left\{ \sum_{i=1}^n \left( x_i^t - x_i^t - \langle X_i^t, D \rangle, x_i^t + \langle X_i^t, \tilde{D} \rangle - \tau_i^t + b_k \alpha_i^t \right) \right\} \right\}, \forall D \leq \mathbf{D}, k \in [K] \quad (b) \\
& \quad r_{n0}^t \geq 0, \quad \forall l \in [L], h \in [H] \\
& \quad T_n^t \geq 0 \\
\end{align*}
\]

The last optimization problem is a semi-infinite linear programming problem with infinite number of constraints. For \( \forall k \in K \), the constraint (a) is equivalent to

\[
\min_{D \leq \mathbf{D} \leq \mathbf{D}} \left\{ \langle S_n^t - a_k X_n^t, D \rangle + \langle T_n^t, |D - \Xi| \rangle + \sum_{l \in [L]} \sum_{h \in [H]} r_{n0}^t f_i \left( \sum_{(i,r) \in S_h} \frac{D_{ir} - \Xi_{ir}}{\Sigma_{ir}} \right) \right\} \geq a_k \left( x_{n0}^t - \tau_{n0}^t \right) + b_k \alpha_{n0}^t
\]
Since function \( f_i(\cdot) \) is a piecewise linear function represented by \( f_i(x) = \max_{k' \in [K_i]} \{a_{ik'}x + p_{ik'}\} \), the left hand side of the above inequality can be written as a linear optimization problem:

\[
\min \langle S^t_n - a_k X^t_n, D \rangle + \langle T^t_n, |D - \Xi| \rangle + \sum_{l \in [L]} \sum_{h \in [H]} \sum_{k' \in [K_i]} r^t_{nhk} \max_{k' \in [K_i]} \left( \sum_{(i, r) \in S_h} \frac{D_{ir} - \Xi_{ir}}{\sum_{i',r}^t} \right) + p_{ik'}
\]

s.t. \( D \geq \underline{D} \),
\( D \leq \overline{D} \).

Since strong duality holds, it is equivalent to:

\[
\max \langle D, \nabla_n^t \rangle - \langle D, \nabla_n^t \rangle + \langle \Xi, \nabla_n^t - \nabla_{nk}^t \rangle + \sum_{l \in [L]} \sum_{h \in [H]} \sum_{k' \in [K_i]} \left( p_{ik'} - a_{ik'} \sum_{(i, r) \in S_h} \frac{\Xi_{ir}}{\sum_{i',r}^t} \right) \nabla_{nkhk'}^t
\]

s.t. \( \nabla_{nkhk'}^t = T^t_n, \)
\( \sum_{k' \in [K_i]} \nabla_{nkhk'}^t = r^t_{nh}, \forall l \in [L], h \in [H] \)
\( \nabla_{nkhk'}^t \geq 0, \forall l \in [L], h \in [H], k' \in [K_i] \)

Consequently, constraint (a) is equivalent to the following constraints:

\[
\langle D, \nabla_n^t \rangle - \langle D, \nabla_n^t \rangle + \langle \Xi, \nabla_n^t - \nabla_{nk}^t \rangle + \sum_{l \in [L]} \sum_{h \in [H]} \sum_{k' \in [K_i]} \left( p_{ik'} - a_{ik'} \sum_{(i, r) \in S_h} \frac{\Xi_{ir}}{\sum_{i',r}^t} \right) \nabla_{nkhk'}^t \geq a_k (x^t_n - \tau_n^t) + b_k X^t_n - s^t_{n0}
\]

\[
\left( \nabla_{nkhk'}^t - \nabla_{nkhk'}^t + \nabla_{nk}^t \right)_{ir} = \sum_{h \in [H]} \sum_{l \in [L]} \sum_{k' \in [K_i]} \sum_{(i, r) \in S_h} \frac{a_{ik'} \nabla_{nkhk'}^t}{\sum_{i',r}^t} = (S^t_n - a_k X^t_n)_{ir}, \forall l \in [N], \tau \in [T]
\]

\[
\nabla_{nk}^t + \nabla_{nk}^t = T^t_n,
\]
\( \sum_{k' \in [K_i]} \nabla_{nk}^t = r^t_{nh}, \forall l \in [L], h \in [H] \)
\( \nabla_{nk}^t, \nabla_{nk}^t, \nabla_{nk}^t, \nabla_{nk}^t \geq 0, \forall l \in [L], h \in [H], k' \in [K_i] \)
\( \nabla_{nk}^t \geq 0, \forall l \in [L], h \in [H], k' \in [K_i] \)
Following the same analysis, the constraint (b) is equivalent to the following constraints:

\[
\begin{align*}
&\langle D, U^t_{nk} \rangle - \langle D, V^t_{nk} \rangle + \langle E, G^t_{nk} - F^t_{nk} \rangle + \sum_{l \in [L]} \sum_{h \in [H]} \sum_{k' \in [K]} \left( p_{lk'} - o_{lk'} \sum_{(i, \tau) \in S_h} \frac{\Xi}{\Sigma_{i\tau}} \right) w^t_{nkhkk'} \\
&\geq a_k (x_n^t - x_{n0}) + b_k o_n^t - s_n^t \\
&\sum_{h \in [H]} (U^t_{nk} - V^t_{nk} - F^t_{nk} + G^t_{nk})_{i\tau} - \sum_{h \in [H]} \sum_{(i, \tau) \in S_h} \sum_{l \in [L]} \sum_{k' \in [K]} \frac{o_{lk'}}{\Sigma_{i\tau}} w^t_{nkhkk'} = (S^t_{n} + a_k X^t_{n})_{i\tau}, \\
&\forall i \in [N], \tau \in [T] \\
&F^t_{nk} + G^t_{nk} = T^t_n \\
&\sum_{k' \in [K]} w^t_{nkhkk'} = r_{nkh}, \\
&\forall l \in [L], h \in [H] \\
&U^t_{nk}, V^t_{nk}, F^t_{nk}, G^t_{nk} \geq 0, \\
&w^t_{nkhkk'} \geq 0, \\
&\forall l \in [L], h \in [H], k' \in [K]
\end{align*}
\]

Combining all of these analysis parts together, we can derive the constraints stated in the proposition 3.

Q.E.D.

Appendix E: Proof of Proposition 4

When the distributional uncertain set \( \mathcal{P} \) is specified as the set (11), and \( \mathcal{W} = \{ D | D \leq D \leq D \} \), for any \( n \in [N], t \in [T] \), we can transform the constraint (8b) as

\[
q_{n0} + \langle D, Q^t_n \rangle \geq 0, \quad \forall D \in \mathcal{W},
\]

\[
\Leftrightarrow q_{n0} + \min_{D \in \mathcal{W}} \langle D, Q^t_n \rangle \geq 0,
\]

\[
\Leftrightarrow q_{n0} + \min_{D \leq D \leq D} \langle D, Q^t_n \rangle \geq 0,
\]

\[
\begin{cases}
-q_{n0}^t + \langle D, L^t_n \rangle - \langle D, H^t_n \rangle \leq 0, \\
Q^t_n + L^t_n - H^t_n = 0,
\end{cases}
\]

\[
L^t_n, H^t_n \geq 0.
\]

The last equivalence holds due to strong duality for linear optimization problems. The constraint (8c) can be formulated as
\[ q_{n0}^i + \langle D, Q_n^i \rangle \leq M y_n^i, \quad \forall D \in \mathcal{W}, \]
\[ \iff q_{n0}^i + \max_{D \in \mathcal{W}} \langle D, Q_n^i \rangle \leq M y_n^i, \]
\[ \iff q_{n0}^i + \max_{\mathcal{D} \subseteq D \subseteq \mathcal{D}} \langle D, Q_n^i \rangle \leq M y_n^i, \]
\[ \iff \begin{cases} q_{n0}^i + \langle \overline{D}, \overline{L}_n^i \rangle - \langle \overline{D}, \overline{H}_n^i \rangle \leq M y_n^i, \\ \overline{L}_n^i - \overline{L}_n^i + \overline{H}_n^i = 0, \\ \overline{L}_n^i, \overline{H}_n^i \geq 0. \end{cases} \]

Q.E.D.
Appendix F: Proof of Corollary 1

\[
\begin{align*}
\inf & \sum_{n \in [N]} \sum_{t \in [T]} \alpha_n^t \\
\text{s.t.} & \quad s_{n0} + \langle S_n^t, \Xi \rangle + \langle T_n^t, \Sigma \rangle + \sum_{l \in [L]} \sum_{h \in [H]} \epsilon_{lk} r_{nli}^t \leq 0, \quad \forall t \in [T], n \in [N] \\
& \quad s_{n0} + \langle \textbf{D}, \textbf{U}_{nk} \rangle - \langle \textbf{D}, \textbf{V}_{nk} \rangle + \langle \Xi, G_n^t - F_n^t \rangle + \sum_{l \in [L]} \sum_{h \in [H]} \sum_{k' \in [K]} \left( p_{lk'} - o_{lk'} \sum_{(i, r) \in \mathcal{S}_h} \frac{\Xi_{ir}}{\Sigma_{ir}} \right) w_{nkll'h}^t \\
& \hspace{1cm} - a_k \left( \sum_{n=1}^t q_{n0}^m \right) - b_k \alpha_n^t \geq -a_k \tau_n^t, \quad \forall t \in [T], n \in [N], k \in [K] \\
& \quad s_{n0} + \langle \textbf{D}, \textbf{U}_{nk} \rangle - \langle \textbf{D}, \textbf{V}_{nk} \rangle + \langle \Xi, G_n^t - F_n^t \rangle + \sum_{l \in [L]} \sum_{h \in [H]} \sum_{k' \in [K]} \left( p_{lk'} - o_{lk'} \sum_{(i, r) \in \mathcal{S}_h} \frac{\Xi_{ir}}{\Sigma_{ir}} \right) w_{nkll'h}^t \\
& \hspace{1cm} + a_k \left( \sum_{n=1}^t q_{n0}^m \right) - b_k \alpha_n^t \geq a_k \tau_n^t, \quad \forall t \in [T], n \in [N], k \in [K] \\
& \quad \left( U_{nk}^t - V_{nk}^t - F_{nk}^t + G_n^t \right) = \sum_{i \in [N], \tau \in [T], t \in [T], n \in [N], k \in [K]} \left( e_a e_m' \right) \tau_i \\
& \quad \left( U_{nk}^t - V_{nk}^t - F_{nk}^t + G_n^t \right) = \sum_{i \in [N], \tau \in [T], t \in [T], n \in [N], k \in [K]} \left( e_a e_m' \right) \tau_i \\
& \quad F_{nk}^t + G_{nk}^t - T_{nk}^t = 0, \quad \forall t \in [T], n \in [N], k \in [K] \\
& \quad F_{nk}^t + G_{nk}^t - T_{nk}^t = 0, \quad \forall t \in [T], n \in [N], k \in [K] \\
& \quad \sum_{k' \in [K]} w_{nkll'h}^t - r_{nli}^t = 0, \quad \forall t \in [T], n \in [N], l \in [L], h \in [H], k \in [K] \\
& \quad \sum_{k' \in [K]} w_{nkll'h}^t - r_{nli}^t = 0, \quad \forall t \in [T], n \in [N], l \in [L], h \in [H], k \in [K] \\
& \quad -q_{n0}^m + \langle \textbf{D}, \textbf{L}_{nk}^t \rangle - \langle \textbf{D}, \textbf{H}_{nk}^t \rangle \leq 0, \quad \forall t \in [T], n \in [N] \\
& \quad q_{n0}^m + \langle \textbf{D}, \textbf{L}_{nk}^t \rangle - \langle \textbf{D}, \textbf{H}_{nk}^t \rangle - My_{nt}^t \leq 0, \quad \forall t \in [T], n \in [N] \\
& \quad Q_{n0}^t - L_{nk}^t - H_{nk}^t = 0, \quad \forall t \in [T], n \in [N] \\
& \quad Q_{n0}^t - L_{nk}^t + H_{nk}^t = 0, \quad \forall t \in [T], n \in [N] \\
& \quad \langle Q_{n}^t \rangle \leq 0, \quad \forall n \in [N], l \geq t, l, t \in [T] \\
& \quad \alpha_n^t \geq \epsilon, \quad \forall t \in [T], n \in [N] \\
& \quad r_{nli}^t \geq 0, \quad \forall t \in [T], n \in [N], l \in [L], h \in [H] \\
& \quad T_{nk}^t, L_{nk}^t, H_{nk}^t, \tilde{L}_{nk}^t, \tilde{H}_{nk}^t \geq 0, \quad \forall t \in [T], n \in [N] \\
& \quad U_{nk}^t, V_{nk}^t, F_{nk}^t, G_{nk}^t, \tilde{U}_{nk}^t, \tilde{V}_{nk}^t, \tilde{F}_{nk}^t, \tilde{G}_{nk}^t \geq 0, \quad \forall t \in [T], n \in [N], k \in [K] \\
& \quad \tilde{w}_{nkll'h}^t, w_{nkll'h}^t \geq 0, \quad \forall t \in [T], n \in [N], l \in [L], h \in [H], k \in [K], k' \in [K] \\
& \quad (y_{nt}^t, z_{ij}^t, n \in [N], t \in [T], (i, j) \in A) \in Z_R
\end{align*}
\]
Appendix G: Proof of Proposition 5

It is similar to that for Proposition 4.

Appendix H: Proof of Proposition 6

For completeness, here we also present a full proof customized to our problem. For any node $n$ and period $t$, we reformulate the constraints (8a)–(8c) as follows. With a slight abuse of notation, we drop the subscript $n$ and superscript $t$ in the rest of the proof. We start from the constraint (8a) and consider the left-hand-side (LHS) as a primal optimization problem where the decision variable is $\mathbb{P}$ and lies in the ambiguity set (14). Then it can be formulated as follow:

\[
\begin{align*}
\sup & \ E_\theta \left( \max_{k \in [K]} \left\{ a_k \max \left\{ \tau - x_0 - \langle X, \hat{D} \rangle, x_0 + \langle X, \hat{D} \rangle - \tau \right\} + b_k \alpha \right\} \right) \\
\text{s.t.} & \ E_{\pi}\left( \| \hat{D} - \hat{D}^+ \|_1 \right) \leq \theta \\
 & \hat{D}^+ \mathbb{P} = \mathbb{P} \\
 & \hat{D}^\dagger \mathbb{P} = \mathbb{P}^\dagger \\\n & \mathbb{P} \left( (\hat{D}, \hat{D}^\dagger) \in \mathcal{W} \times \mathcal{W} \right) = 1.
\end{align*}
\]  

(19)

Let $\mathbb{P}_m$ be the conditional probability distribution of $\mathbb{P}$ on $\hat{D}$, conditioning on that $\hat{D}^\dagger = \hat{D}^{(m)}$, $m \in [M_r]$. In another word, $\mathbb{P}_m \left( \hat{D} = D \right) = \mathbb{P} \left( \hat{D} = D | \hat{D}^\dagger = \hat{D}^{(m)} \right)$. With the law of total probability, we have

\[
\mathbb{P} \left( \hat{D} = D \right) = \sum_{m \in [M_r]} \mathbb{P}_m \left( \hat{D} = D \right) \mathbb{P}^\dagger \left( \hat{D}^\dagger = \hat{D}^{(m)} \right) = \frac{1}{M_r} \sum_{m \in [M_r]} \mathbb{P}_m \left( \hat{D} = D \right). 
\]

Hence, the problem (19) can be reformulated as follows:

\[
\begin{align*}
\sup & \frac{1}{M_r} \sum_{m \in [M_r]} \mathbb{E}_{\mathbb{P}_m} \left( \max_{k \in [K]} \left\{ a_k \max \left\{ \tau - x_0 - \langle X, \hat{D} \rangle, x_0 + \langle X, \hat{D} \rangle - \tau \right\} + b_k \alpha \right\} \right) \\
\text{s.t.} & \frac{1}{M_r} \sum_{m \in [M_r]} \mathbb{E}_{\mathbb{P}_m} \left( \| \hat{D} - \hat{D}^{(m)} \|_1 \right) \leq \theta \\
 & \mathbb{P}_m \left( \hat{D} \in \mathcal{W} \right) = 1, \quad m \in [M_r].
\end{align*}
\]
The strong duality of this problem holds (see, Esfahani and Kuhn 2017, Gao and Kleywegt 2016) and its dual problem is

$$\begin{align*}
\text{inf} \quad & r \theta + \sum_{m \in [M_r]} s_m \\
n\text{s.t.} \quad & \frac{r}{M_r} \left\| D - \hat{D}^{(m)} \right\|_1 + s_m \geq \frac{1}{M_r} \left( \max_{k \in [K]} \left\{ a_k \max \{ \tau - x_0 - \langle X, D \rangle, x_0 + \langle X, D \rangle - \tau \} + b_k \alpha \right\} \right) \\
& \quad \forall D \in \mathcal{W}, m \in [M_r] \\
& r \geq 0, s \in \mathbb{R}^{M_r} \\
& = \text{inf} \quad r \theta + 1's \\
n\text{s.t.} \quad & r \left\| D - \hat{D}^{(m)} \right\|_1 + M_r s_m \geq a_k \max \{ \tau - x_0 - \langle X, D \rangle, x_0 + \langle X, D \rangle - \tau \} + b_k \alpha, \\
& \quad \forall D \in \mathcal{W}, m \in [M_r], k \in [K] \\
& r \geq 0, s \in \mathbb{R}^{M_r}. 
\end{align*}$$

For any $m \in [M_r], k \in [K]$, the first constraint in the above optimization problem is equivalent to the following two inequalities

$$\begin{align*}
& r \left\| D - \hat{D}^{(m)} \right\|_1 + M_r s_m \geq a_k (\tau - x_0 - \langle X, D \rangle) + b_k \alpha \quad \forall D \in \mathcal{W} \\
& r \left\| D - \hat{D}^{(m)} \right\|_1 + M_r s_m \geq a_k (x_0 + \langle X, D \rangle - \tau) + b_k \alpha \quad \forall D \in \mathcal{W}. 
\end{align*}$$

Then can be further equivalently reformulated as

$$\begin{align*}
& \sup_{D \in \mathcal{W}} \left\{ -a_k \langle X, D \rangle - r \left\| D - \hat{D}^{(m)} \right\|_1 \right\} \leq M_r s_m - a_k (\tau - x_0) - b_k \alpha \quad (a) \\
& \sup_{D \in \mathcal{W}} \left\{ a_k \langle X, D \rangle - r \left\| D - \hat{D}^{(m)} \right\|_1 \right\} \leq M_r s_m - a_k (x_0 - \tau) - b_k \alpha \quad (b) \quad (20)
\end{align*}$$

By the definition of the dual norm, the 1-norm in (20) can be written as

$$r \left\| D - \hat{D}^{(m)} \right\|_1 = \max_{\|R_{mk}\|_\infty \leq r} \langle \hat{D}^{(m)} - \hat{D}, R_{mk} \rangle.$$
with the auxiliary variable $\mathbf{R}_{mk} \in \mathbb{R}^{N \times T}$ and $\| \cdot \|_\infty$ being the entrywise infinity-norm. Therefore, the constraint (20a) is equivalent to

$$
\sup_{D \in \mathcal{W}} \left\{ -a_k(\mathbf{X}, D) - \max_{\|\mathbf{R}_{mk}\|_\infty \leq r} \left( \mathbf{D} - \hat{\mathbf{D}}^{(m)} \right) \cdot \mathbf{R}_{mk} \right\} \leq M_r s_m - a_k (\tau - x_0) - b_k \alpha
$$

$$
\iff
\sup_{D \in \mathcal{W}} \left\{ \min_{\|\mathbf{R}_{mk}\|_\infty \leq r} \left\{ -a_k(\mathbf{X}, D) - \left( \mathbf{D} - \hat{\mathbf{D}}^{(m)} \right) \cdot \mathbf{R}_{mk} \right\} \right\} \leq M_r s_m - a_k (\tau - x_0) - b_k \alpha
$$

$$
\iff
\min_{\|\mathbf{R}_{mk}\|_\infty \leq r} \left\{ \sup_{D \in \mathcal{W}} \left\{ -a_k(\mathbf{X}, D) - \left( \mathbf{D} - \hat{\mathbf{D}}^{(m)} \right) \cdot \mathbf{R}_{mk} \right\} \right\} \leq M_r s_m - a_k (\tau - x_0) - b_k \alpha
$$

$$
\iff
\sup_{D \in \mathcal{W}} \left\{ \left\langle -a_k \mathbf{X} - \mathbf{R}_{mk}, D \right\rangle \right\} + \left\langle \hat{\mathbf{D}}^{(m)} \mathbf{R}_{mk} \right\rangle \leq M_r s_m - a_k (\tau - x_0) - b_k \alpha
$$

Here the second equivalence follows from Esfahani and Kuhn (2017), which proves that the maximization over $\mathbf{D}$ and minimization over $\mathbf{R}_{mk}$ are interchangable. By strong duality, we have

$$
\sup_{D \in \mathcal{W}} \left\langle -a_k \mathbf{X} - \mathbf{R}_{mk}, D \right\rangle
$$

$$
= \sup \left\langle -a_k \mathbf{X} - \mathbf{R}_{mk}, D \right\rangle
$$

s.t. $\mathbf{D} \geq \mathcal{D}$

$$
\mathbf{D} \leq \hat{\mathcal{D}}
$$

$$
= \inf \left\langle \mathcal{D}, \mathbf{V}_{mk} \right\rangle - \left\langle \hat{\mathcal{D}}, \mathbf{U}_{mk} \right\rangle
$$

s.t. $-\mathbf{U}_{mk} + \mathbf{V}_{mk} = -a_k \mathbf{X} - \mathbf{R}_{mk}$

$$
\mathbf{U}_{mk}, \mathbf{V}_{mk} \geq 0,
$$

where $\mathbf{U}_{mk}, \mathbf{V}_{mk}$ are the dual variables. Moreover, $\|\mathbf{R}_{mk}\|_\infty = \max_{i \in [N], r \in [T]} |(\mathbf{R}_{mk})_{ir}|$. Therefore, the constraint $\|\mathbf{R}_{mk}\|_\infty \leq r$ is equivalent to $-r \leq (\mathbf{R}_{mk})_{ir} \leq r$, $\forall i \in [N], r \in [T]$. So the constraint (20a) is equivalent to

$$
\begin{cases}
\left\langle \mathcal{D}, \mathbf{V}_{mk} \right\rangle - \left\langle \mathcal{D}, \mathbf{U}_{mk} \right\rangle + \left\langle \hat{\mathcal{D}}^{(m)} \mathbf{R}_{mk} \right\rangle \leq M_r s_m - a_k (\tau - x_0) - b_k \alpha \\
-\mathbf{U}_{mk} + \mathbf{V}_{mk} = -a_k \mathbf{X} - \mathbf{R}_{mk} \\
(\mathbf{R}_{mk})_{ir} \leq r, \forall i \in [N], r \in [T] \\
(\mathbf{R}_{mk})_{ir} \geq -r, \forall i \in [N], r \in [T] \\
\mathbf{U}_{mk}, \mathbf{V}_{mk} \geq 0.
\end{cases}
$$
By the same analysis, the constraint (20b) can be formulated as

\[
\begin{align*}
&\langle \mathbf{D}, \mathbf{V}_{mk} \rangle - \langle \mathbf{D}, \mathbf{U}_{mk} \rangle + \langle \mathbf{\hat{D}}^{(m)}, \mathbf{R}_{mk} \rangle \leq M_r s_m - a_k (x_0 - \tau) - b_k \alpha \\
&- \mathbf{U}_{mk} + \mathbf{V}_{mk} = a_k \mathbf{X} - \mathbf{R}_{mk} \\
&(\mathbf{R}_{mk})_{i\tau} \leq r, \; \forall i \in [N], \tau \in [T] \\
&(\mathbf{E}^*_i, \mathbf{R}_{mk})_{i\tau} \geq -r, \; \forall i \in [N], \tau \in [T] \\
&\mathbf{U}_{mk}, \mathbf{V}_{mk} \geq 0.
\end{align*}
\]

By combining the analysis above, we get the results of proposition 6.  

Q.E.D.
References


