SEG 5520: Optimization I		**2007–08 Second Term**
	Homework Set 2	
Instructor: Anthony Man–Cho So		Due: February 5, 2008

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (10pts). Let $k \geq 1$ be an integer. Recall that for a chi–squared random variable with k degrees of freedom, the probability density function is given by:

$$f_k(x) = \frac{x^{k/2-1}\exp(-x/2)}{2^{k/2}\Gamma(k/2)} \qquad \text{for } x > 0$$

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ is the Gamma function. Show that $\ln f_k$ is convex when $k = 1, 2$ and concave when $k \geq 2$.

Remarks: In general, we say that a function $f : \mathbb{R}^n \to \mathbb{R}_{++}$ is **logarithmically convex** (resp. **logarithmically concave**) if $\ln f$ is convex (resp. concave). Note that $\ln f$ is well–defined whenever $f(x) > 0$ for all $x \in \mathbb{R}^n$.

Problem 2 (25pts). Let $p, q > 1$ be such that $1/p + 1/q = 1$. Define:

$$C_p = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t \geq 0,\ \|x\|_p \leq t\}$$

(a) Show that C_p is a convex cone.

(b) Show that $C_p^* = C_q$ (recall that C_p^* is the dual cone of C_p). (*Hint: Hölder's inequality.*)

Problem 3 (20pts). Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ be given. Consider the function $v : \mathbb{R}^n \to \mathbb{R}$ given by:

$$\begin{array}{rrl} v(c) & := & \text{minimize} \quad c^T x \\ & & \text{subject to} \quad Ax = b \\ & & \qquad\qquad x \geq \mathbf{0} \end{array}$$

Show that $v(\cdot)$ is a concave function on the set $S = \{c \in \mathbb{R}^n : v(c) > -\infty\}$. (*Hint: Consider the dual LP.*)

Problem 4 (25pts). Consider a **production game** defined as follows. Let $\mathcal{N} = \{1, \ldots, n\}$ be the set of players, each of whom is given a vector $b^i = (b_1^i, \ldots, b_m^i)$ $(i = 1, \ldots, n)$ of resources. These resources can be used to produce goods, which in turn can be sold at a given market price. Specifically, we assume the following production model: for player $i \in \mathcal{N}$, a unit of the j–th good $(j = 1, \ldots, p)$ requires a_{kj}^i units of the k–th resource $(k = 1, \ldots, m)$ to produce. Furthermore, the j–th good can be sold at a price c_j, where $j = 1, \ldots, p$.

Now, let $S \subset \mathcal{N}$ be a coalition of players. Such a coalition will possess:

$$b_k(S) = \sum_{i \in S} b_k^i$$

units of the k–th resource, where $k = 1, \ldots, m$. Using all of their resources, the coalition S can produce any vector $x = (x_1, \ldots, x_p) \in \mathbb{R}^p_+$ of goods that satisfies $A(S)x \leq b(S)$, where:

$$A(S)_{kj} = \min_{i \in S} \left\{ a^i_{kj} \right\} \quad \text{for } k = 1, \ldots, m;\ j = 1, \ldots, p \qquad \text{and} \qquad b(S) = (b_1(S), \ldots, b_m(S))$$

Naturally, a coalition S would like to maximize its revenue, and the optimization problem it faces can be formulated as the following LP:

$$\begin{array}{rrl} v(S) \;:= & \text{maximize} & c^T x \\ & \text{subject to} & A(S)x \leq b(S) \\ & & x \geq \mathbf{0} \end{array} \tag{1}$$

The function v will be the value function of this game. Recall that an allocation vector $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ is in the core iff $\sum_{i \in \mathcal{N}} z_i = v(\mathcal{N})$ and $\sum_{i \in S} z_i \geq v(S)$ for all $S \subset \mathcal{N}$.

(a) Consider the LP faced by the grand coalition $\mathcal{N}$. Write down its dual LP.

(b) Suppose that the dual LP given in (a) is feasible, and let y^* be one of its optimal solutions. Show that the allocation vector:

$$z^* = \left(\left(b^1\right)^T y^*, \left(b^2\right)^T y^*, \ldots, \left(b^n\right)^T y^* \right) \in \mathbb{R}^n$$

belongs to the core.

Problem 5 (20pts). Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m_+$. Consider the following system:

$$Ax = b, \; x \geq \mathbf{0} \tag{2}$$

and the related LP:

$$\begin{array}{rl} \text{minimize} & e^T y \\ \text{subject to} & Ax + Iy = b \\ & x \geq \mathbf{0},\, y \geq \mathbf{0} \end{array} \tag{3}$$

Here, $e = (1, \ldots, 1) \in \mathbb{R}^m$, and I is the $m \times m$ identity matrix. The LP (3) is usually known as the **Phase One Problem.**

(a) Write down the dual of (3).

(b) Show that (3) always has an optimal solution.

(c) Show that the system (2) has a solution iff the optimal value of (3) is zero.