SEG 5520: Optimization I **2007–08 Second Term**

Homework Set 3

Instructor: Anthony Man–Cho So Due: March 4, 2008

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (45pts). Suppose that we want to allocate m items to n players, where $m, n \geq 2$. Each player $i \in \{1, \ldots, n\}$ has a utility function w_i that depends only on the set S of items she receives. Furthermore, the utility function w_i (where $i = 1, \ldots, n$) satisfies the following properties:

1. *(Non–negativity)*: $w_i(S) \geq 0$ for all $S \subset \{1, \ldots, m\}$.
2. *(Monotonicity)*: $w_i(S) \leq w_i(S \cup S')$ for all $S, S' \subset \{1, \ldots, m\}$.
3. *(Subadditivity)*: $w_i(S \cup S') \leq w_i(S) + w_i(S')$ for all $S, S' \subset \{1, \ldots, m\}$.

Our goal is then to decide which set $S_i \subset \{1, \ldots, m\}$ of items to allocate to player $i \in \{1, \ldots, n\}$ such that (i) $S_i \cap S_j = \emptyset$ for $1 \leq i < j \leq n$ and (ii) the **social welfare** $\sum_{i=1}^n w_i(S_i)$ is maximized.

Note that the above problem can be formulated as follows:

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{i=1}^n \sum_{S:S\subset\{1,\ldots,m\}} w_i(S)x_{i,S} & \\ \text{subject to} & \displaystyle\sum_{i=1}^n \sum_{\substack{S:j\in S \\ S\subset\{1,\ldots,m\}}} x_{i,S} = 1 & \text{for } j = 1, \ldots, m \\ & \displaystyle\sum_{S:S\subset\{1,\ldots,m\}} x_{i,S} = 1 & \text{for } i = 1, \ldots, n \\ & x_{i,S} \in \{0, 1\} & \text{for } i = 1, \ldots, n; \;\; S \subset \{1, \ldots, m\} \end{array} \tag{1}$$

Here, $x_{i,S}$ is the indicator decision variable indicating whether the set S is allocated to player i. The first constraint states that each item is allocated to exactly one player, and the second constraint states that each player is allocated exactly one set of items. Now, problem (1) becomes an LP problem if we relax the binary constraint $x_{i,S} \in \{0, 1\}$ to $x_{i,S} \geq 0$. Let $\{\bar{x}_{i,S}\}_{i,S}$ be an optimal fractional solution to the LP, and define:

$$w(LP) = \sum_{i=1}^n \sum_{S:S\subset\{1,\ldots,m\}} w_i(S)\bar{x}_{i,S}$$

In words, $w(LP)$ is the objective value of the fractional solution $\{\bar{x}_{i,S}\}_{i,S}$. The purpose of this problem is to design a randomized rounding procedure that converts the fractional solution $\{\bar{x}_{i,S}\}_{i,S}$ into an integral solution $\{x'_{i,S}\}_{i,S}$ such that *with constant probability*, the objective value of the integral solution is at least $\Omega(1/\log m) \cdot w(LP)$.

To begin, let us consider the following three–step randomized rounding procedure:

1. **(Tentative Allocation).** Since $\sum_{S:S\subset\{1,\ldots,m\}} \bar{x}_{i,S} = 1$ and $\bar{x}_{i,S} \geq 0$, we may think of $\{\bar{x}_{i,S}\}_S$ as a distribution over the sets of items. Each player $i \in \{1,\ldots,n\}$ independently picks the set S with probability $\bar{x}_{i,S}$. Let $\{\tilde{x}_{i,S}\}_{i,S}$ be the resulting (random) integral solution. Note that:
$$\mathbb{E}\left[\sum_{i=1}^{n} \sum_{S:S\subset\{1,\ldots,m\}} w_i(S)\tilde{x}_{i,S}\right] = w(LP)$$
However, the integral solution $\{\tilde{x}_{i,S}\}_{i,S}$ may not be feasible, since two or more players may pick the same set. We require that the solution $\{\tilde{x}_{i,S}\}_{i,S}$ satisfies the following properties:
    - Each item $j \in \{1,\ldots,m\}$ belongs to at most k players, where $k = 6\log_2 m$.
    - Suppose that $\tilde{S}_i$ is the set allocated to player i, i.e. $\tilde{x}_{i,\tilde{S}_i} = 1$. Then, we have:
$$\sum_{i=1}^{n} w_i(\tilde{S}_i) \equiv \sum_{i=1}^{n} \sum_{S:S\subset\{1,\ldots,m\}} w_i(S)\tilde{x}_{i,S} \geq \frac{1}{3}w(LP) \tag{2}$$
If any of the above properties is violated, then we discard the solution $\{\tilde{x}_{i,S}\}_{i,S}$ and repeat the procedure. Otherwise, we proceed to the second step.

2. **(Set Partition).** For each player $i \in \{1,\ldots,n\}$ and $r = 1,\ldots,k$, define:
$$\tilde{S}_i^r = \left\{ j \in \tilde{S}_i : j \text{ appears in exactly } r-1 \text{ of the sets } \tilde{S}_1,\ldots,\tilde{S}_{i-1} \right\}$$
As you will show in (c), the sets $\tilde{S}_i^1,\ldots,\tilde{S}_i^r$ form a partition of $\tilde{S}_i$, where $i = 1,\ldots,n$. Furthermore, for $1 \leq i_1 < i_2 \leq n$ and $r = 1,\ldots,k$, we have $\tilde{S}_{i_1}^r \cap \tilde{S}_{i_2}^r = \emptyset$.

3. **(Contention Resolution)** Let $r^* = \arg\max \sum_{i=1}^{n} w_i(\tilde{S}_i^r)$, and allocate the set $S_i' = \tilde{S}_i^{r^*}$ to player $i \in \{1,\ldots,n\}$, i.e. set $x'_{i,S_i'} = 1$ and $x'_{i,S} = 0$ for all $S \neq S_i'$. Let $\{x'_{i,S}\}_{i,S}$ be the resulting allocation. By the property of the set partition step, it is a feasible integral solution to (1).

We are now interested in comparing the following quantity with $w(LP)$:
$$\sum_{i=1}^{n} \sum_{S:S\subset\{1,\ldots,m\}} w_i(S)x'_{i,S}$$
Towards that end, we proceed as follows.

(a) For each item $j \in \{1,\ldots,m\}$, let E_j be the indicator random variable such that $E_j = 1$ if item j is allocated to more than k players. Prove that $\Pr\left(\sum_{j=1}^{m} E_j \geq 1\right) < 1/20$ by using the following version of the Chernoff bound:

Fact 1 (Chernoff Bound) *Let $X_1,\ldots,X_l$ be independent random variables such that for $i = 1,\ldots,l$, we have $\Pr(X_i = 1) = p_i = 1 - \Pr(X_i = 0)$. Set $\Gamma = X_1 + \cdots + X_l$. Then, for any $\mu \geq p_1 + \cdots + p_l$ and $\delta \geq 2e - 1$, we have $\Pr(\Gamma > (1+\delta)\mu) < 2^{-\mu\delta}$.*

(b) Let B be the indicator random variable such that $B = 1$ if $\sum_{i=1}^{n} w_i(\tilde{S}_i) < w(LP)/3$. Note that because of scaling we may assume without loss that $\max_{1 \le i \le n} w_i(M) = 1$, where $M = \{1, \ldots, m\}$. Now, if $w(LP) \le 3$, then by allocating the set M to the player $i \in \{1, \ldots, n\}$ who has the largest $w_i(M)$, we see that condition (2) is satisfied. Hence, we may assume that $w(LP) > 3$. Now, prove that $\Pr(B = 1) \le 3/4$ by using the following version of Chebyshev's inequality:

> **Fact 2 (Chebyshev's Inequality)** *Let Z be the sum of independent random variables, each of which lies in $[0, 1]$. Set $\mu = \mathbb{E}[Z]$. Then, for any $\alpha > 0$, we have* $\Pr(|Z - \mu| \ge \alpha) \le \mu/\alpha^2$.

Note that (a) and (b) imply that with probability at least $1 - 1/20 - 3/4 = 1/5$, the two conditions in the tentative allocation step of the randomized rounding procedure are satisfied.

(c) Show that in the set partition step, the sets $\tilde{S}_i^1, \ldots, \tilde{S}_i^r$ form a partition of $\tilde{S}_i$, where $i = 1, \ldots, n$. Furthermore, show that for $1 \le i_1 < i_2 \le n$ and $r = 1, \ldots, k$, we have $\tilde{S}_{i_1}^r \cap \tilde{S}_{i_2}^r = \emptyset$.

(d) Using the subadditivity of the utility functions, show that whenever the two conditions in the tentative allocation step are satisfied, the contention resolution step will produce a feasible integral solution $\{x'_{i,S}\}_{i,S}$ to (1) such that:

$$\sum_{i=1}^{n} \sum_{S : S \subset \{1, \ldots, m\}} w_i(S) x'_{i,S} \ge \Omega\left(\frac{1}{\log m}\right) \cdot w(LP)$$

Problem 2 (15pts). Give an equivalent SOCP formulation of the following quadratic programming problem:

$$\begin{array}{rl} \text{minimize} & x^T Q x + c^T x \\ \text{subject to} & Ax = b \\ & x \ge \mathbf{0} \end{array}$$

where $Q \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Justify your answer.

Problem 3 (25pts). Consider the following problem:

$$\begin{array}{rll} \text{minimize} & g_0(x) & \\ \text{subject to} & g_i(x) \le 0 & \text{for } i = 1, \ldots, m \end{array} \tag{3}$$

where $g_0, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ are *convex* functions. Suppose that (3) is feasible.

(a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be an arbitrary convex function. Suppose that there exists an $\bar{x} \in \mathbb{R}^n$ such that $f(\bar{x}) \le 0$. Define:

$$C = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t > 0,\ t \cdot f(x/t) \le 0\}$$

Show that C is a pointed cone.

(b) Hence, or otherwise, show that (3) can be formulated as an CLP problem. Justify your answer.

Remarks. Problem (3) is a so–called **convex optimization problem**. As shown in (b), *any* convex optimization problem of the form (3) can be formulated as an CLP problem.

Problem 4 (15pts). Let $X, S \in \mathcal{S}_+^n$. Show that $X \bullet S = 0$ iff $XS = \mathbf{0}$.