SEG 5520: Optimization I **2007–08 Second Term**

Homework Set 4

Instructor: Anthony Man–Cho So Due: March 18, 2008

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (30pts). Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ be given. Consider the following problem:

$$\begin{array}{ll} \text{minimize} & \|b - Ax\|_2^2 \\ \text{subject to} & x_i^2 = 1 \qquad \text{for } i = 1, \ldots, n \end{array} \tag{1}$$

Problem (1) is the so–called *discrete least squares problem* and arises in many engineering applications, e.g. maximum likelihood detection in Multiple Input Multiple Output (MIMO) channels. It is known to be a computationally intractable problem, and our goal is to study a particular SDP relaxation of it. To begin, observe that we can homogenize the objective function of (1) and obtain the following equivalent problem:

$$\begin{array}{ll} \text{minimize} & \|tb - Ax\|_2^2 \\ \text{subject to} & x_i^2 = 1 \qquad \text{for } i = 1, \ldots, n \\ & t^2 = 1 \end{array} \tag{2}$$

Here, $t \in \{-1, 1\}$ is a dummy *scalar* variable.

(a) Show that (2) can be written in the form:

$$\begin{array}{ll} \text{minimize} & y^T Q y \\ \text{subject to} & y_i^2 = 1 \quad \text{for } i = 1, \ldots, n+1 \end{array} \tag{3}$$

where Q is some $(n+1) \times (n+1)$ matrix and $y = (y_1, \ldots, y_{n+1})$ is the decision vector.

(b) Following the procedure given in Handout 6, write down the SDP relaxation of (3). *(Hint: $y^T Q y = Q \bullet yy^T$.)*

(c) Construct an example to show that it is possible for the optimal value of (2) to be non–zero while the optimal value of the corresponding SDP relaxation (as given in (b)) is zero. *(Hint: It suffices to consider $A \in \mathbb{R}^{3 \times 3}$ and $b \in \mathbb{R}^3$.)*

Remarks: The result in part (c) shows that one cannot derive *any* worst–case approximation ratio bound for problem (2) using the SDP relaxation given in (b). This of course is in sharp contrast with the case of, say, the Maximum Cut problem.

Problem 2 (15pts). Consider the following problem:

$$\begin{array}{ll} \text{minimize} & x_1^4 + x_2^4 + 12x_1^2 + 6x_2^2 - x_1x_2 - x_1 - x_2 \\ \text{subject to} & x_1 + x_2 \geq 6 \\ & 2x_1 - x_2 \geq 3 \\ & x_1, x_2 \geq 0 \end{array} \tag{4}$$

(a) Write down the KKT conditions of (4).

(b) Show that $(x_1, x_2) = (3, 3)$ is the unique optimal solution to (4).

Problem 3 (25pts). Let X be a non–empty open subset of $\mathbb{R}^n$, and let $f, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable functions. Consider the following problem:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \le 0 \quad \text{for } i = 1, \ldots, m \\ & x \in X \end{array} \tag{5}$$

Let $S = \{x \in X : g_i(x) \le 0 \text{ for } i = 1, \ldots, m\}$ be the feasible region, and let $\bar{x} \in S$. Define $I = \{i \in \{1, \ldots, m\} : g_i(\bar{x}) = 0\}$ be the index set of the active constraints, and let

$$\begin{aligned} D &= \left\{ d \in \mathbb{R}^n : \nabla f(\bar{x})^T d < 0 \right\} \\ F &= \{ d \in \mathbb{R}^n \backslash \{\mathbf{0}\} : \text{there exists an } \alpha_0 > 0 \text{ such that } \bar{x} + \alpha d \in S \text{ for all } \alpha \in (0, \alpha_0) \} \\ G &= \left\{ d \in \mathbb{R}^n : \nabla g_i(\bar{x})^T d < 0 \text{ for } i \in I \right\} \end{aligned}$$

(a) Suppose that g_i, where $i \in I$, are *strictly convex*[1] at $\bar{x}$. Show that $F = G$.

(b) Suppose that $\bar{x}$ is an FJ point of (5). Suppose further that f is convex at $\bar{x}$, and that g_i, where $i \in I$, are strictly convex at $\bar{x}$. Show that $\bar{x}$ is a global minimum of problem (5).

REMARKS: The result in part (b) is known as the **Fritz John sufficient condition**.

Problem 4 (30pts). Let $f, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable functions. Consider the following problem:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \le 0 \quad \text{for } i = 1, \ldots, m \\ & x \in \mathbb{R}^n \end{array} \tag{6}$$

Let $\bar{x}$ be a feasible solution, and let $I = \{i \in \{1, \ldots, m\} : g_i(\bar{x}) = 0\}$. Suppose that g_i, where $i \in I$, are concave at $\bar{x}$. Consider the following LP problem:

$$\begin{array}{ll} \text{minimize} & \nabla f(\bar{x})^T d \\ \text{subject to} & \nabla g_i(\bar{x})^T d \le 0 \quad \text{for } i \in I \\ & -e \le d \le e \end{array} \tag{7}$$

where $e = (1, 1, \ldots, 1)$. Let d^* be an optimal solution to (7), and set $v^* = \nabla f(\bar{x})^T d^*$.

(a) Show that $v^* \le 0$.

(b) Show that if $v^* < 0$, then there exists an $\alpha_0 > 0$ such that $\bar{x} + \alpha d^*$ is feasible and $f(\bar{x} + \alpha d^*) < f(\bar{x})$ for all $\alpha \in (0, \alpha_0)$.

(c) Show that if $v^* = 0$, then $\bar{x}$ satisfies the KKT conditions of (6).

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[1] Let S be a non–empty convex subset of $\mathbb{R}^n$. A function $f : S \to \mathbb{R}$ is **strictly convex** on S if $f(\alpha x_1 + (1-\alpha) x_2) < \alpha f(x_1) + (1-\alpha) f(x_2)$ for all $x_1, x_2 \in S$ and $\alpha \in [0, 1]$. It is known that f is differentiable and strictly convex on S iff $f(x_1) > f(x_2) + \nabla f(x_2)^T (x_1 - x_2)$ for all $x_1, x_2 \in S$ with $x_1 \ne x_2$.