SEG 5520: Optimization I **2007–08 Second Term**

Homework Set 5

Instructor: Anthony Man–Cho So Due: April 3, 2008

SOLVE THE FOLLOWING PROBLEMS:

Problem 1 (20pts). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function. Consider the following problem:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \geq 0 \end{array} \tag{1}$$

Show that $\bar{x} \in \mathbb{R}^n$ is an optimal solution to (1) iff it satisfies the following system:

$$\begin{array}{rcl} \nabla f(\bar{x}) & \geq & 0 \\ \bar{x} & \geq & 0 \\ \bar{x}^T \nabla f(\bar{x}) & = & 0 \end{array}$$

Problem 2 (25pts). Let $\kappa \in \mathbb{R}$ be given, and consider the following quadratic program:

$$\begin{array}{ll} \text{minimize} & \dfrac{1}{2}x_1^2 + \dfrac{1}{2}x_2^2 - x_1 - 2x_2 \\ \text{subject to} & x_1 + x_2 - \kappa \geq 0 \\ & x_1, x_2 \geq 0 \end{array} \tag{2}$$

(a) Show that problem (2) always has an optimal solution.

(b) Using the KKT conditions, verify that $(3/2, 5/2)$ is an optimal solution to (2) when $\kappa = 4$.

(c) For certain values of κ, the optimal solution to (2) lies on the boundary of the feasible region. What are those values, and what are the corresponding optimal solutions, Lagrange multipliers, and optimal values?

Problem 3 (30pts). Recall that $\mathcal{S}^n$ is the set of $n \times n$ symmetric matrices, and $\mathcal{S}^n_{++}$ is the set of $n \times n$ symmetric positive definite matrices.

(a) Let $u \in \mathbb{R}^n$ be fixed, and define the function $g : \mathcal{S}^n \to \mathbb{R}_+$ by $g(X) = \|Xu\|_2^2$. Find $\nabla g(X)$.

(b) Let $V = \{v^1, \ldots, v^m\} \subset \mathbb{R}^n$ be a set of vectors that span $\mathbb{R}^n$. Consider the following problem:

$$\begin{array}{ll} \inf & -\log\det X \\ \text{subject to} & \|Xv^i\|_2^2 \leq 1 \quad i = 1, \ldots, m \\ & X \in \mathcal{S}^n_{++} \end{array} \tag{3}$$

Let $\bar{X}$ be an optimal solution to (3) (it can be shown that such an $\bar{X}$ exists). Write down the KKT conditions that $\bar{X}$ must satisfy. *(Hint: The Matrix Reference Manual.)*

REMARKS: The set $\{y \in \mathbb{R}^n : \|\bar{X}y\| \leq 1\}$ is the **minimum volume ellipsoid** (centered at the origin) containing the vectors $v^1, \ldots, v^m$.

(c) Suppose that $m = n$ and $v^i = e_i$ for $i = 1, \ldots, n$, where e_i is the i–th standard basis vector. Using the result of (b), determine the optimal solution to (3) and find the corresponding Lagrange multipliers. *(Hint: The function $X \mapsto -\log\det X$ is convex.)*

Problem 4 (25pts). In this problem we study the convergence property of the method of steepest descent with **diminishing step sizes**. To begin, let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function. Recall that the method of steepest descent is defined via the following update rule:

$$x^{k+1} = x^k - \alpha^k \nabla f(x^k) \qquad \text{for } k = 0, 1, \ldots$$

(a) Show that for any $y \in \mathbb{R}^n$ and $k \geq 0$, we have:

$$\|x^{k+1} - y\|_2^2 \leq \|x^k - y\|_2^2 - 2\alpha^k \left(f(x^k) - f(y) \right) + \left(\alpha^k \|\nabla f(x^k)\|_2 \right)^2$$

(b) Suppose that:

$$\sum_{k=0}^{\infty} \alpha^k = \infty \quad \text{and} \quad \alpha^k \|\nabla f(x^k)\|_2^2 \to 0$$

Show that[1]:

$$\liminf_{k \to \infty} f(x^k) = \inf_{x \in \mathbb{R}^n} f(x)$$

(Hint: Argue by contradiction. Suppose that for some $\delta > 0$, there exists an $y \in \mathbb{R}^n$ such that $f(y) < f(x^k) - \delta$ for all sufficiently large k. Use the result of (a).)

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[1] Recall that given a sequence $\{a_k\}_k$, its **limit inferior** is defined as:

$$\liminf_{k \to \infty} a_k = \lim_{k \to \infty} \inf_{n \geq k} a_n = \sup_{k \to \infty} \inf_{n \geq k} a_n$$