SEG 5520: Optimization I	**2007–08 Second Term**
Homework Set 6	
Instructor: Anthony Man–Cho So	Due: April 17, 2008

SOLVE THE FOLLOWING PROBLEMS (TOTAL 80pts):

Problem 1 (20pts). Prove Theorem 2(b) of Handout 10.

Problem 2 (60pts). Consider the following pair of standard form LPs:

$$(P)\quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq \mathbf{0} \end{array} \qquad\qquad (D)\quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c \\ & s \geq \mathbf{0} \end{array}$$

Here, $A \in \mathbb{R}^{m \times n}$ with $m \leq n$ and $\text{rank}(A) = m$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. In Handout 10 we developed and analyzed a path following algorithm for solving (P) and (D) with the aid of the following *primal* barrier sub–problem:

$$(P_\mu)\quad \begin{array}{ll} \text{minimize} & B(x,\mu) \equiv c^T x - \mu \sum_{i=1}^{m} \ln(x_i) \\ \text{subject to} & Ax = b \end{array}$$

The goal of this problem is to develop and analyze a similar algorithm using the *dual* barrier sub–problem, which is defined as:

$$(D_\mu)\quad \begin{array}{ll} \text{maximize} & D(y,\mu) \equiv b^T y + \mu \sum_{i=1}^{n} \ln\left(c_i - a_i^T y\right) \\ \text{subject to} & A^T y + s = c \end{array}$$

Here, $\mu > 0$ denotes the barrier parameter, and a_i denotes the i–th column of the matrix A, where $i = 1, \ldots, n$. We assume that problem (D) has a strictly feasible solution. In other words, we assume that the set $\mathring{\mathcal{F}}_d = \{(y,s) \in \mathbb{R}^m \times \mathbb{R}^n : A^T y + s = c,\ s > \mathbf{0}\}$ is non–empty.

(a) Compute $\frac{\partial D}{\partial y_i}$ for $i = 1, \ldots, m$ and $\frac{\partial^2 D}{\partial y_i \partial y_j}$ for $1 \leq i \leq j \leq m$. Hence, or otherwise, show that:

$$\nabla D(y,\mu) = b - \mu A S^{-1} e \quad \text{and} \quad \nabla^2 D(y,\mu) = -\mu A S^{-2} A^T$$

where $s = c - A^T y$ and $S = \text{diag}(s_1, \ldots, s_n)$.

(b) To solve the barrier sub–problem (D_μ), we use Newton's method. Suppose that we are given a point $(\bar{y}, \bar{s}) \in \mathring{\mathcal{F}}_d$. To find the next iterate $(\bar{y}', \bar{s}')$, we first form the quadratic approximation of $y \mapsto D(y,\mu)$ at $\bar{y}$, which is given by:

$$\tilde{D}(y,\mu) = D(\bar{y},\mu) + (b - \mu A \bar{S}^{-1} e)^T (y - \bar{y}) - \frac{1}{2}\mu (y - \bar{y})^T A \bar{S}^{-2} A^T (y - \bar{y})$$

and solve for the Newton step $\bar{d}$ via:

$$\bar{d} = \arg\max_{d \in \mathbb{R}^m} \left\{ -\frac{1}{2}\mu d^T A\bar{S}^{-2}A^T d + (b - \mu A\bar{S}^{-1}e)^T d \right\} \tag{1}$$

Then, the next iterate $(\bar{y}', \bar{s}')$ is given by $\bar{y}' = \bar{y} + \bar{d}$ and $\bar{s}' = c - A^T\bar{y}'$. Write down the optimality condition of (1). Hence, or otherwise, express the Newton step $\bar{d}$ in terms of $A, b, \bar{s}$ and μ.

(c) Given the point $(\bar{y}, \bar{s}) \in \overset{\circ}{\mathcal{F}}_d$, we would like to know how close it is to the central path $\mathcal{C}$ associated with (P) and (D). Towards that end, we would like to find a vector $x \in \mathbb{R}^n$ such that $Ax = b$ and $\|\bar{S}x - \mu e\|_2$ is minimized. Equivalently, we are interested in solving the following problem:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T\bar{S}^2x - \mu x^T\bar{s} \\ \text{subject to} & Ax = b \end{array} \tag{2}$$

Let $\theta \in \mathbb{R}^m$ be the Lagrange multiplier associated with the constraint $Ax = b$. Write down the KKT conditions associated with (2). Hence, or otherwise, express the optimal solution $x(\bar{s}, \mu)$ and the optimal Lagrange multiplier $\bar{\theta}$ in terms of $A, b, \bar{s}$ and μ.

(d) To quantify the closeness of the point $(\bar{y}, \bar{s}) \in \overset{\circ}{\mathcal{F}}_d$ to the central path $\mathcal{C}$, let us use the following proximity measure:

$$\delta(s, \mu) = \left\| \frac{Sx(s, \mu)}{\mu} - e \right\|_2$$

Here, $x(s, \mu)$ is the optimal solution to (2). In general, we may not have $x(\bar{s}, \mu) \geq \mathbf{0}$ and $\bar{s}' = c - A^T\bar{y}' > \mathbf{0}$. However, show that if $\delta(\bar{s}, \mu) < 1$, then we have (i) $(\bar{y}', \bar{s}') \in \overset{\circ}{\mathcal{F}}_d$, (ii) $\delta(\bar{s}', \mu) \leq \delta(\bar{s}, \mu)^2$, and (iii) $x(\bar{s}, \mu) \geq \mathbf{0}$.