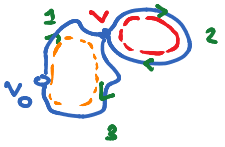
Walks in Graphs (cont'd)

Thursday, November 25, 2021 10:28 AM

Theorem: Every closed walk (i.e., a walk where the starting and ending vertices are the same) that is odd in length contains an odd cycle.

Idea:



1) If the walk has no repeated vertices (except the start and end), then we are done.

2) Suppose that v is repeated in the walk. This creates two v-v walks, observe that one of them must be odd in length → apply induction

Proof: By induction on the length of the walk n.

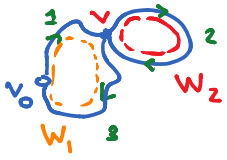
Predicate:

P(n): Every closed odd walk of length n has an odd cycle.

Base case:

n=1 : cycle of length 1

Inductive Step:



Suppose that P(k) is true for k<n.

Case 1: The walk has no repeated vertex → done

Case 2: Suppose that v is repeated in the walk. Then, we partition the original walk into two v-v walks W_1 and W_2. Observe that

odd = length of the original walk = length of W_1 + length of W_2

⇒ one of W_1, W_2, say W_1, is odd.

Note that W_1 is closed odd walk. Also, it is shorter

than the original walk. because W_2 is not empty.

$\Rightarrow$ By IH, W_1 has an odd cycle. This odd cycle is also present in the original walk.

Definition: Consider a multi-graph $G=(V,E)$ (i.e., E can contain duplicated elements). An Eulerian walk is a walk in G that uses each edge exactly once. An Eulerian tour is a closed Eulerian walk.

Theorem: A connected graph has an Eulerian tour iff every vertex has even degree.

Proof: ($\Rightarrow$) Suppose that $G=(V,E)$ is connected and has an Eulerian tour $v_0 - v_1 - \cdots - v_k$ "v_0".

1) Consider an intermediate vertex v_i $(i=1,\ldots,k-1)$

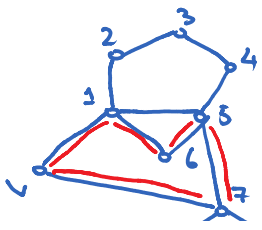
The tour enters v_i and then leaves it $\Rightarrow$ tour traverses exactly 2 edges.

2) The tour uses every edge exactly once

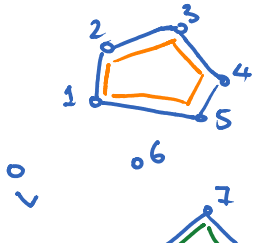
$\Rightarrow$ every intermediate vertex has even degree.

3) Same is true for v_0, because first edge leaving v_0 can be paired up with last edge entering v_0.

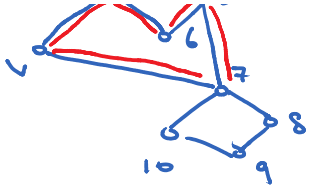
($\Leftarrow$) Suppose that every vertex in G has even degree.



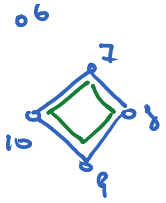
Delete red edges $\rightarrow$



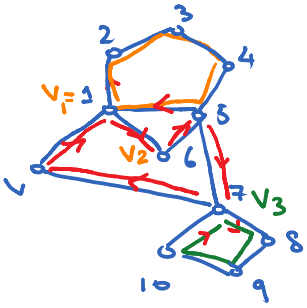
By induction, each piece



join the Eulerian tours to form a big one



each piece has an Eulerian tour.



By induction on the number of edges in G.

Predicate:

P(n): A connected multi-graph with n edges in which every vertex has even degree has an Eulerian tour

Base case:

n = 2 :

(n = 1 :)

Inductive step:

Suppose that P(k) is true for $k < n$.

1) Start at any vertex v and traverse unused edges in an arbitrary manner until returning to v. This gives a v-v walk W. Why is this always possible?

Since the degrees are even, incoming edge to a vertex can always be paired with an outgoing one.

2) Let $C_1, \cdots, C_k$ be the connected components of $G - W$.

If C_i is not an isolated vertex, then by IH, it has an Eulerian tour. (fill in the details on why IH can be applied)

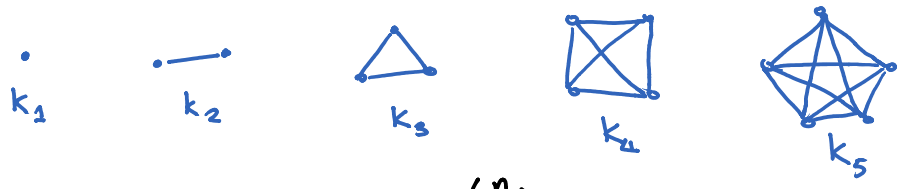
3) Since G is connected, there is a vertex v_i in C_i that is also on W. Assuming that W encounters the vertices $v_1, v_2, \cdots, v_k$ in that order, we can form an Eulerian tour in G as follows:

Start at v, follow W until it hits v_1,
traverse C_1 using the Eulerian tour in it
and return to v_1,
then continue on W until it hits v_2,
and so on, until it returns to v.

Definitions:

① A complete graph on n vertices, denoted by K_n, is a graph where there is an edge between every pair of vertices.

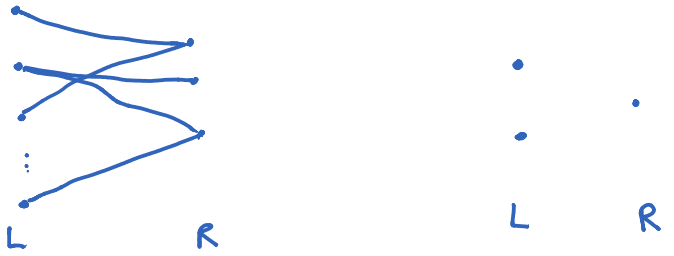
e.g.



number of edges in $K_n = \binom{n}{2}$

② A graph is bipartite if V can be partitioned as $V = L \cup R$, and each edge e takes the form $e = (u, v)$ with $u \in L$, $v \in R$.

e.g.



Note: K_2 is bipartite, but K_n is not bipartite for $n \geq 3$