- In the past lectures, we focused on regularized loss minimization problems that have non-convex loss functions and/or non-convex regularizers. By stipulating that the loss function satisfies RSC and that the regularizer is not "too non-convex", we essentially turned the problem into a convex one, for which standard techniques apply.
- In this lecture, we depart from the regularized loss minimization problem and consider another quite different estimation problem: the phase synchronization problem. In this problem, one is interested in recovering a collection of phases $\{e^{i\theta_k}\}$ based on noisy measurements of relative phases $\{e^{i(\theta_j-\theta_k)}\}$.

Specifically, let

$$z^* \in \mathbb{T}^n \triangleq \{ w \in \mathbb{C}^n : |w_1| = \cdots = |w_n| = 1 \}$$

be the unknown phase vector we wish to recover. We have noisy measurements of the form

$$C_{jl} = z_j^* \bar{z}_l^* + \Delta_{jl} \qquad 1 \le j < l \le n, \qquad (*)$$

where $\bar{(\cdot)}$ is the complex conjugate and Δ_{jl} is the noise associated with the (j,l)-th measurement. To recover z^*, it is natural to formulate the following least-squares problem:

$$\hat{z} \in \underset{z \in \mathbb{T}^n}{\arg\min} \sum_{1 \le j < l \le n} | C_{jl} - z_j \bar{z}_l |^2 .$$

Using the fact that $|z_j \bar{z}_l|^2 = 1$ for any $z \in \mathbb{T}^n$, the above is equivalent to

$$\hat{z} \in \arg\max_{z \in \mathbb{T}^n} z^H C z, \qquad (P)$$

where $C_{jj} = 1$ for $j = 1, \dots, n$.

- Note that neither the objective function nor the constraint set of (P) is convex. Moreover, there will be multiple optimal solutions to (P), because whenever $\hat{z}$ is optimal, so is $e^{i\theta}\hat{z}$ for any $\theta \in [0, 2\pi)$. This also suggests that we can only recover z^* up to a global phase. Hence, we define the following distance metric to measure the closeness of any $z \in \mathbb{T}^n$ to the target vector z^*:

$$d_2(z, z^*) = \min_{\theta \in [0,2\pi)} \| z - e^{i\theta} z^* \|_2$$

- Following our earlier studies, a natural first question is to study the estimation performance of $\hat{z}$ wrt the metric d_2. As it turns out, this is rather straightforward. To begin, observe from $(*)$ that

$$C = z^*(z^*)^H + \Delta,$$

where Δ is a Hermitian matrix whose diagonal is zero and its above-diagonal entries are given by Δ_{jk}. Then, we have the following:

<u>Proposition 1</u>: Let $z \in \mathbb{C}^n$ be such that $\|z\|_2^2 = n$ and $(z^*)^H C z^* \le z^H C z$ (in particular, these conditions are satisfied by an optimal solution $\hat{z}$ to (P)). Then,

$$d_2(z, z^*) = \sqrt{2\left(n - |z^H z^*|\right)} \le \frac{4\|\Delta\|}{\sqrt{n}}.$$

Proof: By definition,

$$d_2(z, z^*)^2 = \min_{\theta \in [0, 2\pi)} \| z - e^{i\theta} z^* \|_2^2$$

$$= 2\left(n - \max_{\theta \in [0, 2\pi)} \mathrm{Re}\left(e^{i\theta} z^H z^* \right) \right)$$

$$= 2\left(n - |z^H z^*| \right)$$

Now, without loss, suppose that $z^H z^* = |z^H z^*|$. By assumption,

$$z^H C z = |z^H z^*|^2 + z^H \Delta z \geq (z^*)^H C z^* = n^2 + (z^*)^H \Delta z^*.$$

This gives

$$n^2 - |z^H z^*|^2 \leq z^H \Delta z - (z^*)^H \Delta z^*.$$

Dividing both sides by n and observing that $|z^H z^*| \leq n$ and

$$n^2 - |z^H z^*|^2 = (n - |z^H z^*|)(n + |z^H z^*|),$$

we have

$$n - |z^H z^*| \leq \frac{1}{n}\left(z^H \Delta z - (z^*)^H \Delta z^* \right)$$

$$= \frac{1}{n} \mathrm{Re}\left((z - z^*)^H \Delta (z + z^*) \right)$$

$$\leq \frac{1}{n} \|\Delta\| \cdot \| z - z^* \|_2 \cdot \| z + z^* \|_2$$

$$\leq \frac{2}{\sqrt{n}} \|\Delta\| \cdot \| z - z^* \|_2 .$$

Note that $d_2(z, z^*) = \| z - z^* \|_2$, since we assume that $z^H z^* = |z^H z^*|$. It follows that

$$d_2(z, z^*) \leq \frac{4\|\Delta\|}{\sqrt{n}},$$

as desired.

- In view of Proposition 1, it is natural to ask if we can find an optimal solution $\hat{z}$ to (P) efficiently. Noting that projection onto the non-convex set T^n is still efficiently computable, let us consider the following simple projected gradient scheme:

$$\begin{aligned} w^k &\leftarrow z^k + \frac{\alpha_k}{n} C z^k \\ z^{k+1} &\leftarrow \frac{w^k}{|w^k|}, \end{aligned} \qquad (A)$$

where $\alpha_k > 0$ is the step size, and for a vector $w \in \mathbb{C}^n$, $\frac{w}{|w|}$ is the vector given by
$$\left(\frac{w}{|w|}\right)_j = \begin{cases} \frac{w_j}{|w_j|} & \text{if } w_j \neq 0 \\ \text{any unit modulus complex number} & \text{otherwise.} \end{cases}$$

- A priori, it is not clear the above scheme will converge in general given an arbitrary initial point. To proceed, let us consider initializing the above scheme by the so-called eigenvector estimator $V_c \in T^n$, which is computed as follows. Let $u \in \mathbb{C}^n$ be a leading eigenvector of C and $a \in \mathbb{C}^n$ be any vector satisfying $a^H u \neq 0$. Then, set

$$(V_c)_j = \begin{cases} \frac{u_j}{|u_j|} & \text{if } u_j \neq 0 \\ \frac{a^H u}{|a^H u|} & \text{if } u_j = 0 \end{cases}$$

The advantage of using V_c can in part be seen from the following result.

<u>Proposition 2</u>: We have $d_2(V_c, z^*) \leq \frac{8\|\Delta\|}{\sqrt{n}}$.

The proof of Proposition 2 relies on the following result, which states that the projection onto T^n is not "too expansive":

<u>Proposition 3</u>: For any $w \in \mathbb{C}^n$, $z \in T^n$, we have

$$\left\| \frac{w}{|w|} - z \right\|_2 \leq 2\|w - z\|_2.$$

Armed with Proposition 3, let us prove Proposition 2:

Proof of Proposition 2: Without loss, we may choose u s.t. $\|u\|_2^2 = n$ and $u^H z^* = |u^H z^*|$. Then, by definition of v_c,

$$d_2(v_c, z^*) \le \|v_c - z^*\|_2 \le 2\|u - z^*\|_2 \le \frac{8\|\Delta\|}{\sqrt{n}},$$

where the second inequality is by Proposition 3, and the last inequality is due to Proposition 1 and the fact that u is the leading eigenvector.

Proof of Proposition 3: Without loss, we may assume that $z = e$, the vector of all-ones. It suffices to show that

$$\left| \left(\frac{w}{|w|}\right)_j - 1 \right| \le 2\,|w_j - 1| \quad \text{for } j = 1, \dots, n.$$

The inequality is trivial if $w_j = 0$. Thus, suppose that $w_j \ne 0$. Then, $\frac{w_j}{|w_j|} = e^{i\phi}$ for some $\phi \in [0, 2\pi)$. We claim that

$$|e^{i\phi} - 1| \le 2\,|re^{i\phi} - 1| \quad \text{for any } \phi \in [0, 2\pi),\ r \ge 0.$$

Indeed, we have $|re^{i\phi} - 1|^2 = r^2 - 2r\cos\phi + 1$. Thus,

$$\operatorname*{argmin}_{r \ge 0} |re^{i\phi} - 1|^2 = \begin{cases} 0 & \text{if } \phi \in [\pi/2, 3\pi/2] \\ \cos\phi & \text{if } \phi \in [0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi). \end{cases}$$

This yields

$$\min_{r \ge 0} |re^{i\phi} - 1|^2 = \begin{cases} 1 & \text{if } \phi \in [\frac{\pi}{2}, \frac{3\pi}{2}] \\ \sin^2\phi & \text{if } \phi \in [0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi) \end{cases}$$

Now, for $\phi \in [\frac{\pi}{2}, \frac{3\pi}{2}]$,

$$|e^{i\phi} - 1| \le 2 \le 2\,|re^{i\phi} - 1| \quad \text{for any } r \ge 0.$$

On the other hand, for $\phi \in [0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$, we use the half-angle formula to get

$$|e^{i\phi} - 1| = \sqrt{2(1-\cos\phi)} = 2|\sin \phi/2| \le 2|\sin\phi| \le 2|re^{i\phi} - 1|$$

for any $r \ge 0$. This completes the proof.

- Based on Propositions 1 and 2, the next question is whether Algorithm (A) decreases the estimation error in each iteration. As the following result shows, the answer is affirmative under some mild assumptions:

<u>Theorem 1</u>: Suppose that $\|\Delta\| \le n/16$. If $z^0 = v_c$ and $\alpha_k \ge 2$ in Algorithm (A), then we have

$$d_2(z^{k+1}, z^*) \le \mu^{k+1} \cdot d_2(z^0, z^*) + \frac{\nu}{1-\mu} \cdot \frac{8\|\Delta\|}{\sqrt{n}} \quad \text{for } k \ge 0,$$

where

$$\mu = \frac{16(\alpha \cdot \|\Delta\| + n)}{(7\alpha + 8)n} < 1 \quad \text{and} \quad \nu = \frac{2\alpha}{7\alpha + 8}.$$

Before we prove Theorem 1, some remarks are in order. First, Theorem 1 does not claim the convergence of Algorithm (A). Nevertheless, it shows that the bound on the distance between z^k and z^* decreases in k. Second, if we let $\alpha \to \infty$, which can be interpreted as using the update $z^{k+1} \leftarrow \frac{Cz^k}{|Cz^k|}$, we have

$$d_2(z^{k+1}, z^*) \le \left[\left(\frac{1}{7}\right)^{k+1} + \frac{1}{3}\right] \cdot \frac{8\|\Delta\|}{\sqrt{n}}.$$

The update in this case is akin to the power method for computing the leading eigenvector of a matrix. Thus, Algorithm (A) is also known as the Generalized Power Method (GPM).

- To prove Theorem 1, we need the following auxiliary result:

<u>Proposition 4:</u> Let $\{z^k\}_{k\geq 0}$ be the iterates generated by the GPM with $\alpha_k = \alpha \geq 0$. Define

$$\theta_k = \underset{\theta\in[0,2\pi)}{\operatorname{argmin}} \| z^k - e^{i\theta} z^*\|_2 ,$$

$$\varepsilon^k = e^{-i\theta_k}(z^k - e^{i\theta_k} z^*),$$

$$\beta_k = 1 + \alpha + \frac{\alpha}{n}(z^*)^H \varepsilon^k .$$

Then, for any $y \in \mathbb{C}$ and $k \geq 0$, we have

$$d_2(z^{k+1}, z^*) \leq 2\| y g^k - z^*\|_2 ,$$

where $g^k = \beta_k z^* + (I + \frac{\alpha}{n}\Delta)\varepsilon^k + \frac{\alpha}{n}\Delta z^*$.

<u>Proof:</u> By definition of the GPM iterates and the model $C = z^*(z^*)^H + \Delta$,

$$w^k = (I + \tfrac{\alpha}{n} C) z^k$$

$$= e^{i\theta_k}\left(I + \tfrac{\alpha}{n}(z^*(z^*)^H + \Delta)\right)(z^* + \varepsilon^k)$$

$$= \left[(1 + \alpha + \tfrac{\alpha}{n}(z^*)^H \varepsilon^k) z^* + (I + \tfrac{\alpha}{n}\Delta)\varepsilon^k + \tfrac{\alpha}{n}\Delta z^*\right] e^{i\theta_k}$$

$$= g^k e^{i\theta_k}$$

Since $z^{k+1} = \frac{w^k}{|w^k|}$, by Proposition 3, for any $y = re^{i\phi} \in \mathbb{C}\setminus\{0\}$,

$$d_2(z^{k+1}, z^*) \leq \left\| e^{i\phi}\left(\frac{w^k}{|w^k|} - e^{i(\theta_k - \phi)} z^*\right)\right\|_2 = \left\| \frac{yg^k}{|yg^k|} - z^*\right\|_2 \leq 2\|yg^k - z^*\|_2 .$$

Since the above inequality holds for all $y \in \mathbb{C}\setminus\{0\}$, by letting $y \to 0$, it holds for $y = 0$ as well.

- <u>Proof of Theorem 1:</u> We will prove by induction that

$$\|\varepsilon^k\|_2 \leq \frac{\sqrt{n}}{2}$$

and

$$d_2(z^{k+1}, z^*) \leq \mu \cdot d_2(z^k, z^*) + \nu \cdot \frac{8\|\Delta\|}{\sqrt{n}} .$$

The result will then follow by unrolling the recurrence.

For the base case, we have $z^0 = v_c$. By Proposition 2, we have

$$\|\varepsilon^0\|_2 = \|z^0 - e^{i\theta_0} z^*\|_2 = d_2(v_c, z^*) \le \frac{8\|\Delta\|}{\sqrt{n}} \le \frac{\sqrt{n}}{2}.$$

This implies

$$|\beta_0| \ge \left|1 + \alpha + \frac{\alpha}{n} \mathrm{Re}\left((z^*)^H \varepsilon^0\right)\right|$$

$$= \left|1 + \alpha + \frac{\alpha}{2n}\left(\|z^* + \varepsilon^0\|_2^2 - \|z^*\|_2^2 - \|\varepsilon^0\|_2^2\right)\right|$$

$$= \left|1 + \alpha + \frac{\alpha}{2n}\left(\|z^0\|_2^2 - \|z^*\|_2^2 - \|\varepsilon^0\|_2^2\right)\right|$$

$$\ge 1 + \frac{7\alpha}{8}$$

as $\|z^0\|_2^2 = \|z^*\|_2^2 = n$ and $\|\varepsilon^0\|_2 \le \frac{\sqrt{n}}{2}$. Hence, taking $r = \frac{1}{\beta_0}$ in Proposition 4, we get

$$d_2(z^1, z^*) \le 2\left\| \frac{1}{\beta_0}\left(I + \frac{\alpha}{n}\Delta\right)\varepsilon^0 + \frac{1}{\beta_0}\cdot\frac{\alpha}{n}\Delta z^*\right\|_2$$

$$\le 2|\beta_0^{-1}|\cdot\left(\left\|\left(I + \frac{\alpha}{n}\Delta\right)\varepsilon^0\right\|_2 + \frac{\alpha}{n}\|\Delta z^*\|_2\right)$$

$$\le \frac{16}{7\alpha + 8}\left[\left(1 + \frac{\alpha}{n}\|\Delta\|\right)\|\varepsilon^0\|_2 + \frac{\alpha}{\sqrt{n}}\|\Delta\|\right]$$

$$= \mu \cdot d_2(z^0, z^*) + \nu \cdot \frac{8\|\Delta\|}{\sqrt{n}}.$$

Now, for the inductive step, we compute

$$\|\varepsilon^{k+1}\|_2 = d_2(z^{k+1}, z^*) \le \mu \cdot d_2(z^k, z^*) + \nu \frac{8\|\Delta\|}{\sqrt{n}}$$

$$= \mu \cdot \|\varepsilon^k\|_2 + \nu \frac{8\|\Delta\|}{\sqrt{n}}$$

$$\le \frac{8(\alpha\|\Delta\| + n)}{(7\alpha + 8)\sqrt{n}} + \frac{16\alpha\|\Delta\|}{(7\alpha + 8)\sqrt{n}} \quad \text{(plug in } \mu \text{ and } \nu\text{)}$$

$$\le \sqrt{n}/2,$$

where the last step uses the assumption that $\|\Delta\| \le \frac{n}{16}$ and $\alpha \ge 2$.

Now, using the same argument as above, one can show that $|\beta_{k+1}| \geq 1 + \frac{7\alpha}{8}$ and

$$d_2(z^{k+2}, z^*) \leq \mu \cdot d_2(z^{k+1}, z^*) + \nu \frac{8\|\Delta\|}{\sqrt{n}}.$$

This completes the proof.

- As mentioned earlier, Theorem 1 does not establish the convergence of GPM. To study this issue, we need a better understanding of the stationary points of (P). Towards that end, let us view T^n as a smooth manifold embedded in $\mathbb{C}^n$. Given $z \in T \subseteq \mathbb{C}$, the tangent space at z is given by

$$\mathcal{T}_z T = \{ w \in \mathbb{C} : \mathrm{Re}(z\bar{w}) = 0 \}.$$

In view of the product structure of T^n, we then conclude that the tangent space at $z \in T^n \subseteq \mathbb{C}^n$ is given by

$$\begin{aligned} \mathcal{T}_z T^n &= \mathcal{T}_{z_1} T \times \mathcal{T}_{z_2} T \times \cdots \times \mathcal{T}_{z_n} T \\ &= \{ w \in \mathbb{C}^n : \mathrm{Re}(z_j \bar{w}_j) = 0 \quad \forall j \}. \end{aligned}$$

Now, it is not hard to verify that the projector onto $\mathcal{T}_z T^n$ is given by

$$\Pi_{\mathcal{T}_z T^n}(w) = w - \mathrm{Diag}(\mathrm{Re}(z_j \bar{w}_j)) z.$$

Let $f(z) = -z^H C z$ be the objective function to be minimized. The Riemannian gradient of f on T^n is then defined as

$$\begin{aligned} \mathrm{grad}\, f(z) &= \Pi_{\mathcal{T}_z T^n}(\nabla f(z)) \\ &= 2\left(\mathrm{Diag}(\mathrm{Re}((Cz)_j \bar{z}_j)) - C\right) z. \end{aligned}$$

Let $S(z) = \mathrm{Diag}(\mathrm{Re}((Cz)_j \bar{z}_j)) - C$. Then, we can express the first-order optimality condition of (P) as

$$0 = \operatorname{grad} f(z) = S(z)z.$$

In the sequel, we shall also consider the second-order optimality condition of (P), which involves the Riemannian Hessian of f on T^n, denoted $\operatorname{Hess} f(x)$. The Riemannian Hessian is defined by the projection of the directional derivatives of the Riemannian gradient onto the tangent space; i.e.,

$$(\operatorname{Hess} f(z))(w) = \Pi_{\mathcal{T}_z T^n}(D \operatorname{grad} f(x))(w) = \Pi_{\mathcal{T}_z T^n}(2S(z)w).$$

Then, using the fact that $\Pi_{\mathcal{T}_z T^n}$ is self-adjoint, the second-order optimality condition is given by

$$w^H(\operatorname{Hess} f(z))(w) = 2w^H S(z) w \geq 0 \quad \forall w \in \mathcal{T}_z T^n.$$

- Now, it can be shown that if $\hat{z}$ is an optimal solution to (P), then it satisfies both the first- and second-order optimality conditions. To gain further insight into the structure of $\hat{z}$, we establish the following result:

<u>Proposition 5.</u>

(i) $z \in T^n$ satisfies the first-order optimality condition iff $(Cz)_j \bar{z}_j$ is real for all j.

(ii) If $\operatorname{diag}(C) \geq 0$ and $z \in T^n$ satisfies both the first- and second-order optimality conditions, then $(Cz)_j \bar{z}_j \geq 0$ and $z^H C z = \|Cz\|_1$

<u>Proof</u>: (i) Observe that since $z \in T^n$,

$$S(z)z = 0 \iff \operatorname{Re}((Cz)_j \bar{z}_j) z_j = (Cz)_j \quad \forall j$$

$$\iff \operatorname{Re}((Cz)_j \bar{z}_j) = (Cz)_j \bar{z}_j \quad \forall j.$$

(ii) Let e_j be the j^{th} basis vector of $\mathbb{R}^n$. Then, it is clear that $w = (i z_j) e_j \in T_z T^n$. Hence, we have

$$0 \leq w^H S(z) w = |z_j|^2 \cdot e_j^H S(z) e_j = S(z)_{jj}$$
$$= (Cz)_j \bar{z}_j - C_{jj}.$$

It follows that if $C_{jj} \geq 0$, then $(Cz)_j \bar{z}_j \geq 0$. Moreover, we have $(Cz)_j \bar{z}_j = |(Cz)_j|$.

- Note that we can also express $S(z)$ as

$$S(z) = \mathrm{Diag}(\mathrm{Re}((\tilde{C}z)_j \bar{z}_j)) - \tilde{C},$$

where $\tilde{C} = \frac{n}{\alpha}(I + \frac{\alpha}{n} C) = C + \frac{n}{\alpha} I$. Hence, the proof of Proposition 5 also shows that if $\mathrm{diag}(\tilde{C}) \geq 0$ and $z \in T^n$ satisfies both the first- and second-order optimality conditions, then $(\tilde{C}z)_j \bar{z}_j \geq 0$ and $z^H \tilde{C} z = \|\tilde{C} z\|_1$. In particular, we have

$$(\mathrm{Diag}(|\tilde{C}z|) - \tilde{C}) z = (\mathrm{Diag}(|Cz|) - C) z = 0.$$