Recovering objects from pairwise measurements

1) Localization

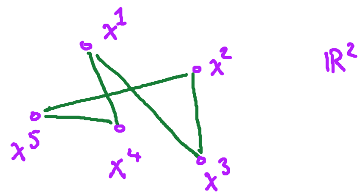
Let $x^1, \dots, x^n$ be points in a metric space $(\mathcal{M}, \rho)$.

Given noisy measurements

$$d_{ij} \simeq \rho(x^i, x^j), \quad (i,j) \in E \subseteq [n] \times [n]$$

Recover $x^1, \dots, x^n$.

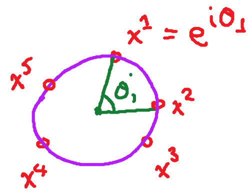
e.g.: $\mathcal{M} = \mathbb{R}^n$, $\rho(x,y) = \|x-y\|_2$



Note: The solution will not be unique because any rigid motion will preserve distances.

Applications: Sensor network localization, Statistics (multidimensional scaling)

e.g. $\mathcal{M} = S^1$, $\rho(x,y) = \|x-y\|_2$



Note: Again, the solution will not be unique because rotation will preserve distance.

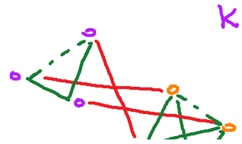
Application: Phase Synchronization

2) Community Detection

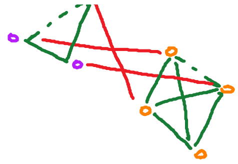
Suppose that we have n vertices. Assume that they can be partitioned into $K \geq 2$ groups (communities).

Let $h_j \in \{0,1\}^K$ be the membership indicator of vertex j; i.e.,

$K=2, n=7$



indicator of vertex j; i.e.,



$$h_j = e_k \quad \text{if vertex } j \text{ belongs to community } k.$$

Given noisy measurements

$$A_{j\ell} \simeq h_j^T h_\ell \qquad (A_{j\ell} \text{ is a noisy adjacency matrix of the graph})$$

recover $h_1, \dots, h_n$.

Phase Synchronization

Goal: Recover a collection of phases $\{ \underbrace{e^{i\theta_k}}_{= z_k} \}_{k=1}^{n}$ from a set of noisy measurements of relative phases $\{ e^{i(\theta_j - \theta_k)} \}$, $1 \le j < k \le n$.

Setup: Let z^* be the ground truth with

$$z^* \in \mathbb{T}^n \triangleq \{ w \in \mathbb{C}^n : |w_1| = \cdots = |w_n| = 1 \}.$$

Consider measurements of the form (additive noise model)

$$C_{j\ell} = z_j^* \bar{z}_\ell^* + \Delta_{j\ell}, \quad 1 \le j < \ell \le n.$$

where $\overline{(\cdot)}$ is the complex conjugate and $\Delta_{j\ell}$ is the noise associated with the (j,ℓ)-th measurement.

Least-squares formulation:

$$\hat{z} \in \underset{z \in \mathbb{T}^n}{\operatorname{argmin}} \sum_{1 \le j < \ell \le n} |C_{j\ell} - z_j \bar{z}_\ell|^2$$

Using the fact that for $z \in \mathbb{T}^n$, $|z_j \bar{z}_\ell|^2 = 1$, we can simply the above to

$$\hat{z} \in \underset{z \in \mathbb{T}^n}{\arg\max} \; z^H C z \qquad \text{— (P)}$$

where $C \in \mathbb{H}^n$ is obtained from the measurements with

↳ Set of $n \times n$ Hermitian matrices ($C = C^H$)

$C_{jj} = 1$ $\forall j$. Note that there can be multiple optimal solutions to (P). Indeed, if $\hat{z}$ is optimal, then so is $e^{i\theta}\hat{z}$ for any $\theta \in [0, 2\pi)$. Hence, we can only recover z^* up to a common phase. This motivates the following metric to measure the closeness of $z \in \mathbb{T}^n$ to the ground truth $z^* \in \mathbb{T}^n$:

$$d_2(z, z^*) = \min_{\theta \in [0, 2\pi)} \| z - e^{i\theta} z^* \|_2$$

<u>Next:</u> Bound the estimation error $d_2(\hat{z}, z^*)$