Recall the least-squares formulation:

$$\hat{z} \in \operatorname*{argmax}_{z \in \mathbb{T}^n} z^H C z,$$

where $\mathbb{T}^n \triangleq \{ w \in \mathbb{C}^n : |w_1| = \cdots = |w_n| = 1 \}$, the measurement model is given by

$$C_{j\ell} = z_j^* \bar{z}_\ell^* + \Delta_{j\ell}, \quad 1 \le j < \ell \le n; \quad C_{jj} = 1, \ \forall j,$$

and the performance metric is given by

$$d_2(z, z^*) = \min_{\theta \in [0, 2\pi)} \| z - e^{i\theta} z^* \|_2 .$$

Remark: Another more stringent metric is

$$d_\infty(z, z^*) = \min_{\theta \in [0, 2\pi)} \| z - e^{i\theta} z^* \|_\infty$$

This measures the componentwise discrepancy.

Proposition 1: Let $z \in \mathbb{C}^n$ be such that $\|z\|_2^2 = n$ and $(z^*)^H C (z^*) \le z^H C z$ (note that $z = \hat{z}$ satisfies these conditions). Then,

$$d_2(z, z^*) = \sqrt{2(n - |z^H z^*|)} \le \frac{4\|\Delta\|}{\sqrt{n}}$$

Proof: By definition,

$$\begin{aligned} d_2(z,z^*)^2 &= \min_{\theta \in [0,2\pi)} \| z - e^{i\theta} z^* \|_2^2 \\ &= 2\left(n - \max_{\theta \in [0,2\pi)} \operatorname{Re}\left(e^{i\theta} z^H z^* \right) \right) \\ &= 2(n - |z^H z^*|). \end{aligned}$$

Without loss, we can assume $z^H z^* = |z^H z^*|$, so that $d_2(z,z^*) = \|z - z^*\|_2$

By assumption,

$$\underset{\parallel}{z^H C z} = |z^H z^*|^2 + z^H \Delta z \ge (z^*)^H C (z^*) = n^2 + (z^*)^H \Delta z^*$$

$z^* z^{*H} + \Delta$

$$\Rightarrow \quad \underbrace{n^2 - |z^H z^*|^2}_{\substack{\shortparallel \\ (n - |z^H z^*|)(n + |z^H z^*|)}} \leq z^H \Delta z - (z^*)^H \Delta (z^*)$$

Dividing by n and noting $\frac{1}{n}(n + |z^H z^*|) \geq 1$, $|z^H z^*| \leq n$,

$$n - |z^H z^*| \leq \frac{1}{n}\left(z^H \Delta z - (z^*)^H \Delta (z^*)\right)$$

$$= \frac{1}{n} \operatorname{Re}\left[(z - z^*)^H \Delta (z + z^*)\right]$$

$$\leq \frac{1}{n} \underbrace{\|\Delta\|}_{\substack{\text{operator norm} \\ = \text{ largest singular value of } \Delta}} \cdot \|z - z^*\|_2 \cdot \underbrace{\|z + z^*\|_2}_{\leq \|z\|_2 + \|z^*\|_2 = 2\sqrt{n}}$$

$$\leq \frac{2}{\sqrt{n}} \|\Delta\| \, \|z - z^*\|_2$$

Now, note that without loss of generality, we may assume that $z^H z^* = |z^H z^*|$. Consequently, $d_2(z, z^*) = \|z - z^*\|_2$.

This gives

$$d_2(z, z^*)^{\cancel{2}} \leq \frac{4}{\sqrt{n}} \|\Delta\| \cdot \cancel{d_2(z, z^*)}$$

Which is the desired result.

Observe that $\hat{z}$ is not the only vector that satisfies the assumptions of the proposition. Consider a leading eigenvector u of C, properly scaled. This also satisfies the assumptions (verify). However, u need not be in T^n. This leads us to consider the following the spectral estimator $V_c \in T^n$:

$$\text{Set} \quad (V_c)_j = \begin{cases} \dfrac{u_j}{|u_j|} & \text{if } u_j \neq 0 \\[2ex] \dfrac{a^H u}{|a^H u|} & \text{if } u_j = 0, \quad \text{where } a \in \mathbb{C}^n \text{ is s.t. } a^H u \neq 0 \end{cases}$$

However, v_c may not satisfy $(z^*)^H C (z^*) \le (v_c)^H C (v_c)$. To establish the estimation error of v_c, we need to link u and v_c.

Proposition 2: For any $w \in \mathbb{C}^n$, $z \in \mathbb{T}^n$, we have

$$\Big\| \underbrace{\overset{\Pi_{\mathbb{T}^n}(w)}{\overset{\shortparallel}{\frac{w}{|w|}}}} - \overset{\Pi_{\mathbb{T}^n}(z)}{z} \Big\|_2 \le 2 \| w - z \|_2$$

apply componentwise projection onto $\mathbb{T}^n$

With Proposition 2, we can prove

Proposition 3: $d_2(v_c, z^*) \le \dfrac{8 \|\Delta\|}{\sqrt{n}}$.

Proof: Let $u \in \mathbb{C}^n$ be a leading eigenvector of C with $\|u\|_2^2 = n$. Then, u satisfies the requirements of Proposition 1. Without loss, assume that $u^H z^* = |u^H z^*|$, s.t. $d_2(u, z^*) = \| u - z^* \|_2$.

By definition,

$$d_2(v_c, z^*) \le \| v_c - \overset{\mathbb{T}^n}{\overset{\cup}{z^*}} \|_2 \underset{\text{Proposition 2}}{\le} 2 \| u - z^* \|_2 = 2\, d_2(u, z^*)$$

$$\underset{\text{Proposition 1}}{\le} \frac{8 \|\Delta\|}{\sqrt{n}}$$