Recall the measurement model

$$C = z^* z^{*H} + \Delta,$$

where $z^* \in \mathbb{T}^n \triangleq \{ w \in \mathbb{C}^n : |w_i| = 1 \ \forall i \}$ is the ground truth,

$\Delta \in \mathbb{H}^n$ is Hermitian with a zero diagonal,

Consider the eigenvector estimator

$$(E) \qquad u \in \operatorname{argmax} \ z^H C z \quad \text{s.t.} \ \|z\|_2^2 = n$$

Proposition 1: $d_2(u, z^*) \le \frac{4\|\Delta\|}{\sqrt{n}}$, $\quad d_2(v_c, z^*) \le \frac{8\|\Delta\|}{\sqrt{n}}$

$d_2(u, z^*) = \min_{\theta \in [0,2\pi)} \|u - e^{i\theta} z^*\|_2$, $\quad v_c = \Pi_{\mathbb{T}^n}(u)$

Today: We assume that $\Delta = \sigma W$, W is a Hermitian matrix with zero diagonal and independent standard complex Gaussian random variables above the diagonal.

(Recall: $w \sim CN(0,1) \iff \mathrm{Re}(w) \sim N(0,\frac{1}{2})$, $\mathrm{Im}(w) \sim N(0, \frac{1}{2})$ and $\mathrm{Re}(w) \perp \mathrm{Im}(w)$)

Fact: whp, $\|W\| = O(\sqrt{n})$ (Exercise)

Together with Proposition 1, we have $d_2(v_c, z^*) = O(\sigma)$.

Q: How about $d_\infty(u, z^*)$ and $d_\infty(v_c, z^*)$?

$$= \min_{\theta \in [0,2\pi)} \|u - e^{i\theta} z^*\|_\infty$$

Challenge: Assume without loss $u^H z^* = |u^H z^*|$ (why?). Then,

$$|u_j - z_j^*| = \left| \frac{(Cu)_j}{\lambda_1(C)} - z_j^* \right| \qquad (\because Cu = \lambda_1(C) u)$$

$$\le \left| \frac{(z^{*H} u) z_j^*}{\lambda_1(C)} - z_j^* \right|$$

$$\le \left| \frac{(z^{*}u)\, z_j}{\lambda_1(C)} - z_j^{*} \right| + \frac{\sigma\, |(Wu)_j|}{\lambda_1(C)} \qquad (\because\ C = z^* z^{*H} + \sigma W)$$

$$= \left| \frac{|z^{*H}u|}{\lambda_1(C)} - 1 \right| + \frac{\sigma\, |(Wu)_j|}{\lambda_1(C)} \qquad \forall j.$$

The difficulty lies in $(Wu)_j$, since W and u are not independent. If they were independent, then if we let w^{j^H} be the j^{th} row of W, we have

$$(Wu)_j = (w^j)^H u = \sum_k \underbrace{\bar{w}^j_k u_k}_{\mathrm{var} = |u_k|^2} \sim \mathcal{CN}\Big(0, \underbrace{\sum_k |u_k|^2}_{n}\Big) \Rightarrow \mathbb{E}\left[|(w^j)^H u|^2\right] = n$$

By some standard calculation, we get

$$\|Wu\|_\infty = \max_{1 \le j \le n} |(w^j)^H u| = O(\sqrt{n \log n})$$

whp (Exercise)

To tackle the possible dependence between W and u, a powerful idea is the leave-one-out technique. Consider

$$(E) \quad u \in \operatorname{argmax}\ z^H \overset{z^* z^{*H} + \sigma W}{\overset{\parallel}{C}}\, z \quad \text{s.t. } \|z\|_2^2 = n \qquad \xrightarrow{\text{Perturb}} \qquad (E^{(j)}) \quad u^{(j)} \in \operatorname{argmax}\ z^H C^{(j)} z \quad \text{s.t. } \|z\|_2^2 = n,$$

where $C^{(j)} = z^* z^{*H} + \sigma W^{(j)}$, $W^{(j)}$ is the same as W except that its j^{th} row and j^{th} col are zero

$$W = \begin{array}{cc} \begin{array}{ccc} & w^j & \end{array} & \\ \begin{bmatrix} * & \boxed{*} & * \\ \boxed{*} & \boxed{*} & \boxed{*} \\ * & \boxed{*} & * \end{bmatrix} & j \\ \begin{array}{ccc} & j & \end{array} & \end{array} \longrightarrow W^{(j)} = \begin{bmatrix} * & 0 & * \\ 0 & 0 & 0 \\ * & 0 & * \end{bmatrix}$$

Observe: $C^{(j)}$ is independent of w^j
↳ j^{th} col of W

$\Rightarrow$ $u^{(j)}$ is independent of w^j.

For any $\theta \in [0, 2\pi)$, let us write

$$
\begin{aligned}
|(Wu)_j| = |(w^j)^H u| &\le |(w^j)^H (u - e^{i\theta} u^{(j)})| + |(w^j)^H u^{(j)}| \\
&\le \|w^j\|_2 \, \|u - e^{i\theta} u^{(j)}\|_2 + \underbrace{|(w^j)^H u^{(j)}|}_{\substack{\text{can analyze} \\ \text{b/c of independence}}}
\end{aligned}
$$

Hence,

$$
|(Wu)_j| \le \underbrace{\|w^j\|_2}_{\checkmark} \cdot \underbrace{d_2(u, u^{(j)})}_{?} + \underbrace{|(w^j)^H u^{(j)}|}_{\checkmark}.
$$

Idea: Since we perturb only one row and one col of C to get $C^{(j)}$, we expect $d_2(u, u^{(j)})$ should be small.

Here is the tool to formalize this:

Theorem (Davis-Kahan)

Suppose that $A, E \in \mathbb{H}^n$ are given and set $\hat{A} = A + E$.

Let $\delta(A) \triangleq \lambda_1(A) - \lambda_2(A)$ be the top eigengap of A, $v, \tilde{v}$ be the leading eigenvectors of $A, \tilde{A}$, respectively, with $\|v\|_2^2 = \|\tilde{v}\|_2^2 = n$. If $\delta(A) > \|E\|$, then

$$
d_2(v, \tilde{v}) \le \frac{\sqrt{2}\, \|Ev\|_2}{\delta - \|E\|}
$$

Intuition: Two ways of interpreting C as a perturbation.

1°:
$$
\underset{\substack{\shortparallel \\ \tilde{A}}}{C} = \underset{\substack{\shortparallel \\ A}}{z^* z^{*H}} + \underset{\substack{\shortparallel \\ E}}{\sigma W}
$$

Then, $\|Ev\|_2 \le \|E\| \cdot \|v\|_2 = \sigma \underbrace{\|W\|}_{O(\sqrt{n})} \cdot \underbrace{\|v\|_2}_{\sqrt{n}} = \underbrace{O(\sigma n)}_{\text{Seems too large}}$ whp.

2°:
$$
\underset{\shortparallel}{C} = \underset{\shortparallel}{C^{(j)}} + \underset{\shortparallel}{\sigma (W - W^{(j)})} \qquad (C^{(j)} = z^* z^{*H} + \sigma W^{(j)})
$$

$\underline{2^\circ}$:

$$\underset{\tilde{A}}{\underset{\shortparallel}{C}} = \underset{A}{\underset{\shortparallel}{C^{(j)}}} + \underset{E}{\underset{\shortparallel}{\sigma (W - W^{(j)})}} \qquad \left(C^{(j)} = z^* z^{*H} + \sigma W^{(j)} \right)$$

Then, $\| E v \|_2 = \sigma \underbrace{\| (W - W^{(j)}) u^{(j)} \|_2}_{\text{expect to be small because } W \text{ and } W^{(j)} \text{ are very similar}}$ — leading eigenvector of $A = C^{(j)}$

Apply Davis-Kahan to $A = C^{(j)}$, $E = \sigma (W - W^{(j)})$

We compute $\delta(A)$ using Weyl's inequality :

If $\tilde{A} = A + E$

$\mu_i \quad \nu_i \quad \rho_i \quad \longleftarrow$ ordered eigenvalues

then $\nu_i + \rho_n \le \mu_i \le \nu_i + \rho_1 \quad \forall i$.

By Weyl's inequality, $\qquad C^{(j)} = z^* z^{*H} + \sigma W^{(j)}$

$$\delta(C^{(j)}) = \lambda_1 (C^{(j)}) - \lambda_2 (C^{(j)})$$

$$\ge \underbrace{\lambda_1 (z^* z^{*H})}_{= n} - \sigma \| W^{(j)} \| - \underbrace{\lambda_2 (z^* z^{*H})}_{= 0} - \sigma \| W^{(j)} \|$$

$$= n - 2 \sigma \| W^{(j)} \| .$$

Then, by Davis-Kahan,

$$d_2 (u, u^{(j)}) \le \frac{\sqrt{2}\, \sigma \, \| (W - W^{(j)}) u^{(j)} \|_2}{\delta (C^{(j)}) - \sigma \| W - W^{(j)} \|}$$

leading eigenvector of C

leading eigenvector of $C^{(j)}$