Phase Synchronization (cont'd)

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Recall

$$(E) \quad u \in \arg\max_{\text{s.t. } \|z\|_2^2 = n} z^H \underset{\overset{\parallel}{z^* z^{*H} + \sigma W}}{C} z \quad \xrightarrow{\text{Perturb}} \quad (E^{(j)}) \quad u^{(j)} \in \arg\max_{\text{s.t. } \|z\|_2^2 = n} z^H \underset{\overset{\parallel}{z^* z^{*H} + \sigma W^{(j)}}}{C^{(j)}} z ,$$

$$W = \begin{array}{cc} & \begin{array}{ccc} & W^j & \end{array} \\ & \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} j \\ & \begin{array}{ccc} & j & \end{array} \end{array} \rightarrow \quad W^{(j)} = \begin{bmatrix} * & 0 & * \\ 0 & 0 & 0 \\ * & 0 & * \end{bmatrix}$$

Under the assumption $u^H z^* = |u^H z^*|$, we have $d_\infty(u, z^*) = \|u - z^*\|_\infty$ $\left(= \min_{\theta \in [0,2\pi)} \|u - e^{i\theta} z^*\|_\infty\right)$

Now,

$$|u_j - z_j^*| \leq \left| \frac{|z^{*H} u|}{\lambda_1(C)} - 1 \right| + \frac{\sigma |(Wu)_j|}{\lambda_1(C)} \quad \forall j \qquad — (\dagger)$$

and we have

$$|(Wu)_j| \leq \underbrace{\|W^j\|_2}_{\checkmark} \cdot \underbrace{d_2(u, u^{(j)})}_{?} + \underbrace{|\overbrace{(W^j)^H u^{(j)}}^{\text{independent}}|}_{\checkmark} . \qquad — (\Delta)$$

By the Davis-Kahan theorem,

$$d_2(u, u^{(j)}) \leq \frac{\sqrt{2}\,\sigma \,\|(W - W^{(j)}) u^{(j)}\|_2}{\delta(C^{(j)}) - \sigma \|W - W^{(j)}\|} \qquad — (*)$$

leading eigenvector of C

leading eigenvector of $C^{(j)}$

and $\delta(C^{(j)}) \triangleq \lambda_1(C^{(j)}) - \lambda_2(C^{(j)}) \geq n - 2\sigma \|W^{(j)}\|$.

Weyl's inequality

Fact: Let $W \in \mathbb{H}^n$ be a Wigner matrix (i.e., a Hermitian matrix with zero on the diagonal and independent standard complex Gaussian entries above the diagonal). Then, whp,

(i) $\|W\| \leq c_1 \sqrt{n}$, (ii) $\|W^{(j)}\| \leq c_2 \sqrt{n}$, (iii) $\|W - W^{(j)}\| \leq c_3 \sqrt{n}$,

(iv) $\|w^j\|_2 \le C_4\sqrt{n}$.

Assuming $\sigma < c\sqrt{n}$ for some sufficiently small $c > 0$, by (*) and the fact, we have

$$d_2(u, u^{(j)}) \le \frac{\sqrt{2}\,\sigma\,\|(W - W^{(j)})u^{(j)}\|_2}{n - 2\sigma\|W^{(j)}\| - \sigma\|W - W^{(j)}\|}$$

$$\overset{\text{whp}}{\lesssim} c' \frac{\sigma}{n - \sigma\sqrt{n}} \cdot \|(W - W^{(j)})u^{(j)}\|_2$$

$$\overset{\sigma < c\sqrt{n}}{\lesssim} \frac{c''}{\sqrt{n}} \|(W - W^{(j)})u^{(j)}\|_2$$

Putting this into (Δ) and applying the fact,

$$|(Wu)_j| \le \underbrace{\|w^j\|_2}_{\sim\sqrt{n}} \cdot d_2(u, u^{(j)}) + |(w^j)^H u^{(j)}|.$$

$$\overset{\text{whp}}{\lesssim} c''' \left[\|(W - W^{(j)})u^{(j)}\|_2 + |(w^j)^H u^{(j)}| \right]$$

$$\Rightarrow \|Wu\|_\infty \lesssim c''' \cdot \max_j \left[\|(W - W^{(j)})u^{(j)}\|_2 + |(w^j)^H u^{(j)}| \right]$$

$$\lesssim c'''' \cdot \max_j \|\overbrace{(W - W^{(j)})\,u^{(j)}}^{\text{independent}}\|_2$$

$$(\because (w^j)^H u^{(j)} = j^{\text{th}} \text{ coord of } (W - W^{(j)})u^{(j)})$$

Idea: $W - W^{(j)}$ involves independent standard complex Gaussian entries, and since $u^{(j)}$ is independent of $W - W^{(j)}$, we can think of $[(W - W^{(j)})u^{(j)}]_j$ as a sum of independent complex Gaussian random variables

Fact: Whp,

$$\max_j \|(W - W^{(j)})u^{(j)}\|_2 \lesssim c\left[\sqrt{n\log n} + \sqrt{n}\max_j |u^{(j)}_j|\right]$$

$$\max_j \|(W - W^{(j)})u^{(j)}\|_2 \le c\left[\sqrt{n\log n} + \sqrt{n}\,\max_j |u_j^{(j)}|\right]$$

$$(W - W^{(j)})\,u^{(j)} = \begin{array}{c} j \\ \left[\qquad\qquad\right] \end{array} \underbrace{\begin{bmatrix} u_1^{(j)} \\ \vdots \\ u_j^{(j)} \\ \vdots \\ u_n^{(j)} \end{bmatrix}}_{=\,u^{(j)}}$$

Claim: $|u_j^{(j)}| \le 2 \quad \forall j$, under the assumption that $\sigma < c\sqrt{n}$.

Sketch: By definition,

$$W^{(j)} = \begin{array}{c} j \\ \begin{bmatrix} & 0 & \\ 0 & \cdots & 0 \\ & 0 & \end{bmatrix} \end{array} j$$

$$\underbrace{\lambda_1(C^{(j)})}_{\text{scalar}}\, u^{(j)} = \underbrace{C^{(j)}}_{= z^* z^{*H} + \sigma W^{(j)}} u^{(j)} = (z^{*H} u^{(j)})\, z^* + \underbrace{\sigma W^{(j)} u^{(j)}}_{j^{\text{th}}\text{ coord} = 0}$$

Hence, whp, $\|Wu\|_\infty \le c\cdot\sqrt{n\log n}$. Now, recall from (+) that

$$|u_j - z_j^*| \le \left|\frac{|z^{*H}u|}{\lambda_1(C)} - 1\right| + \frac{\sigma|(Wu)_j|}{\lambda_1(C)} \qquad \forall j$$

To bound $\|u - z^*\|_\infty$, it remains to bound $|z^{*H}u|$ and $\lambda_1(C)$

1°: Since $u^H z^* = |u^H z^*|$, we have $d_2(u, z^*)^2 = 2(n - |u^H z^*|)$.

Also, whp, $d_2(u, z^*) = O(\sigma)$. Hence,

$$n \underset{\substack{\uparrow \\ \text{by Cauchy} \\ \text{Schwarz}}}{\ge} |u^H z^*| \ge n - O(\sigma^2).$$

2°: Since $C = z^* z^{*H} + \sigma W$, by Weyl's inequality,

$$|\lambda_1(C) - \underbrace{\lambda_1(z^* z^{*H})}_{= n}| \le \sigma\|W\| \underset{\text{whp}}{\le} c\cdot\sigma\sqrt{n}.$$

Thus,

$$\|u - z^*\|_\infty \le \frac{||u^H z^*| - n| + |n - \lambda_1(C)|}{\lambda_1(C)} + \frac{\sigma\|Wu\|_\infty}{\lambda_1(C)}$$

$$\overset{\text{whp}}{\le} c\cdot\left[\frac{\sigma^2 + \sigma\sqrt{n}}{n} + \frac{\sigma\sqrt{n\log n}}{n}\right]$$

$$\overset{\text{whp}}{\leq} \; c \cdot \left[\frac{\sigma^2 + \sigma\sqrt{n}}{n} + \frac{\sigma\sqrt{n \log n}}{n} \right]$$

$$\overset{\sigma < c' \cdot \sqrt{n}}{=} \; O\left(\sigma \sqrt{\frac{\log n}{n}} \right)$$

Algorithmic aspect of phase synchronization

Recall we are interested in

$$(P) \qquad \hat{z} \in \begin{array}{ll} \operatorname{argmax} & z^H C z \\ \text{s.t.} & z \in \mathbb{T}^n \end{array}$$

Note that the eigenvector estimator $v_c = \Pi_{\mathbb{T}^n}(u)$ is just a feasible solution to (P)

leading eigenvector of C

Q: Can we refine v_c to get a "better" solution?

What do we mean by "refining" v_c?

Idea: How about projected gradient ascent?

initialize: $z^0 = v_c$

repeat: $w^k \leftarrow \underset{\text{current iterate}}{z^k} + \underset{\text{step size}}{\frac{\alpha_k}{n}} \underbrace{C z^k}_{\text{gradient}}$

$$z^{k+1} \leftarrow \frac{w^k}{|w^k|}$$