https://www.se.cuhk.edu.hk/~manchoso/2122/x2te2109

$$(P) \quad v^* = \inf_{x \in X} f(x)$$

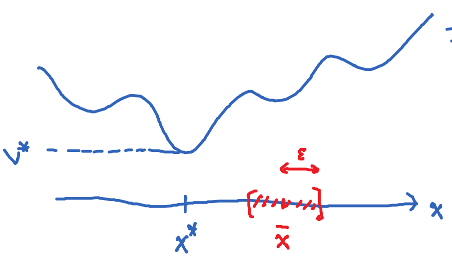
infimum "minimize"

- $f: \mathbb{R}^n \to \mathbb{R}$ objective function
- $X \subseteq \mathbb{R}^n$ feasible region ; $x \in \mathbb{R}^n$ decision variable
- v^* : optimal value of (P)

(P)
- infeasible. $X = \emptyset$; $v^* = +\infty$ by convention
- feasible $X \neq \emptyset$
  - $v^* = -\infty$ e.g. $n=1$ $\inf\ x$ s.t. $x \le 0$
  - $v^* > -\infty$

Def: optimal solution
$x^* \in X$ s.t. $v^* = f(x^*)$

- Optimal solution exists
  e.g. $n=1$
  $\inf\ x$ s.t. $x \geq 0$
  $v^* = 0$, attained by $x^* = 0$
- optimal Solution does not exist
  e.g. $n=1$
  $\inf\ 1/x$ s.t. $x \geq 0$
  $v^* = 0$; $\exists\, x^* \geq 0$ s.t. $1/x^* = 0$?
  No!



Def. local minimizer

$\bar{x}$ is local minimizer if $\bar{x} \in X$ and
$\exists\, \varepsilon > 0$ s.t. $\forall x \in X \cap B(\bar{x}, \varepsilon)$, we have
$$f(\bar{x}) \le f(x)$$

$$\left(B(\bar{x}, \varepsilon) = \{ y \in \mathbb{R}^n : \|y - \bar{x}\|_2 \le \varepsilon \} \right)$$

center radius

Simple Examples of (P)

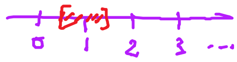
(1) Unconstrained : $X = \mathbb{R}^n$ $\qquad \inf_{x \in \mathbb{R}^n} f(x)$

If f is differentiable, then $\nabla f(x) = 0$ is necessary for optimality.

gradient $\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$

(2) Discrete : X is a discrete set; i.e.,

$$\forall x \in X, \ \exists \varepsilon > 0 \text{ s.t. } X \cap B(x, \varepsilon) = \{x\}$$



e.g. $X = \mathbb{Z}_+$, $X = \{0,1\}^n$

$X = [0,1]$ not discrete

Note: For discrete optimization problems, local minimality is meaningless! This is because every feasible solution is a local minimizer.

(3) Linear Programming (LP)

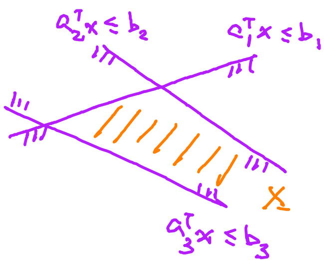
$$f(x) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = c^T x \quad \text{(linear function)}$$

$$c = (c_1, \cdots, c_n) \ ; \ x = (x_1, \cdots, x_n)$$

X : defined by a finite number of linear inequalities

$$X = \{ x \in \mathbb{R}^n : \underbrace{a_i^T x}_{\text{linear function}} \le b_i \ ; \ i = 1, \cdots, \underbrace{m}_{\text{finite}} \} \qquad a_i \in \mathbb{R}^n \ ; \ b_i \in \mathbb{R}$$

$$= \{ x \in \mathbb{R}^n : Ax \le b \} \qquad A = \begin{bmatrix} - a_1^T - \\ \vdots \\ - a_m^T - \end{bmatrix} \in \mathbb{R}^{m \times n}$$



Note: $u \le v$ $\iff$ $u_i \le v_i \ \forall i$

$$b = (b_1, \cdots, b_m) \in \mathbb{R}^m$$

What if we want $a_i^T x = b_i$? Simply consider

$$a_i^T x = b_i \iff \begin{cases} a_i^T x \le b_i \\ -a_i^T x \le -b_i \end{cases}$$

(4) Quadratic Programming (QP)

$$f(x) = \sum_{i,j=1}^{n} Q_{ij} x_i x_j = x^T Q x \qquad Q = [Q_{ij}] \in \mathbb{R}^{n \times n}$$

homogeneous

$f(\alpha x) = \alpha^2 f(x)$

X : Same as LP

Note: In the above definition, Q need not be symmetric.

However, observe that

$$x^T Q x = x^T \underbrace{\left(\frac{Q + Q^T}{2} \right)}_{\text{Symmetric}} x$$

Hence, we can assume without loss that Q is symmetric.