Definition: Let $P \subseteq \mathbb{R}^n$ be a non-empty polyhedron. We say that $\bar{x} \in P$ is a vertex of P if there exist n linearly independent active constraints at $\bar{x}$.

Consider

$$\min_{x \in P} \; h^T x \qquad \text{— (LP)}$$

where $h \in \mathbb{R}^n$ and $\phi \neq P \subseteq \mathbb{R}^n$ be a polyhedron.

Q: Does an optimal solution to (LP) exist?

① optimal value can be $-\infty$

e.g. $\inf \; x$ s.t. $x \le 0$

0

No optimal solution in this case.

② What if optimal value $> -\infty$?

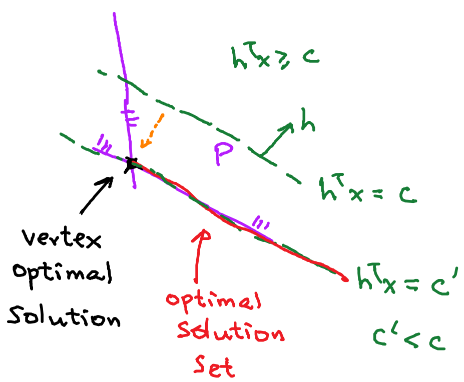
Can situation like $\inf_{x \ge 0} 1/x$ arise?

↳ opt. val. = 0, no opt. soln.

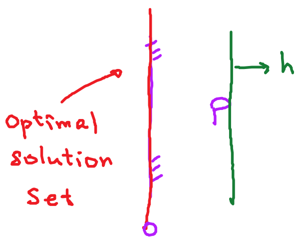
Theorem: Consider (LP). Then, either the optimal value is $-\infty$, or there exists an optimal solution. Moreover, if optimal solution exists and P has a vertex, then there exists a vertex optimal solution.

Example 1:

Note: There can be non-vertex optimal solutions.



Example 2:

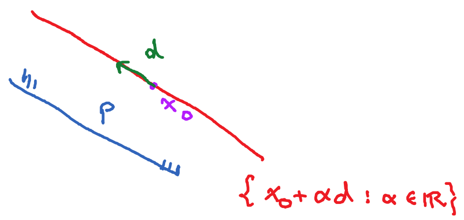


$\min\ x_1$

s.t. $x_1 \geq 0$

Q: How do we know if P has a vertex?

A: **Definition:** A polyhedron $P \subseteq \mathbb{R}^n$ contains a line if $\exists\, x_0 \in P$ and $d \in \mathbb{R}^n \setminus \{0\}$ s.t. $x_0 + \alpha d \in P \quad \forall \alpha \in \mathbb{R}$

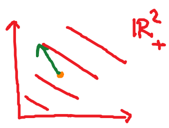


Theorem: Let $P \subseteq \mathbb{R}^n$ be a non-empty polyhedron.

Then, the following are equivalent:

1. P has a vertex
2. P does not contain a line.

Example: Consider $P = \{ x \in \mathbb{R}^n : \underbrace{Ax = b}_{\begin{cases} Ax \leq b \\ Ax \geq b \end{cases}},\ \underbrace{x \geq 0}_{x_i \geq 0\ \forall i} \} \neq \emptyset$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ are given.



1. Does P have a vertex?

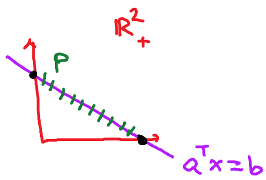
   Yes: Observe that $P = \{ x \in \mathbb{R}^n : Ax = b \} \cap \underbrace{\mathbb{R}^n_+}_{= \{ x \in \mathbb{R}^n : x_i \geq 0\ \forall i \}}$.

   Claim (Exercise): $\mathbb{R}^n_+$ does not contain a line.

   Since $P \subseteq \mathbb{R}^n_+$, P does not contain a line either.

2. Is $x = 0$ necessarily a vertex of P?

   No, unless $A(0) = b$



3. By definition, the m constraints $Ax = b$ are active.

   If $m < n$, then we need at least $n - m$ active constraints from the n constraints $x \geq 0$ to define a vertex.

Consider again (LP):

$$\min_{x \in P}\ h^T x \quad , \quad P = \{ x \in \mathbb{R}^n : g_i^T x \leq t_i,\ i = 1, \cdots, m \}$$

We don't know if P has a vertex. However, we can transform

(LP) into the following form:

$$(S) \qquad \begin{array}{ll} \min & c^T z \\ \text{s.t.} & Az = b \\ & z \geq 0 \end{array} \Bigg\} \text{ has a vertex}$$

① Observe that we can write $x = x^+ - x^-$, $x^+, x^- \geq 0$. Then, (LP) is the same as

$$\begin{array}{ll} \min & h^T(x^+ - x^-) \\ \text{s.t.} & g_i^T(x^+ - x^-) \leq t_i, \quad i = 1, \dots, m. \qquad \text{—} (\Delta) \\ & x^+, x^- \geq 0 \end{array}$$

② Add slack variables:

$$g_i^T(x^+ - x^-) \leq t_i \iff \underbrace{g_i^T(x^+ - x^-) + s_i = t_i}, \quad s_i \geq 0$$

slack variable

Then, we can write (Δ) as

$$\min \quad \underbrace{[h^T \;\; -h^T \;\; 0]}_{c^T} \underbrace{\begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix}}_{z}$$

$$\text{s.t.} \quad \underbrace{[g_i^T \;\; -g_i^T \;\; e_i^T]}_{a_i^T} \underbrace{\begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix}}_{z} = t_i, \quad i = 1, \dots, m, \qquad e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th}$$

$$e_i^T s = s_i$$

$$\begin{array}{c} n \\ n \\ m \end{array} \underbrace{\begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix}}_{z} \geq 0.$$

which is in the form of (S). Here, $z \in \mathbb{R}^{2n+m}$.