LP Duality Theory

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<u>Recall</u>: Every LP can be transformed into the following

Standard form LP:

$$V_p^* = \min \ c^T x \qquad c \in \mathbb{R}^n$$
$$\text{s.t. } Ax = b, \qquad A \in \mathbb{R}^{m\times n}, \ b \in \mathbb{R}^m \quad (S)$$
$$x \geq 0$$

Q: Given a feasible solution $\bar{x}$ to (S), how to certify its optimality?

Note: If $\exists x'$ feasible for (S) and $c^T x' < c^T \bar{x}$, then $\bar{x}$ cannot be optimal.

Idea: Find a lower bound on V_p^*. $(A^T y)_i \leq c_i \ \forall i$

Consider a vector $y \in \mathbb{R}^m$ s.t. $A^T y \leq c$. Then, for any $x \in \mathbb{R}^n$ feasible for (S),

$$b^T y \underset{Ax=b}{=} (Ax)^T y = \underbrace{x^T}_{x \geq 0} \underbrace{(A^T y)}_{A^T y \leq c} \leq c^T x$$

In particular, $b^T y$ is a lower bound on the objective value $c^T x$. This implies that $b^T y \leq V_p^*$.

To get the best lower bound using this approach, we should solve

$$V_d^* = \max \ b^T y \qquad \text{— (D)}$$
$$\text{s.t. } A^T y \leq c.$$

Observe: This is also an LP. It is called the dual of (S).

<u>Theorem</u> (Weak Duality)

Let $\bar{x} \in \mathbb{R}^n$ be feasible for (S) and $\bar{y}$ be feasible for (D). Then,

$$b^T \bar{y} \leq c^T \bar{x}.$$

In particular, we have $V_d^* \leq V_p^*$.

Corollary:

(1) If the optimal value of (S) is $-\infty$, then (D) is infeasible.

(2) If the optimal value of (D) is $+\infty$, then (S) is infeasible.

(3) Let $\bar{x}$ be feasible for (S) and $\bar{y}$ be feasible for (D). If the duality gap $\Delta(\bar{x},\bar{y}) \triangleq c^T\bar{x} - b^T\bar{y}$ is zero, then $\bar{x}$ is optimal for (S) and $\bar{y}$ is optimal for (D)

Q:

(1) Suppose that (S) is infeasible What can we say about (D)?

(2) Suppose that $\bar{x}$ is optimal for (S) and $\bar{y}$ is optimal for (D). Is it true that $\Delta(\bar{x},\bar{y}) = 0$?

A: For (1), it is possible that both (S) and (D) are infeasible.

e.g. $A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $c = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

(S): $Ax = b,\ x \geq 0$

$\Updownarrow$

$-x_1 + x_2 = 1,$

$-x_1 + x_2 = 2,$

$x_1, x_2 \geq 0$

(D): $A^T y \leq c$

$\Updownarrow$

$-y_1 - y_2 \leq -1,$

$y_1 + y_2 \leq -1$

For (2), we have

Theorem (Strong Duality)

Suppose that (S) has an optimal solution x^*. Then, (D) has an optimal solution y^* and $\Delta(x^*, y^*) = \underbrace{c^T x^*}_{= v_p^*} - \underbrace{b^T y^*}_{= v_d^*} = 0$.

Corollary

Suppose that both (S) and (D) are feasible. Then, both (S) and (D) have optimal solutions and $v_p^* = v_d^*$.

Proof of Corollary

1) Both (S) and (D) are feasible $\overset{\text{Why?}}{\underset{\text{(Exercise)}}{\Longrightarrow}}$ (S) has an optimal solution.

2) The result then follows from the theorem.