LP Duality Theory (cont'd)

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Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ be given. Recall the standard primal-dual pair

$$(S)\quad \begin{aligned} v_p^* = \min\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq 0 \end{aligned}, \qquad (D)\quad \begin{aligned} v_d^* = \max\ & b^T y \\ \text{s.t. } & A^T y \leq c \end{aligned}$$

(primal) (dual)

Theorem (Strong Duality)

Suppose that (S) has an optimal solution x^*. Then, (D) has an optimal solution y^* and $\Delta(x^*, y^*) = c^T x^* - b^T y^* = 0$.

Remark:

(Linear Optimization)

Find optimal solutions to (S) / (D)

(Linear Feasibility)

Find a feasible solution to a linear system, e.g., $Ax = b,\ x \geq 0$.

Observe that finding optimal solutions to (S)/(D) is equivalent to the following linear feasibility problem:

$$(x,y)\ \left\{\begin{array}{ll} Ax = b,\ x \geq 0 & \text{(primal feasibility)} \\ A^T y \leq c & \text{(dual feasibility)} \\ c^T x = b^T y & \text{(zero duality gap)} \end{array}\right\}\ \text{optimality conditions of LP}$$

Theorem (Complementarity)

Let $\bar{x}$ be feasible for (S) and $\bar{y}$ be feasible for (D). Then, they are optimal for their respective problems iff

$$\overset{\geq 0}{\bar{x}_i}\, \underbrace{\overset{\geq 0}{(c - A^T\bar{y})_i}} = 0 \quad \forall i \qquad (*)$$

$$\underset{\geq 0}{\bar{x}_i}\underbrace{(c - A^T\bar{y})_i}_{\geq 0} = 0 \quad \forall i \qquad \text{——} (*)$$

i^{th} primal variable $\longleftrightarrow$ i^{th} dual constraint

<u>Proof</u>: We compute

$$c^T\bar{x} - b^T\bar{y} \underset{A\bar{x}=b}{=} c^T\bar{x} - (A\bar{x})^T\bar{y} = \bar{x}^T(c - A^T\bar{y}) = \sum_{i=1}^{n} \bar{x}_i (c - A^T\bar{y})_i .$$

If $(*)$ holds, then $c^T\bar{x} = b^T\bar{y}$, so $\bar{x}$ is optimal for (S) and $\bar{y}$ is optimal for (D) by weak duality.

Conversely, we have $c^T\bar{x} = b^T\bar{y}$ by strong duality. Hence,

$$\sum_{i=1}^{n} \underbrace{\bar{x}_i (c - A^T\bar{y})_i}_{\geq 0} = 0 \quad \Longrightarrow \quad \bar{x}_i (c - A^T\bar{y})_i = 0 \quad \forall i$$

This gives us another set of optimality conditions for LP:

$$(x, y, s) \quad \begin{cases} Ax = b, \; x \geq 0 & \text{(primal feasibility)} \\ A^Ty + s = c, \; s \geq 0 & \text{(dual feasibility)} \\ x^Ts = 0 & \text{(complementarity)} \end{cases}$$

$$\text{(S)} \quad \begin{aligned} V_p^* = \min \;& c^Tx \\ \text{s.t.} \;& Ax = b \\ & x \geq 0 \end{aligned} , \qquad \text{(D)} \quad \begin{aligned} V_d^* = \max \;& b^Ty \\ \text{s.t.} \;& A^Ty \leq c \end{aligned}$$

<u>Example</u>: Consider

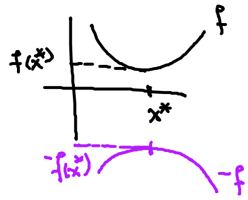
$$\begin{aligned} \min \quad & x_1 + 2x_2 + x_3 \\ \text{s.t.} \quad & x_1 - 2x_2 + x_3 \geq 2, \\ & -x_1 + x_3 \geq 4, \\ & 2x_1 + x_3 \geq 6, \\ & x_1 + x_2 + x_3 \geq 2, \\ & x_1 \geq 0, \\ & x_2 \geq 0, \end{aligned}$$

$$c = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 2 \end{bmatrix}$$

$$x_3 \geq 0$$

Hence, the above is the same as

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$



Convert into the form of (S)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax - s = b \\ & x, s \geq 0 \end{aligned} \quad \longleftrightarrow \quad \begin{aligned} \min \quad & (c,0)^T (x,s) \\ \text{s.t.} \quad & \begin{bmatrix} A & -I \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = b, \\ & (x,s) \geq 0 \end{aligned}$$

take the dual

$$\begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & \begin{bmatrix} A^T \\ -I \end{bmatrix} y \leq \begin{bmatrix} c \\ 0 \end{bmatrix} \end{aligned} \quad \longleftrightarrow \quad \begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T y \leq c \\ & y \geq 0 \end{aligned}$$

Convert into the form (D)

$$-\left[\begin{aligned} \max \quad & -c^T x \\ \text{s.t.} \quad & \begin{bmatrix} -A \\ -I \end{bmatrix} x \leq \begin{bmatrix} -b \\ 0 \end{bmatrix} \end{aligned} \right]$$

take the dual

$$-\left[\begin{aligned} \min \quad & (-b,0)^T (y,w) \\ \text{s.t.} \quad & \begin{bmatrix} -A^T & -I \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix} = -c, \\ & (y,w) \geq 0 \end{aligned} \right]$$

$$\updownarrow$$

$$\begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T y + w = c \\ & y, w \geq 0 \end{aligned} \quad \longleftrightarrow \quad -\left[\begin{aligned} \min \quad & -b^T y \\ \text{s.t.} \quad & -A^T y - w = -c \\ & y, w \geq 0 \end{aligned} \right]$$

Hence, the dual is given by

$$\begin{aligned} \max \quad & 2y_1 + 4y_2 + 6y_3 + 2y_4 \\ \text{s.t.} \quad & y_1 - y_2 + 2y_3 + y_4 \leq 1 & (x_1) \\ & -2y_1 + y_4 \leq 2 & (x_2) \\ & y_1 + y_2 + y_3 + y_4 \leq 1 & (x_3) \\ & -y_1 \leq 0 & (s_1) \\ & -y_2 \leq 0 & (s_2) \\ & -y_3 \leq 0 & (s_3) \\ & -y_4 \leq 0 & (s_4) \end{aligned}$$