Applications of LP

2022年3月22日 13:40

Vertex Cover

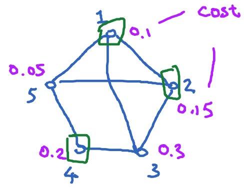
Problem: Given a graph $G = (V, E)$, a cost function $c: V \to \mathbb{R}_+$,

vertex set, edge set

find a vertex cover S that has a minimum cost.

Here, we say that $S \subseteq V$ is a vertex cover if each edge has at least one endpoint in S.

e.g.:



$S = \{1, 2, 4\}$ is a vertex cover.

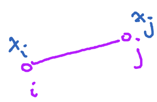
Cost $= 0.1 + 0.15 + 0.2 = 0.45$

Let $x_i \in \{0, 1\}$ be an indicator variable defined as

$$x_v = \begin{cases} 1 & \text{if vertex } i \text{ is chosen to be in the cover} \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have the following formulation:

$$(P) \qquad \begin{aligned} V^* = \min \quad & \sum_{i \in V} c_i x_i \quad (= c^T x) \\ \text{s.t.} \quad & x_i + x_j \geq 1, \quad (i,j) \in E \\ & x_i \in \{0, 1\}, \quad i \in V. \quad \leftarrow \text{binary constraint} \end{aligned}$$



This is an integer linear program. This is NP-complete.

To get a more tractable formulation, one idea is LP relaxation:

$$(LR) \qquad \begin{aligned} V^*_{LR} = \min \quad & c^T x \\ \text{s.t.} \quad & x_i + x_j \geq 1, \quad (i,j) \in E \\ & 0 \leq x_i \boxed{\leq 1}, \quad i \in V. \end{aligned}$$

↳ can be removed (why?)

Observe: $V^* \geq V^*_{LR}$

After solving the LP, we need to "round" the solution to obtain a feasible solution for (P) with an objective value not too far from V^*.

Observe: $V^* \leq V_{rd} \leq \alpha V^*$

$\exists ? \alpha \geq 1$

approximation ratio

↳ cost of rounded solution

Q: Can we develop a rounding process s.t. $\exists \alpha \geq 1$ satisfying the inequality for any problem instance?

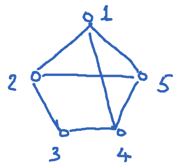
Theorem: Let

$$Q = \{ x \in \mathbb{R}^{|V|} : x_i + x_j \geq 1, \ (i,j) \in E ; \ x_i \geq 0, \ i \in V \}$$

be the feasible region of (LR). Suppose that $\bar{x}$ is a vertex of Q. Then,

$$\forall i : \quad \bar{x}_i \in \{ 0, \tfrac{1}{2}, 1 \}.$$

e.g.:



$$Q : \left\{ \begin{array}{l} x_1 + x_2 \geq 1 \ \checkmark \ \checkmark \\ x_2 + x_3 \geq 1 \ \checkmark \ \checkmark \\ x_3 + x_4 \geq 1 \ \checkmark \\ x_4 + x_5 \geq 1 \ \checkmark \\ x_1 + x_5 \geq 1 \ \checkmark \\ x_1 + x_4 \geq 1 \ \checkmark \\ x_2 + x_5 \geq 1 \ \checkmark \\ x_1, \underset{=0}{\underline{x_2}}, x_3, x_4, x_5 \geq 0 \ \checkmark \end{array} \right.$$

$$\begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ 1 & & & & 1 \\ 1 & & & 1 & \\ & 1 & & & 1 \end{bmatrix} \Rightarrow x_1 = \cdots = x_5 = \tfrac{1}{2}.$$

$$\begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ & 1 & & & \end{bmatrix} \Rightarrow x_1 = 1, \ x_2 = 0, \ x_3 = 1, \ x_4 = 0, \ x_5 = 1,$$

How to use the theorem to design the rounding algorithm?

Idea:

1) Solve (LR) to get an optimal vertex solution $\hat{x}$.

2) By the theorem, set

$$\bar{x}_i = \begin{cases} 1 & \text{if } \hat{x}_i \in \{ \tfrac{1}{2}, 1 \}, \\ 0 & \text{otherwise.} \end{cases} \quad \text{(rounding step)}$$

3) Note that $\bar{x}$ is feasible for (P). Moreover, its cost is

$$V_{rd} = c^T \bar{x} = \sum_{i \in V} c_i \bar{x}_i \leq 2 \underbrace{\sum_{i \in V} c_i \hat{x}_i}_{= c^T \hat{x} = V^*_{LR}} = 2 V^*_{LR} \leq \underset{\alpha}{2} V^*$$