# Conic Linear Programming

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Recall the standard form LP:

$$\min \; c^T x \quad \text{s.t.} \quad Ax = b, \quad \boxed{x \geq \vec{0}}$$

need definition:

$x \geq \vec{0} \iff x_i \geq 0 \;\; \forall i$

$x \geq y \iff x - y \geq \vec{0}$

Q: Is this the only way to define the vector inequality, so that the algebraic properties of "$\geq$" are preserved?

To answer this, let us study the properties of "$\geq$".

## Algebraic View

"$\geq$" satisfies the following:

partial order

1) (Reflexivity) $u \geq u, \;\; \forall u \in \mathbb{R}^n$

2) (Anti-Symmetry)

$$\left.\begin{array}{l} u \geq v \\ v \geq u \end{array}\right\} \Rightarrow u = v, \;\; \forall u, v \in \mathbb{R}^n$$

3) (Transitivity)

$$\left.\begin{array}{l} u \geq v \\ v \geq w \end{array}\right\} \Rightarrow u \geq w, \;\; \forall u, v, w \in \mathbb{R}^n$$

4) (Homogeneity)

$$\left.\begin{array}{l} u \geq v \\ \alpha \geq 0 \end{array}\right\} \Rightarrow \alpha u \geq \alpha v, \;\; \begin{array}{l} \forall \alpha \in \mathbb{R} \\ u, v \in \mathbb{R}^n \end{array}$$

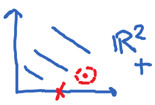
5) (Additivity)

$$\left.\begin{array}{l} u \geq v \\ w \geq z \end{array}\right\} \Rightarrow u + w \geq v + z, \;\; \forall \begin{array}{l} u, v, \\ w, z \end{array} \in \mathbb{R}^n$$

Def: Any order "$\succeq$" satisfying (1)-(5) above is called a good order.

## Geometric View

Consider $K = \{ x \in \mathbb{R}^n : x \geq 0 \} = \mathbb{R}^n_+$.




This is a closed set and

$$\operatorname{int}(K) \triangleq \{ y \in K : \exists \varepsilon > 0 \text{ s.t. } B(y, \varepsilon) \subseteq K \}$$

$$= \mathbb{R}^n_{++} = \{ x \in \mathbb{R}^n : x_i > 0 \;\; \forall i \}$$

Moreover, $K$ is a pointed cone:

a) (Closure under addition)

$K \neq \emptyset$, $u, v \in K \Rightarrow u + v \in K$

b) (Conic)

$\alpha > 0$, $u \in K \Rightarrow \alpha u \in K$

c) (pointedness)

$u, -u \in K \Rightarrow u = 0$

Remark: A pointed cone is always convex: Take $u, v \in K$, $\alpha \in (0,1)$. Then,

$$\left.\begin{array}{l} \alpha u \in K \;\; (b) \\ (1-\alpha) v \in K \;\; (b) \end{array}\right\} \Rightarrow \alpha u + (1-\alpha) v \in K \;\; (a)$$

Let us rephrase our original question:

Q: Is "$\geq$" defined in LP the only good order?

or

Is $\mathbb{R}^n_+$ the only closed pointed cone (with non-empty interior)?

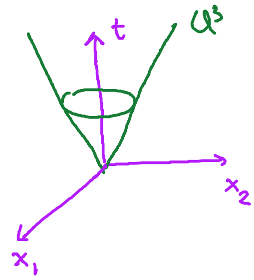
A: Interestingly, no to both!

## Examples.

1) Lorentz cone / second-order cone / ice-cream cone

$$Q^{n+1} \triangleq \{ (t,x) \in \mathbb{R} \times \mathbb{R}^n : t \geq \|x\|_2 \}$$

e.g. $n=2$




Claim: $Q^{n+1}$ is a closed pointed cone with non-empty interior.

Sketch:

1) $\text{int}(Q^{n+1}) = \{ (t,x) \in \mathbb{R} \times \mathbb{R}^n : t > \|x\|_2 \}$

2) conic: $(t,x) \in Q^{n+1}, \ \alpha > 0 \Rightarrow \alpha t \geq \|x\|_2 \cdot \alpha \Rightarrow (\alpha t, \alpha x) \in Q^{n+1}$

3) pointedness: $(t,x) \in Q^{n+1}$, $-(t,x) \in Q^{n+1}$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$t \geq 0$ $\qquad\qquad$ $-t \geq 0 \Rightarrow t = 0 \Rightarrow x = 0$

4) closure under addition:

$$\left.\begin{array}{l} (s,x) \in Q^{n+1} \Rightarrow s \geq \|x\|_2 \\ (t,y) \in Q^{n+1} \Rightarrow t \geq \|y\|_2 \end{array}\right\} \Rightarrow s+t \geq \|x\|_2 + \|y\|_2 \underset{\substack{\uparrow \\ \text{triangle ineq.}}}{\geq} \|x+y\|_2$$

$$\Downarrow$$

$$(s+t, x+y) \in Q^{n+1}$$

Where is the good order? Consider the order "$\succeq_{Q^{n+1}}$" defined by

$$(t,y) \succeq_{Q^{n+1}} (s,x) \iff (t-s, y-x) \in Q^{n+1}$$

Exercise: Prove that "$\succeq_{Q^{n+1}}$" is a good order

2) Semidefinite cone

$$S^n_+ = \{ Y \in S^n : u^T Y u \geq 0 \ \forall u \in \mathbb{R}^n \} \subseteq \underbrace{S^n}_{\substack{\text{set of } n\times n \\ \text{symmetric matrices}}} \cong \mathbb{R}^{\frac{n(n+1)}{2}}$$

Claim: $S^n_+$ is a closed pointed cone with non-empty interior.

Sketch:

(1) $\mathrm{int}(S^n_+) = S^n_{++}$ (positive definite matrices)

(2) conic: $Y \in S^n_+$, $\alpha > 0 \Rightarrow \alpha Y \in S^n_+$

(3) pointedness:

$Y, -Y \in S^n_+ \Rightarrow$ eigenvalues of $Y \geq 0$
eigenvalues of $-Y \geq 0$

eigenvalues of $Y = -$(eigenvalues of $-Y$)

$\Rightarrow$ eigenvalues of $Y = 0 \Rightarrow Y = 0$.

(4) Closure of addition:

$$\begin{matrix} X \in S^n_+ \\ Y \in S^n_+ \end{matrix} \Rightarrow u^T(X+Y)u = \underbrace{u^TXu}_{\geq 0} + \underbrace{u^TYu}_{\geq 0} \geq 0 \quad \forall u \in \mathbb{R}^n$$

Consider the order "$\succcurlyeq_{S^n_+}$" defined by

$$X \succcurlyeq_{S^n_+} Y \iff X - Y \in S^n_+$$

Exercise: Prove that "$\succcurlyeq_{S^n_+}$" is a good order.