Conic Linear Programming (cont'd)

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Examples (cont'd)

3) New cones from old ones via Cartesian product

Let $K_1, \cdots, K_\ell$ be closed pointed cones with non-empty interior. Then,

$$K \triangleq K_1 \times \cdots \times K_\ell = \{ (x_1, x_2, \cdots, x_\ell) : x_i \in K_i, \forall i \}$$

is a closed pointed cone with non-empty interior.

e.g. $\mathbb{R}^5_+ = \mathbb{R}^2_+ \times \mathbb{R}^3_+$; $K = \mathcal{Q}^{n_1+1} \times \cdots \times \mathcal{Q}^{n_\ell+1}$

$K = \mathbb{R}^n_+ \times \mathcal{Q}^{m+1} \times \mathcal{S}^\ell_+$

Conic LP

Recall the standard form LP:

$$\begin{aligned} \min \quad & \langle c, x \rangle \\ \text{s.t.} \quad & \langle a_i, x \rangle = b_i, \quad i = 1, \cdots, m, \\ & x \in \mathbb{R}^n_+ \end{aligned}$$

$K \triangleq \mathbb{R}^n_+ \subseteq \mathbb{R}^n \triangleq E$

$c, a_i \in E$, $b_i \in \mathbb{R}$

$\langle \cdot, \cdot \rangle$: inner product on E

This gives us a natural way to extend LP to conic LP:

Standard form CLP (P)

$$\begin{aligned} \inf \quad & \langle c, x \rangle \\ \text{s.t.} \quad & \langle a_i, x \rangle = b_i, \quad i = 1, \cdots, m, \\ & x \in K \end{aligned}$$

K closed pointed cone with non-empty interior in the Euclidean space E (i.e., $K \subseteq E$), $c, a_i \in E$; $\langle \cdot, \cdot \rangle$ inner product on E; $b_i \in \mathbb{R}$

Note: This is a convex optimization problem

Examples

1) Second-order cone programming (SOCP)

$E = \mathbb{R}^{n+1}$, $K = \mathcal{Q}^{n+1} = \{ (t, x) \in \mathbb{R} \times \mathbb{R}^n : t \geq \|x\|_2 \}$

$\langle u, v \rangle = u^T v$ (usual inner product on E); $c, a_i \in E$; $b_i \in \mathbb{R}$

$$\begin{aligned} \inf \quad & \langle c, x \rangle \\ \text{s.t} \quad & \langle a_i, x \rangle = b_i, \\ & x \in Q^{n+1} \end{aligned}$$

2) Semidefinite Programming (SDP)

$E = S^n, \quad K = S^n_+ = \{ Y \in S^n : u^T Y u \geq 0 \ \forall u \in \mathbb{R}^n \}$

$\langle A, B \rangle = tr(AB) = \sum_{i,j} A_{ij} B_{ij}$ inner product on E

$C, A_i \in E$; $b_i \in \mathbb{R}$

$$\begin{aligned} \inf \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \langle A_i, X \rangle = b_i, \\ & X \in S^n_+ \end{aligned}$$

How to derive the dual of (P)?

Recall that for LP, we derive the dual as follows:

1) Choose y s.t. $A^T y \leq c$ $\qquad A = \begin{bmatrix} - a_1^T - \\ \vdots \\ - a_m^T - \end{bmatrix}$

(i.e., $c - \sum_{i=1}^{m} y_i a_i \geq 0$)

2) Observe

$$\underbrace{\langle c, x \rangle}_{\substack{\text{primal} \\ \text{obj. val}}} \underset{\substack{x \geq 0 \\ c - A^T y \geq 0}}{\geq} \langle A^T y, x \rangle = \langle y, Ax \rangle \underset{Ax = b}{=} \underbrace{\langle y, b \rangle}_{\substack{\text{dual} \\ \text{obj. val}}}$$

3) This gives the dual

$$\begin{aligned} \sup \quad & b^T y \\ \text{s.t.} \quad & c - \sum_{i=1}^{m} y_i a_i \in \mathbb{R}^n_+ \end{aligned}$$

Let us mimic this strategy to derive the dual of (P):

$$\sum_{i=1}^{m} y_i b_i \underset{\langle a_i, x \rangle = b_i}{=} \sum_{i=1}^{m} y_i \cdot \langle a_i, x \rangle \underset{\text{use linearity}}{=} \left\langle \sum_{i=1}^{m} y_i a_i , x \right\rangle \underset{\text{want}}{\leq} \langle c, x \rangle$$

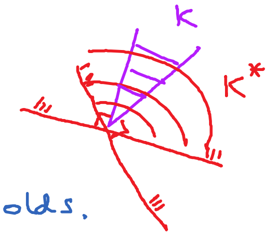
$\langle a_i, x\rangle = b_i$ $\forall i$ ↑ Use linearity of inner product ↑ want

That means we want

$$\langle \underbrace{c - \sum_{i=1}^{m} y_i a_i}_{\in E}, \underbrace{x}_{\in K} \rangle \geq 0 \qquad \text{——} (*)$$

Q: How to choose y to ensure $(*)$ holds?

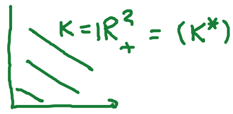
A: Consider the set



$$K^* = \{ w \in E : \langle w, x\rangle \geq 0 \ \ \forall x \in K \}$$

If $c - \sum_{i=1}^{m} y_i a_i \in K^*$, then $(*)$ automatically holds.

The set K^* is called the <u>dual cone</u> of K.



This gives us the following dual of (P):

$$(D) \qquad \begin{aligned} &\sup \ \sum_{i=1}^{m} b_i y_i \\ &\text{s.t.} \quad c - \sum_{i=1}^{m} y_i a_i \in K^* \subseteq E \end{aligned}$$

CLP: optimize a linear function s.t. output of affine map belongs to closed pointed cone (with non-empty interior)

<u>Observations</u>

(1) <u>Fact</u>: If K is a closed pointed cone with nonempty interior, then so is K^*

(2) In (D), the objective function is linear and

$$\mathbb{R}^m \ni y \mapsto M(y) \triangleq c - \sum_{i=1}^{m} y_i a_i \in E$$

is affine (i.e., for $y, z \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$,

$$M(\alpha y + (1-\alpha) z) = \alpha M(y) + (1-\alpha) M(z))$$

→ Both (P) and (D) are CLPs.