Conic LP: Examples

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Recall the standard primal-dual pair of CLPs:

(P) $\inf \ \langle c, x\rangle$

s.t. $\langle a_i, x\rangle = b_i, \quad i=1,\cdots,m,$

$x \in K \subseteq E \quad (x \succeq_K 0)$

(D) $\sup \ \sum_{i=1}^m b_i y_i \ (= b^T y)$

s.t. $\underbrace{c - \sum_{i=1}^m y_i a_i}_{s =} \in K^* . \quad (s \succeq_{K^*} 0)$

Here, E is an Euclidean space, K is a closed pointed cone with non-empty interior in E, $c, a_i \in E$, $b_i \in \mathbb{R}$, $\langle \cdot,\cdot\rangle$ is an inner product on E,

$$K^* = \{ w \in E : \langle w, x\rangle \geq 0 \ \forall x \in K\}$$

is the dual cone of K.

Examples

(1) LP: $E = \mathbb{R}^n$, $K = \mathbb{R}^n_+$, $\langle u,v\rangle = u^T v$

(P) $\inf \ c^T x$

s.t. $a_i^T x = b_i$,

$x \in \mathbb{R}^n_+ \quad (x \geq 0)$

(D) $\sup \ b^T y$

s.t. $\underbrace{c - \sum_{i=1}^m y_i a_i}_{= c - A^T y} \in \underbrace{(\mathbb{R}^n_+)^*}_{= \mathbb{R}^n_+}$ $\quad c - A^T y \geq 0$

$$A = \begin{bmatrix} - a_1^T - \\ \vdots \\ - a_m^T - \end{bmatrix}$$

(Exercise: $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$ — self-dual)

(2) SOCP: $E = \mathbb{R}^{n+1}$, $K = \mathcal{Q}^{n+1}$, $\langle u,v\rangle = u^T v$

(P) $\inf \ c^T x$

s.t. $a_i^T x = b_i$,

$x \in \mathcal{Q}^{n+1}$

(D) $\sup \ b^T y$

s.t. $c - \sum_{i=1}^m y_i a_i \in \underbrace{(\mathcal{Q}^{n+1})^*}_{= \mathcal{Q}^{n+1}}$

(Exercise: $(\mathcal{Q}^{n+1})^* = \mathcal{Q}^{n+1}$)

(3) SDP: $E = S^n$, $K = S^n_+$, $\langle A, B\rangle = \mathrm{tr}(AB)$

(P) $\inf \ \mathrm{tr}(CX)$

$\sup \ b^T y$

s.t. $\mathrm{tr}(A_i X) = b_i$ (D)   s.t. $C - \sum_{i=1}^m y_i A_i \in \underbrace{(S^n_+)^*}_{= S^n_+}$

$X \in S^n_+$

(Exercise: $(S^n_+)^* = S^n_+$)

Sketch: Recall $(S^n_+)^* = \{ W : \langle W, X \rangle \geq 0 \ \forall X \in S^n_+ \}$

Let $W \in S^n_+$. Claim: $W \in (S^n_+)^* \Rightarrow S^n_+ \subseteq (S^n_+)^*$

$\langle W, X \rangle \overset{?}{\geq} 0 \quad \forall X \in S^n_+$

$W \in S^n_+ \Rightarrow W = U \Sigma U^T$ (eigendecomposition)

$$\Rightarrow \langle W, X \rangle = \langle U\Sigma U^T, X \rangle = \langle \Sigma, U^T X U \rangle$$

$$= \sum_{i=1}^n \underbrace{\Sigma_{ii}}_{\geq 0} \cdot \underbrace{(U^T X U)_{ii}}_{\geq 0} \geq 0 \qquad u_i^T X u_i \geq 0$$

(all eigenvalues of W are ≥ 0)

What can be modeled by SOCP, SDP?

Example: (SOCP dual)

Consider the constraint in the SOCP dual:

$$c - \sum_{i=1}^m y_i a_i \in \mathcal{Q}^{n+1}. \qquad \text{—— } (*)$$

Write $c = (v; \underbrace{d_1, \dots, d_n}_{d \in \mathbb{R}^n})$ and $a_i = (u_i, \underbrace{a_{i,1}, \dots, a_{i,n}}_{\bar{a}_i \in \mathbb{R}^n})$

Then,

$$c - \sum_{i=1}^m y_i a_i = \begin{bmatrix} v \\ d \end{bmatrix} - \sum_{i=1}^m y_i \begin{bmatrix} u_i \\ \bar{a}_i \end{bmatrix} = \begin{bmatrix} v \\ d \end{bmatrix} - \begin{bmatrix} u^T y \\ \bar{A}^T y \end{bmatrix},$$

where

$$\bar{A} = \begin{bmatrix} - \bar{a}_1^T - \\ \vdots \\ - \bar{a}_m^T - \end{bmatrix}.$$ Hence,

$$(*) \iff \begin{bmatrix} v - u^T y \\ d - \bar{A}^T y \end{bmatrix} \in \mathcal{Q}^{n+1} \iff \underbrace{v - u^T y}_{\text{affine in } y} \geq \| \underbrace{d - \bar{A}^T y}_{\text{affine in } y} \|_2$$

Example: (Convex quadratic constraints)

Let $Q \in S^n_+$ be given. Then,

(i) $t \geq 0,\ x^T Q x \leq t^2 \iff x^T Q^{1/2} Q^{1/2} x \leq t^2 \iff \|Q^{1/2} x\|_2^2 \leq t^2,\ t \geq 0$
$t \geq 0$

$$\begin{aligned} Q &= U \Sigma U^T \\ &= \underbrace{U \Sigma^{1/2} U^T}_{Q^{1/2}} \underbrace{U \Sigma^{1/2} U^T}_{Q^{1/2}} \end{aligned}$$

$\iff t \geq \|Q^{1/2} x\|_2$ SOC constraint

(ii) $x^T Q x \leq t \iff \|Q^{1/2} x\|_2^2 \leq t \iff \|Q^{1/2} x\|_2^2 + (t - \frac{1}{4})^2 \leq (t + \frac{1}{4})^2$

not an SOC constraint

$\iff \|\underbrace{(t - \tfrac{1}{4},\ Q^{1/2} x)}\|_2 \leq t + \frac{1}{4}$ SOC constraint

$$\begin{bmatrix} t - \frac{1}{4} \\ Q^{1/2} x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ Q^{1/2} & 0 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} + \begin{bmatrix} -1/4 \\ 0 \end{bmatrix}$$

Example (SOCP vs SDP)

Consider the SOC constraint $v - u^T y \geq \|d - \bar{A}^T y\|_2$

Claim: This is equivalent to the SDP constraint

$$\begin{bmatrix} (v - u^T y) I & d - \bar{A}^T y \\ (d - \bar{A}^T y)^T & v - u^T y \end{bmatrix} \in S^n_+$$

Fact: Let $A \in S^m$, $B \in \mathbb{R}^{m \times l}$, $C \in S^l$. If A is invertible, then

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \in S^{m+l}_+ \iff A \in S^m_+,\ C \in S^l_+,\ \underbrace{C - B^T A^{-1} B}_{\text{Schur complement}} \in S^l_+$$