Conic LP Duality

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Recall the standard primal-dual pair of CLPs:

$$(P)\quad v_p^* = \inf \ \langle c, x\rangle \quad \text{s.t.} \ \langle a_i, x\rangle = b_i, \ i=1,\dots,m, \quad x \in K \subseteq E \ \ (x \succeq_K 0)$$

$$(D)\quad v_d^* = \sup \ \sum_{i=1}^m b_i y_i \ \ (= b^T y) \quad \text{s.t.} \ \underbrace{c - \sum_{i=1}^m y_i a_i}_{s\,=} \in K^* . \ \ (s \succeq_{K^*} 0)$$

Here, E is an Euclidean space, K is a closed pointed cone with non-empty interior in E, $c, a_i \in E$, $b_i \in \mathbb{R}$, $\langle \cdot,\cdot\rangle$ is an inner product on E,

$$K^* = \{ w \in E : \langle w, x\rangle \geq 0 \ \forall x \in K\}$$

is the dual cone of K.

Example (Eigenvalue optimization)

Let $A_1, \dots, A_k \in S^n$ be given symmetric matrices.

Goal. Find $y \in \mathbb{R}^k$ to minimize $\lambda_{max}\left(\sum_{i=1}^k y_i A_i\right)$.

Proposition: Let $A \in S^n$ be given. Then,

$$\lambda_{max}(A) \leq t \iff tI \succeq_{S^n_+} A.$$

(i.e., $tI - A \in S^n_+$)

With the above proposition, we have

$$\min_y \lambda_{max}\left(\sum_{i=1}^k y_i A_i\right) \iff \min_{y,t} \ t \ \ \text{s.t.} \ t \geq \lambda_{max}\left(\sum_{i=1}^k y_i A_i\right) \iff \min_{y,t} \ t \ \ \text{s.t.} \ tI - \sum_{i=1}^k y_i A_i \in S^n_+$$

SDP

Proof of Proposition:

($\Leftarrow$) Suppose that $tI - A \in S^n_+$. By definition, for all $u \in \mathbb{R}^n$, $\|u\|_2 = 1$, $u^T(tI - A)u \geq 0$. This implies that

$$t \cancel{\|u\|_2^2}^{\,1} \geq u^T A u \Rightarrow t \geq u^T A u \Rightarrow \underbrace{\max_{\|u\|_2 = 1} u^T A u}\leq t$$

$$\Rightarrow \lambda_{max}(A) \leq t \quad \text{(by Courant-Fischer theorem)}$$

($\Rightarrow$) Reverse the above argument

Theorem (Weak Duality for CLP)

Let $\bar{x}$ be feasible for (P), $\bar{y}$ be feasible for (D). Then,

$$b^T\bar{y} \le \langle c, \bar{x} \rangle.$$

Proof: We compute

$$\langle c, \bar{x} \rangle - b^T\bar{y} = \langle c, \bar{x} \rangle - \sum_{i=1}^m \bar{y}_i \underbrace{\langle a_i, \bar{x} \rangle}_{= b_i}$$

($\because$ $\bar{x}$ feasible for (P))

$$= \langle c, \bar{x} \rangle - \left\langle \sum_{i=1}^m \bar{y}_i a_i , \bar{x} \right\rangle$$ (by linearity of inner product)

$$= \left\langle c - \sum_{i=1}^m \bar{y}_i a_i , \bar{x} \right\rangle$$

$$\geq 0 \qquad (\because \bar{x} \in K, \; c - \sum_{i=1}^m \bar{y}_i a_i \in K^*)$$

Theorem (Strong Duality for CLP)

Suppose that $\begin{matrix}(P)\\(D)\end{matrix}$ is bounded $\begin{matrix}\text{below}\\\text{above}\end{matrix}$ and satisfies Slater's condition; i.e.,

there exists a feasible $\begin{matrix}\bar{x}\\\bar{y}\end{matrix}$ for $\begin{matrix}(P)\\(D)\end{matrix}$ s.t. $\begin{matrix}\bar{x} \in \text{int}(K)\\ c - \sum_{i=1}^m \bar{y}_i a_i \in \text{int}(K^*).\end{matrix}$

Then, $v_p^* = v_d^*$ (zero duality gap) and there exists an optimal

$\begin{matrix}\text{dual solution } y^*\\ \text{primal solution } x^*\end{matrix}$ s.t. $\begin{matrix}b^T y^*\\ \langle c, x^* \rangle\end{matrix} = v_p^* = v_d^*.$

Note: In LP strong duality, we do not need to assume Slater condition. Also, if the duality gap is zero, then both primal and dual have optimal solutions.

Example: Consider the SOCP

$(e_1 = (1,0))$

$$0 = v_d^* = \sup -y_1 \qquad \leftrightarrow \qquad \sup -e_1^T y$$

$(e_1 = (1,0))$

$$0 = v_d^* = \sup \ -y_1 \quad \text{s.t. } (y_1+y_2,\ 1,\ y_1-y_2) \in Q^3 \iff \sup \ \underbrace{-e_1^T}_{b} y \quad \text{s.t. } \underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{c} - y_1 \underbrace{\begin{bmatrix}-1\\0\\-1\end{bmatrix}}_{a_1} - y_2 \underbrace{\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{a_2} \in Q^3$$

$$\Updownarrow$$

$$y_1 + y_2 \geq \sqrt{1^2 + (y_1-y_2)^2}$$

$$\Updownarrow$$

$$4y_1y_2 \geq 1, \quad y_1 + y_2 > 0$$

$$(\Rightarrow y_1, y_2 > 0)$$

$$\downarrow$$

$$y_1 = 1/4y_2$$

Take $y_2 \nearrow \infty$

$\Rightarrow y_1 \searrow 0$

However, there is no optimal solution.

$$\sup \ b^T y \quad \text{s.t. } c - \sum_{i=1}^{2} y_i a_i \in Q^3$$

$\updownarrow$ Take the dual

Q: $\exists \ \bar{y}_1, \bar{y}_2 > 0$ s.t

$$\bar{y}_1 + \bar{y}_2 > \sqrt{1 + (\bar{y}_1 - \bar{y}_2)^2} \ ?$$

A: Yes, e.g., $\bar{y}_1 = \bar{y}_2 = 1$ Thus, $(1,1)$ satisfies Slater's condition.

$$0 = v_p^* = \inf \ \langle c, x\rangle = x_2$$

$$\text{s.t. } \langle a_1, x\rangle = -x_1 - x_3 = b_1 = -1,$$

$$\langle a_2, x\rangle = -x_1 + x_3 = b_2 = 0,$$

$$x = (x_1, x_2, x_3) \in Q^3$$

This has only one feasible (and hence optimal) solution: $(\frac{1}{2}, 0, \frac{1}{2})$.