Example. SDP

$$v_d^* = \sup \ -y_1$$

(D)

$$\text{s.t.} \begin{bmatrix} 0 & y_1 & 0 \\ y_1 & y_2 & 0 \\ 0 & 0 & 1+y_1 \end{bmatrix} \in S_+^3$$

Observe: 1) $(y_1 = 0, y_2 = 1)$ is a feasible solution

2) $v_d^* = 0$ ($\because y_1 = 0$ in any feasible solution)

3) Slater condition is not satisfied, since any feasible solution has rank ≤ 2.

Rewrite (D) as

$$\sup \ \underbrace{(-e_1)^T}_{b} y \qquad (e_1 = (1,0))$$

$$\text{s.t.} \ \underbrace{\begin{bmatrix} & & \\ & & \\ & & 1 \end{bmatrix}}_{C} - y_1 \underbrace{\begin{bmatrix} & -1 & \\ -1 & & \\ & & -1 \end{bmatrix}}_{A_1} - y_2 \underbrace{\begin{bmatrix} & & \\ & -1 & \\ & & \end{bmatrix}}_{A_2} \in S_+^3$$

The dual is

$$v_p^* = \inf \ \langle C, X \rangle = X_{33}$$

(P)

$$\text{s.t.} \ \langle A_1, X \rangle = -2X_{12} - X_{33} = -1,$$

$$\langle A_2, X \rangle = -X_{22} = 0,$$

$$X \in S_+^3.$$

Observe: Any feasible solution to (P) takes the form

$$\begin{bmatrix} * & 0 & * \\ 0 & 0 & 0 \\ * & 0 & 1 \end{bmatrix} \in S_+^3$$

Hence, $v_p^* = 1$. Note that (P) does not satisfy Slater condition.

Conic LP: Applications

2022年4月12日 14:58

Optimization under uncertainty

Consider

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & a_i^T x \le b_i , \quad i = 1, \cdots, m. \end{aligned}$$

The data are (a_i, b_i, c). In some cases, they are uncertain.

How to protect our solution from such uncertainties?

<u>Approach 1</u>: Stochastic optimization

Assume (a_i, b_i, c) follow certain probability distribution.

Then, we can consider, e.g.,

$$\begin{aligned} \min \quad & \mathbb{E}[c^T x] \\ \text{s.t.} \quad & \underbrace{\Pr[a_i^T x \le b_i] \geq 1 - \delta}_{\text{chance constraint}}, \quad i = 1, \cdots, m. \end{aligned}$$

<u>Approach 2</u>: Robust optimization

Assume (a_i, b_i, c) belong to a bounded uncertainty set.

Then, we consider, e.g.

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & \underbrace{[\, a_i^T x \le b_i \quad \forall (a_i, b_i) \in \underbrace{\mathcal{U}_i}_{\text{uncertainty set}} \,]}_{\text{robust constraint}} \end{aligned}$$

<u>Example</u>: Consider the robust LP

$$\text{(RLP)} \qquad \begin{aligned} \min \quad & c^T z \\ \text{s.t.} \quad & a_i^T z \le 0 , \quad \forall a_i \in \mathcal{U}_i = \{ w : \| w - \underbrace{\bar{a}_i}_{\text{given}} \|_2 \le \underbrace{r_i}_{\text{given}} \} \end{aligned}$$

<u>Note</u>: (RLP) is not an LP, because there are infinitely many linear constraints.

<u>Q</u>: How to tackle (RLP)?

<u>A</u>: Observe that

$$a_i^T z \le 0 \quad \forall a_i \in \mathcal{U}_i = \{ \bar{a}_i + r_i u : \|u\|_2 \le 1 \}$$

$$\iff \max_{a_i \in \mathcal{U}_i} a_i^T z \le 0$$

$$\iff \max_{\|u\|_2 \le 1} (\bar{a}_i + r_i u)^T z \le 0$$

$$\iff \bar{a}_i^T z + r_i \cdot \max_{\|u\|_2 \le 1} u^T z \le 0$$

$$\iff \bar{a}_i^T z + r_i \|z\|_2 \le 0 \quad (\because \text{Cauchy-Schwarz})$$

$$\iff r_i \|z\|_2 \le \underbrace{-\bar{a}_i^T z}_{\text{linear in } z} \quad (\text{SOC constraint})$$

linear in z