Semidefinite Relaxation of Quadratic Optimization Problems

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Consider

$$(Q) \qquad \begin{array}{ll} \inf & x^T C x \\ \text{s.t.} & x^T Q_i x \geq b_i, \quad i = 1, \dots, m. \end{array}$$

Data: $C, Q_1, \dots, Q_m \in S^n$

$b_1, \dots, b_m \in \mathbb{R}$

(quadratically constrained quadratic optimization (QCQP))

Remarks:

① No convexity assumption is made

② Problem (Q) can model a wide range of problems, e.g.

$$\begin{array}{ll} \inf & x^T C x \\ \text{s.t.} & x_i \in \{-1, 1\} \end{array} \quad \Longleftrightarrow \quad \begin{array}{ll} \inf & x^T C x \\ \text{s.t.} & x_i^2 \leq 1 \\ & -x_i^2 \leq -1 \end{array}$$

Semidefinite relaxation (SDR) technique for (Q):

Observe

$$x^T C x = \operatorname{tr}(x^T C x) = \operatorname{tr}(\underset{\in S^n}{C}\, \underset{\in S^n}{xx^T}) = \langle C, xx^T \rangle$$

Hence, (Q) is equivalent to

$$\begin{array}{ll} \inf & \langle C, \underbrace{xx^T}_{= X} \rangle \\ \text{s.t.} & \langle Q_i, \underbrace{xx^T}_{= X} \rangle \geq b_i, \quad \forall i. \end{array}$$

Observe that (Q) is linear in X. Moreover,

$$X = xx^T \iff X \in S^n_+, \quad \underbrace{\operatorname{rank}(X) \leq 1}_{\substack{\| \\ \#\text{ of non-zero} \\ \text{eigenvalues of } X}}$$

$x(x^T u)$ eigenvalue

$\|$

$(xx^T) u = \lambda u$

eigenvector

$(\|u\|_2 = 1)$

Take $u = \frac{x}{\|x\|_2}$. Then,

$xx^T u = \|x\| x$

Hence, (Q) is equivalent to

$$\inf \ \langle C, X \rangle \ \text{— linear}$$

Take $u = \frac{x}{\|x\|_2}$. Then,

$x x^T u = \|x\|_2 \, x$

eigenvector $= \|x\|_2^2 \cdot u$

largest eigenvalue

$$
(Q) \quad \begin{aligned} \inf \quad & \langle C, X \rangle \quad \text{— linear} \\ \text{s.t.} \quad & \langle Q_i, X \rangle \geq b_i, \quad \text{— linear} \\ & X \in S^n_+, \quad \underbrace{\operatorname{rank}(X) \leq 1}_{\text{non-convex}} \end{aligned}
$$

psd

The SDR of (Q) is

$$
(SDR) \quad \begin{aligned} \inf \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \langle Q_i, X \rangle \geq b_i, \quad \longleftarrow \text{SDP} \\ & X \in S^n_+ \end{aligned}
$$

Q: Given an optimal solution X^* to (SDR), how do we extract from it a feasible solution to (Q)?

Remark. Idea of SDR: $X = xx^T \longrightarrow X \in S^n_+$

In essence, $x^T C x \longrightarrow \langle C, X \rangle$

$$
\| \qquad\qquad\qquad \|
$$

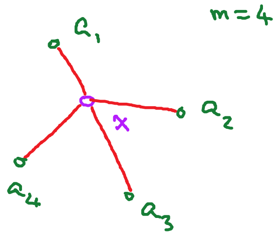
$$
\sum_{i,j} C_{ij} x_i x_j \qquad\qquad \sum_{i,j} C_{ij} X_{ij}
$$

$g_i^2 = g_i g_i \to G_{ii}$

Example: Single-source localization

$x \in \mathbb{R}^n$: source with unknown position

$a_i \in \mathbb{R}^n$: anchor with known position $(i = 1, \cdots, m)$



Measurement model:

$$
d_i = \|x - a_i\|_2 + \varepsilon_i, \quad i = 1, \cdots, m.
$$

measurement   noise

Goal: Recover x from $d_1, \cdots, d_m$.

Least-squares formulation

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{m} \left(d_i - \|x - a_i\|_2 \right)^2$$

$$(d-g)^T(d-g)$$

$$\text{(LS)} \quad \min_{\substack{x \in \mathbb{R}^n \\ g \in \mathbb{R}^m}} \sum_{i=1}^{m} (d_i - g_i)^2 = \|d - g\|_2^2 = [g^T \ 1] \underbrace{\overset{\text{data}}{\begin{bmatrix} I & -d \\ -d^T & \|d\|_2^2 \end{bmatrix}}}_{= D} \begin{bmatrix} g \\ 1 \end{bmatrix}$$

$$\Longleftrightarrow \quad \text{s.t.} \quad g_i^2 = \|x - a_i\|_2^2, \quad i = 1, \dots, m,$$

$$g_i \geq 0$$

$$g_i^2 = [x^T \ 1] \underbrace{\overset{\text{data}}{\begin{bmatrix} I & -a_i \\ -a_i^T & \|a_i\|_2^2 \end{bmatrix}}}_{= A_i} \begin{bmatrix} x \\ 1 \end{bmatrix}$$

Note:

$$[g^T \ 1] D \begin{bmatrix} g \\ 1 \end{bmatrix} = \operatorname{tr}\left([g^T \ 1] D \begin{bmatrix} g \\ 1 \end{bmatrix} \right) = \left\langle D, \begin{bmatrix} g \\ 1 \end{bmatrix} [g^T \ 1] \right\rangle$$

$$= \left\langle D, \begin{bmatrix} \cancel{gg^T} \ G & g \\ g^T & 1 \end{bmatrix} \right\rangle$$

Similarly,

$$[x^T \ 1] A_i \begin{bmatrix} x \\ 1 \end{bmatrix} = \left\langle A_i, \begin{bmatrix} \cancel{xx^T} \ X & x \\ x^T & 1 \end{bmatrix} \right\rangle$$

Hence, the SDR of (LS) is

$$\min_{\substack{G, g \\ X, x}} \left\langle D, \begin{bmatrix} G & g \\ g^T & 1 \end{bmatrix} \right\rangle$$

$$\text{s.t.} \quad G_{ii} = \left\langle A_i, \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \right\rangle,$$

$$g_i \geq 0,$$

$$X \in S_+^n, \quad G \in S_+^m.$$