Introduction (cont'd)

2022年1月13日 15:39

Recall the abstract problem:

$$(P) \qquad v^* = \inf_{x \in X} f(x)$$

Simple examples of (P)

(5) Semidefinite programming (SDP)

Definition: Let $Q \in \underbrace{S^n}_{\text{Set of } n \times n \text{ Symmetric matrices}}$. Then, the following are equivalent:

(a) Q is positive semidefinite (psd)

(b) $\forall x \in \mathbb{R}^n, \quad x^T Q x \geq 0$

(c) All eigenvalues of Q are non-negative.

↳ $\det(Q - \lambda I) = 0$

Let $C, A_1, \dots, A_m \in S^n$; $b_1, \dots, b_m \in \mathbb{R}$ be given.

$$(SDP) \qquad \inf \quad b^T y$$

$$\text{s.t.} \quad C - \sum_{i=1}^{m} y_i A_i \underbrace{\succeq 0}_{\text{psd}}, \quad \text{— } (*)$$

$$y \in \mathbb{R}^m.$$

Remarks:

1) (*) is called linear matrix inequality: Observe that (*) is equivalent to

$$\underbrace{-\sum_{i=1}^{m} y_i A_i}_{M(y)} \succeq -C$$

$$\mathbb{R}^m \ni y \mapsto M(y) \in S^n$$

Then, it can be verified that (Exercise)

$$\forall \alpha, \beta \in \mathbb{R};\ y, z \in \mathbb{R}^m : M(\alpha y + \beta z) = \alpha M(y) + \beta M(z)$$

2) Suppose that $C, A_1, \dots, A_m$ are diagonal. Then, (*) becomes

$$\underbrace{\begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{bmatrix}}_{C} - \sum_{i=1}^{m} y_i \underbrace{\begin{bmatrix} a_{i1} & & \\ & \ddots & \\ & & a_{in} \end{bmatrix}}_{A_i} \succcurlyeq 0$$

$$\iff \begin{bmatrix} c_1 - \sum_{i=1}^{m} y_i a_{i1} & & \\ & \ddots & \\ & & c_n - \sum_{i=1}^{m} y_i a_{in} \end{bmatrix} \succcurlyeq 0$$

$$\iff \quad c_j - \sum_{i=1}^{m} y_i a_{ij} \geq 0 \quad \forall j \qquad \text{linear inequalities}$$

Then, (SDP) becomes an LP.

3) How about more LMIs in the constraint?

e.g.

$$C - \sum_{i=1}^{m} y_i A_i \succcurlyeq 0 \qquad \overset{\text{(Exercise)}}{\iff} \qquad \begin{bmatrix} C & \\ & D \end{bmatrix} - \sum_{i=1}^{m} y_i \begin{bmatrix} A_i & \\ & B_i \end{bmatrix} \succcurlyeq 0$$

$$D - \sum_{i=1}^{m} y_i B_i \succcurlyeq 0$$

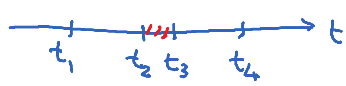
Hint: Let $A = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \in S^n$, Then, $A \succcurlyeq 0 \iff A_1, A_2 \succcurlyeq 0$.

Reformulation Example

Air Traffic control

- n planes arriving
- i^{th} plane arrives within $[a_i, b_i]$
- assume that planes land in order
- Let t_i be the assigned landing time of plane i

For safety, want the shortest metering time



$$\min_{1 \le i \le n-1} \{ t_{i+1} - t_i \}$$

is maximized.

The problem can be formulated as

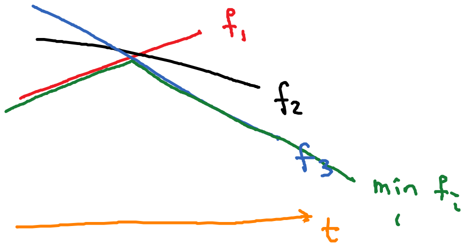
$f_i(t) = t_{i+1} - t_i$ linear

$f(t) = \min\limits_{1\le i\le n-1} f_i(t)$

$$\max \quad f(t) = \min_{1\le i\le n-1} \{ \underbrace{t_{i+1} - t_i}_{f_i(t)} \} \rightarrow \text{piecewise linear}$$

$$\text{s.t.} \quad \left.\begin{aligned} & a_i \le t_i \le b_i \; ; \quad i = 1, \cdots, n, \\ & t_i \le t_{i+1} \; , \quad i = 1, \cdots, n-1. \end{aligned}\right\} \text{linear inequalities}$$

$\Leftrightarrow \; t_i - t_{i+1} \le 0$



$$\Leftrightarrow \quad \max \quad z$$

$$\text{s.t.} \quad z = f(t),$$

can be replaced by $\le$ without changing the optimal solution

$$a_i \le t_i \le b_i \; ; \quad i = 1, \cdots, n,$$

$$t_i \le t_{i+1} \; ; \quad i = 1, \cdots, n-1.$$

$z \le f(t) = \min\limits_i \{ t_{i+1} - t_i \}$

$\Leftrightarrow \quad z \le t_{i+1} - t_i \quad \forall i$

$$\Leftrightarrow \quad \max \quad z \quad \leftarrow \text{linear function}$$

$$\text{s.t.} \quad \left.\begin{aligned} & z \le t_{i+1} - t_i \; ; \quad i = 1, \cdots, n-1 \\ & a_i \le t_i \le b_i \; ; \quad i = 1, \cdots, n \\ & t_i \le t_{i+1} \; ; \quad i = 1, \cdots, n-1 \end{aligned}\right\} \text{linear inequalities}$$

Thus, we get an LP.