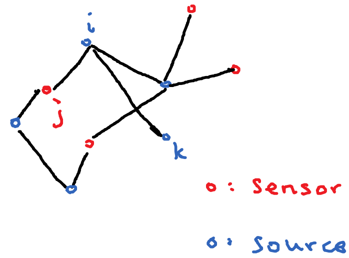
Example: Sensor network localization



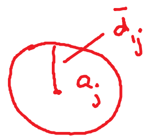
$x_i \in \mathbb{R}^d$: unknown position of i^{th} source

$a_j \in \mathbb{R}^d$: known position of j^{th} sensor

Assuming without measurement noise, we have

$$(L)\quad \begin{cases} \|x_i - x_k\|_2^2 = d_{ik}^2, & \forall (i,k) \in E \quad \text{non-convex} \\ \|x_i - a_j\|_2^2 = \bar{d}_{ij}^2, & \forall (i,j) \in E \quad \text{non-convex} \end{cases}$$

edge set



Goal: Find $x_1, \ldots, x_n \in \mathbb{R}^d$ that satisfy the above quadratic equations.

Apply SDR to (L):

$$\|x_i - x_k\|_2^2 = \underbrace{\overset{Y_{ii}}{x_i^T x_i}}_{=\sum_{l=1}^{d} x_{il}^2} - \underbrace{2\overset{Y_{ik}}{x_i^T x_k}}_{=\sum_{l=1}^{d} x_{il} x_{kl}} + \overset{Y_{kk}}{x_k^T x_k}$$

x_{il} $\quad$ x_{ikl} $\leftarrow$ too tedious

This gives the substitution: $Y_{ik} = x_i^T x_k$. Hence, define

$$\underset{n\times n}{Y} = \underset{n\times d}{X^T} \underset{d\times n}{X}, \quad \text{where} \quad X = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} \in \mathbb{R}^{d\times n}$$

(Compare $\underset{n\times n}{Y} = \underset{n\times 1}{x}\,\underset{1\times n}{x^T}$ in the previous approach)

Hence,

$$\|x_i - x_k\|_2^2 = Y_{ii} - 2Y_{ik} + Y_{kk} = \left\langle \underbrace{\begin{array}{c|cc} & i & k \\ \hline i & 1 & -1 \\ k & -1 & 1 \end{array}}_{E_{ik}}, Y \right\rangle$$

Moreover,

$$\|x_i - a_j\|_2^2 = \overset{Y_{ii}}{x_i^T x_i} - \underbrace{2a_j^T x_i}_{\text{linear in } x_i} + \underbrace{a_j^T a_j}_{\text{constant}}$$

$$= \left\langle \begin{bmatrix} a_j a_j^T & -a_j e_i^T \\ -e_i a_j^T & E_i \end{bmatrix}, \begin{bmatrix} I & X \\ X^T & Y \end{bmatrix} \right\rangle$$

$$E_i = \begin{array}{c} \\ i \end{array}\begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix}, \qquad \begin{bmatrix} I & X \\ X^T & Y \end{bmatrix} = \begin{bmatrix} I \\ X^T \end{bmatrix} \begin{bmatrix} I & X \end{bmatrix}$$

For any $A, B \in \mathbb{R}^{m\times n}$,

$\langle A, B \rangle = \operatorname{tr}(A^T B)$.

$a_j^T x_i = a_j^T X e_i$
$= \operatorname{tr}(a_j^T X e_i)$

$\langle a_j a_j^T, I \rangle$
$= \operatorname{tr}(a_j a_j^T)$
$= \operatorname{tr}(a_j^T a_j)$
$= a_j^T a_j$

$\tilde{a}_j^T\tilde{a}_i = a_j^T X e_i$

$= \mathrm{tr}(a_j^T X e_i)$

$= \mathrm{tr}(X \underbrace{e_i a_j^T})$

$= \langle a_j e_i^T, X \rangle$

$E_i = \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix}$ $\overset{"}{=} \begin{bmatrix} I \\ X^T \end{bmatrix} \begin{bmatrix} I & X \end{bmatrix}$ $\qquad \mathrm{tr}(a_j^T a_j) = a_j^T a_j$

Hence, (L) is equivalent to

$$\left\langle \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & E_{ik} \end{array}\right], \begin{bmatrix} I & X \\ X^T & Y \end{bmatrix} \right\rangle = d_{ik}^2, \quad \leftarrow \text{linear}$$

$$\left\langle \left[\begin{array}{c|c} a_j a_j^T & -a_j e_i^T \\ \hline -e_i a_j^T & E_i \end{array}\right], \begin{bmatrix} I & X \\ X^T & Y \end{bmatrix} \right\rangle = \bar{d}_{ij}^2, \quad \leftarrow \text{linear}$$

$$Y = X^T X \quad \leftarrow \text{non-convex} \quad \{(X,Y) : Y = X^T X\}$$

The SDR approach is to relax $Y = X^T X$ to $Y - X^T X \in S_+^n$.

By Schur complement,

$$Y - X^T X \in S_+^n \iff \begin{bmatrix} I & X \\ X^T & Y \end{bmatrix} \in S_+^{d+n}.$$

$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0 \iff C - B^T A^{-1} B \succeq 0$, $A, C \succeq 0$

Example: Extension of SDR to complex QCQPs

Consider

$$(C) \quad \begin{array}{ll} \inf & z^H C z \\ \text{s.t.} & z^H Q_i z \geq b_i, \\ & i = 1, \dots, m. \\ & z \in \mathbb{C}^n \end{array}$$

always real: $(z^H C z)^H = z^H C^H z = z^H C z$

Data:

$C, Q_1, \dots, Q_m \in \mathbb{H}^n$ ← set of $n \times n$ Hermitian matrices (i.e., $A = A^H$)

$b_1, \dots, b_m \in \mathbb{R}$

The SDR technique still applies:

$$z^H C z = \mathrm{tr}(z^H C z) = \mathrm{tr}(C \underbrace{z z^H}_{=Z}) = \langle C, Z \rangle$$

This gives the following SDR of (C):

$$\inf \langle C, Z \rangle$$
$$\text{s.t.} \ \langle Q_i, Z \rangle \geq b_i,$$

$Z = z z^H \iff Z_{ij} = z_i \bar{z}_j$

$Z \in \mathbb{H}_+^n$

set of $n \times n$ Hermitian psd matrices; i.e.,

$A \in \mathbb{H}_+^n \iff z^H A z \geq 0 \quad \forall z \in \mathbb{C}^n$