Tightness of SDR

2022年4月21日 15:41

Recall that the SDR technique applies to both real and complex QCQPs:

$$(Q_R)\quad \begin{array}{ll} \inf & x^T C x \\ \text{s.t.} & x^T Q_i x \geq b_i, \quad i=1,\dots,m. \\ & x \in \mathbb{R}^n \end{array}$$

Data:

$C, Q_1, \dots, Q_m \in S^n$

$b_1, \dots, b_m \in \mathbb{R}$

$$(SDR_R)\quad \begin{array}{ll} \inf & \langle C, X \rangle \\ \text{s.t.} & \langle Q_i, X \rangle \geq b_i, \\ & X \in S^n_+ \end{array}$$

$$(Q_C)\quad \begin{array}{ll} \inf & z^H C z \\ \text{s.t.} & z^H Q_i z \geq b_i, \quad i=1,\dots,m. \\ & z \in \mathbb{C}^n \end{array}$$

Data:

$C, Q_1, \dots, Q_m \in \mathbb{H}^n$

$b_1, \dots, b_m \in \mathbb{R}$

$$(SDR_C)\quad \begin{array}{ll} \inf & \langle C, Z \rangle \\ \text{s.t.} & \langle Q_i, Z \rangle \geq b_i, \quad \rightarrow Z^* \text{ opt. soln.} \\ & Z \in \mathbb{H}^n_+, \cancel{\operatorname{rank}(Z) \leq 1} \end{array}$$

Observe: If the optimal solution to (SDR_C) Z^* has rank ≤ 1, then it is optimal for (Q_C).

Q: Are there conditions on (SDR_C) s.t. it has an optimal solution of rank ≤ 1? (We say that the relaxation is tight)

Theorem: Consider the SDP

$$(P)\quad \left.\begin{array}{ll} \inf & \langle C, Z \rangle \\ \text{s.t.} & \langle Q_i, Z \rangle \geq b_i, \quad i=1,\dots,m', \\ & \langle Q_i, Z \rangle = b_i, \quad i=m'+1,\dots,m, \\ & Z \in \mathbb{H}^n_+. \end{array}\right\} \text{closed feasible set}$$

Suppose that (P) has an optimal solution. Then, there exists an optimal solution Z^* satisfying $\text{rank}(Z^*) \le \lfloor \sqrt{m} \rfloor$. Moreover, Z^* can be found efficiently.

<u>Remark</u>: (P) has optimal solution if

(i) its dual is bounded above and satisfies Slater's condition; or (strong duality of CLP)

(ii) the feasible set of (P) is compact (closed and bounded) (Weierstrass theorem)

Also, if $m \le 3$, then (P) has a rank-1 optimal solution.

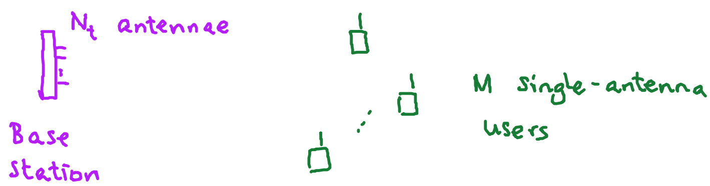
The following is an extension of the above theorem:

<u>Theorem</u>: Consider

$$
(B) \qquad \begin{aligned}
\inf \quad & \sum_{k=1}^{K} \langle C_k, Z_k \rangle \\
\text{s.t.} \quad & \sum_{k=1}^{K} \langle Q_{ik}, Z_k \rangle \ge b_i, \quad i = 1, \dots, m', \\
& \sum_{k=1}^{K} \langle Q_{ik}, Z_k \rangle = b_i, \quad i = m'+1, \dots, m, \\
& Z_k \in \mathbb{H}_+^n, \qquad k = 1, \dots, K.
\end{aligned}
$$

(Note: (P) is a special case of (B) with $K=1$). Suppose that (B) has an optimal solution. Then, there exists an optimal solution $(Z_1^*, \dots, Z_K^*)$ satisfying $\sum_{k=1}^{K} \text{rank}^2(Z_k^*) \le m$.

<u>Example</u>: Unicast transmit beamforming



- Signal transmitted by base station:

$$
x(t) = \sum_{i=1}^{M} s_i(t) w_i ,
$$

$s_i(t) \in \mathbb{C}$: unit-power symbol ; $w_i \in \mathbb{C}^{N_t}$: beamforming vector for user i

- signal received by i^{th} user:

$$y_i(t) = h_i^H x(t) + n_i(t)$$

$h_i \in \mathbb{C}^{N_t}$: channel vector of user i

$n_i(t) \sim \mathcal{CN}(0, \sigma_i^2)$: additive noise

$$= \underbrace{h_i^H w_i s_i(t)}_{\text{signal}} + \underbrace{\sum_{j \neq i} h_i^H w_j s_j(t)}_{\text{interference}} + \underbrace{n_i(t)}_{\text{noise}}$$

- Goal: To minimize transmit power while guaranteeing a certain level of SINR for each user.

Assuming we have the second-order statistics of the channels $R_i = \mathbb{E}[h_i h_i^H]$, we can formulate the problem as

$$\inf \quad \sum_{i=1}^{M} \|w_i\|_2^2 = \sum_{i=1}^{M} w_i^H w_i = \sum_{i=1}^{M} \langle I, w_i w_i^H \rangle$$

$$\text{s.t.} \quad \underbrace{SINR_i = \frac{w_i^H R_i w_i}{\sum_{j \neq i} w_j^H R_i w_j + \sigma_i^2}}_{\Updownarrow} \geq \gamma_i \ ; \quad i = 1, \dots, M.$$

$$\underbrace{w_i^H R_i w_i}_{\langle R_i, w_i w_i^H \rangle} - \gamma_i \sum_{j \neq i} w_j^H R_i w_j \geq \gamma_i \sigma_i^2$$

$$w_i \in \mathbb{C}^{N_t}$$

The SDR of the above problem is

$$\inf \quad \sum_{i=1}^{M} \langle I, W_i \rangle$$

$$(U) \qquad \text{s.t.} \quad \langle R_i, W_i \rangle - \gamma_i \sum_{j \neq i} \langle R_i, W_j \rangle \geq \gamma_i \sigma_i^2 \ ; \quad i = 1, \dots, M,$$

$$W_1, \dots, W_M \in \mathbb{H}_+^{N_t}.$$

Claim: (U) has an optimal solution. (Exercise: Check the dual)

By the Second theorem, there exists an optimal solution $(W_1^*, \dots, W_M^*)$ to (U) s.t. $\sum_{i=1}^{M} \text{rank}^2(W_i^*) \leq M$. This implies that $\text{rank}(W_i^*) \leq 1$ if $W_i^* \neq 0$ $\forall i$. Hence, the SDR is tight.

(verify)