Affine and Convex Sets

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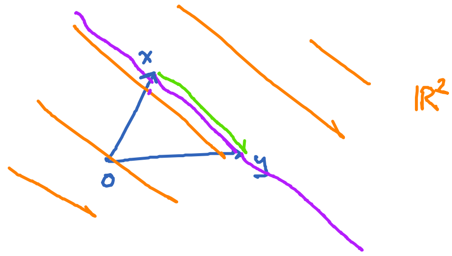
Let $S \subseteq \mathbb{R}^n$ be arbitrary,

Definitions:

① S is a linear subspace if $\forall x, y \in S$, $\alpha, \beta \in \mathbb{R}$,

$$\alpha x + \beta y \in S,$$

linear combination



② S is an affine subspace if $\forall x, y \in S$, $\alpha \in \mathbb{R}$,

$$\alpha x + (1-\alpha) y \in S$$

affine combination

$\alpha x + (1-\alpha) y = y + \alpha (x - y)$

③ S is convex if $\forall x, y \in S$; $\alpha \in [0,1]$,

$$\alpha x + (1-\alpha) y \in S$$

convex combination

Example: Suppose that S is a linear subspace.

Affine? Yes Convex? Yes

Example: Consider $S = \{x\}$. Is this linear? affine? convex?

affine: Yes convex: Yes

linear: if $x = 0$: yes

if $x \neq 0$: no

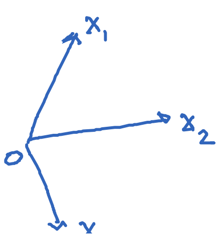
④ Given $x^1, \cdots, x^k \in \mathbb{R}^n$. We say $y = \sum_{i=1}^{k} \alpha_i x^i$ is

a) linear combination of $x^1, \cdots, x^k$ if $\alpha_1, \cdots, \alpha_k \in \mathbb{R}$

b) affine combination of $x^1, \cdots, x^k$ if $\alpha_1, \cdots, \alpha_k \in \mathbb{R}$, $\sum_{i=1}^{k} \alpha_i = 1$

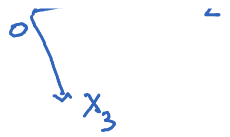
c) convex combination of $x^1, \cdots, x^k$ if $\alpha_1, \cdots, \alpha_k \geq 0$, $\sum_{i=1}^{k} \alpha_i = 1$.

Example:



linear: Space spanned by x^1, x^2, x^3
$= \mathbb{R}^3$

affine: plane through x^1, x^2, x^3



affine: plane through x^1, x^2, x^3

convex: triangle with corners x^1, x^2, x^3

How does an affine set look like?

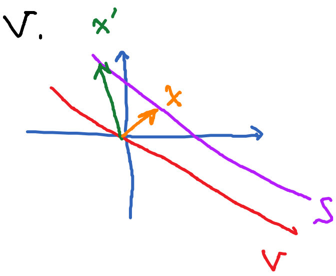
<u>Proposition</u>: The following are equivalent:

1) S is affine

2) Any affine combination of a finite number of points in S belongs to S.

3) S can be written as $S = \{x\} + V \triangleq \{x + v : v \in V\}$

for some $x \in \mathbb{R}^n$ and linear subspace V.



<u>Example</u>. $\mathbb{R}^2$: 3 types of linear subspace

1) 0-dim: $V = \{0\}$

2) 1-dim: Span by a vector $x \neq 0$

$$V = \{\alpha x : \alpha \in \mathbb{R}\}$$

3) 2-dim: Span by 2 linearly independent vectors x, y

$$V = \{\alpha x + \beta y : \alpha, \beta \in \mathbb{R}\} = \mathbb{R}^2$$

<u>Example</u>: Let $S = \{x \in \mathbb{R}^n : Ax = b\}$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.

<u>Claim</u>: S is affine.

<u>Proof</u>: Let $x, y \in S$ and $\alpha \in \mathbb{R}$ be arbitrary.

<u>Want</u>: $\alpha x + (1-\alpha) y \in S$

$x \in S \iff Ax = b$, $\quad y \in S \iff Ay = b$

$$A(\alpha x + (1-\alpha) y) = \alpha \underbrace{Ax}_{=b} + (1-\alpha)\underbrace{Ay}_{=b} = b$$

$$\iff \alpha x + (1-\alpha) y \in S$$

How about convex sets?

Proposition: The following are equivalent:

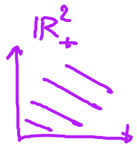
1) S is convex

2) Any convex combination of a finite number of points in S belongs to S.

Examples: (Convex sets)

1) Non-negative orthant

$$\mathbb{R}^n_+ = \{ x \in \mathbb{R}^n : x_i \geq 0 \quad \forall i \}$$

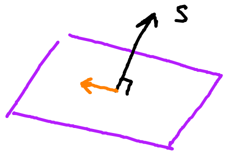


Take $x, y \in \mathbb{R}^n_+$ and $\alpha \in [0,1]$.

Then, $\underbrace{\alpha x}_{\geq 0} + \underbrace{(1-\alpha) y}_{\geq 0} \in \mathbb{R}^n_+$

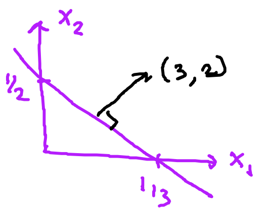
2) Hyperplane

$$H(s,c) = \{ x \in \mathbb{R}^n : \underbrace{s^T x = c}_{\text{linear equation}} \}$$



e.g.: $n=2$

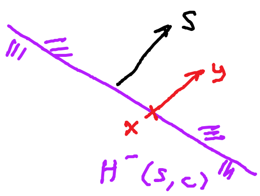
$$H((3,2),1) = \{ (x_1, x_2) : \underbrace{3x_1 + 2x_2 = 1}_{(3,2)^T (x_1, x_2) = 1} \}$$



3) Halfspaces

$$H^+(s,c) = \{ x \in \mathbb{R}^n : s^T x \geq c \}$$

$$H^-(s,c) = \{ x \in \mathbb{R}^n : s^T x \leq c \}$$



Note: $H(s,c) = H^+(s,c) \cap H^-(s,c)$

$$s^T y = s^T (x + \alpha s)$$

$$= \underbrace{s^T x}_{=c} + \underbrace{\alpha s^T s}_{\geq 0} \geq c$$