Convex Sets (cont'd)

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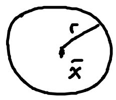
Examples (convex sets)

4) Euclidean balls

$$B(\bar{x}, r) = \{ x \in \mathbb{R}^n : \|x - \bar{x}\|_2 \le r \}$$

center radius



Verify its convexity:

Take $x, y \in B(\bar{x}, r)$; $\alpha \in [0,1]$. Want: $\alpha x + (1-\alpha) y \in B(\bar{x}, r)$

We compute

$$\|\alpha x + (1-\alpha) y - \bar{x}\|_2 = \|\alpha(x - \bar{x}) + (1-\alpha)(y - \bar{x})\|_2$$

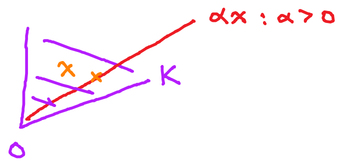
$$\le \alpha \underbrace{\|x - \bar{x}\|_2}_{\le r} + (1-\alpha) \underbrace{\|y - \bar{x}\|_2}_{\le r} \quad \text{(triangle inequality)}$$

$$\le r,$$

Generalization: $B_q(\bar{x}, r) = \{ x \in \mathbb{R}^n : \|x - \bar{x}\|_q \le r \}$ is convex for any $q \ge 1$.

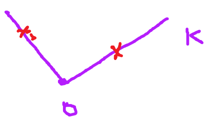
5) Convex cone

Definition: A set $K \subseteq \mathbb{R}^n$ is called a cone if $\forall x \in K$, $\alpha > 0$: $\alpha x \in K$



Q: Must a cone be convex?

A: No. For instance:



Definition: A convex cone is a cone that is convex.

Examples: ① $\mathbb{R}^n_+$ (trivial)

② $S^n_+ = \{ X \in S^n : X \succeq 0 \}$ (Exercise)

S^n: Set of $n \times n$ real symmetric matrices

Convexity-Preserving Operations

Convexity-Preserving Operations

Motivation: So far, we prove convexity by first principles.

Q: Are there other ways?

Examples

① Let $S_1, S_2 \subseteq \mathbb{R}^n$ be convex sets.

- $S_1 \cup S_2$ convex? No.

- $S_1 \cap S_2$ Convex? Yes. (exercise)

② Affine Transformation

Definition: We say that $A : \mathbb{R}^n \to \mathbb{R}^m$ is an affine map if

$$\forall x, y \in \mathbb{R}^n;\ \alpha \in \mathbb{R} \cdot \quad A(\underbrace{\alpha x + (1-\alpha) y}_{\text{affine combination of } x \text{ and } y}) = \alpha A(x) + (1-\alpha) A(y)$$

e.g. $\mathbb{R}^n \ni x \mapsto A(x) = c^T x + d \in \mathbb{R}$; $c \in \mathbb{R}^n$, $d \in \mathbb{R}$

Proposition: Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be an affine map, $S \subseteq \mathbb{R}^n$ be a convex set. Then, $A(S) \triangleq \{ A(x) : x \in S \}$ is convex.

e.g.

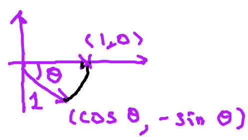
① Translation : $A : \mathbb{R}^n \to \mathbb{R}^n$: $A(x) = x + d$, $d \in \mathbb{R}^n$ given

② Rotation : $A : \mathbb{R}^n \to \mathbb{R}^n$ · $A(x) = Ux$, $U \in \mathbb{R}^{n \times n}$ orthogonal matrix

↳ $U^T U = U U^T = I$

e.g.: $n = 2$

$$U = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$



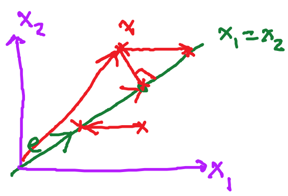
(clockwise rotation by θ)

③ Projection: $A \cdot \mathbb{R}^n \to \mathbb{R}^m$: $A(x) = Px$; P projection matrix

↳ $P^2 = P$

e.g. $P = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$$Px = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix}$$



$$P' = \frac{1}{2} e e^T = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

Note: P' is orthogonal projection.

Mathematically, P is an orthogonal projection if $P^2 = P$ and $P = P^T$

④ Ellipsoid

$$E(\bar{x}, Q) = \{ x \in \mathbb{R}^n : (x-\bar{x})^T Q (x-\bar{x}) \le 1 \}$$

center axes

$\bar{x} \in \mathbb{R}^n$, $Q \succ 0$

positive definite

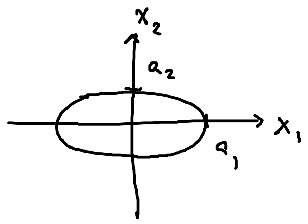
e.g.

a) $Q = \frac{1}{r^2} I \Rightarrow E(\bar{x}, Q) = B(\bar{x}, r)$

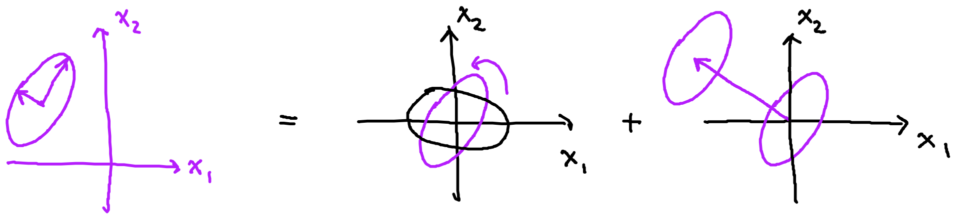
b) $n = 2$:

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} \le 1$$

$$\Leftrightarrow \underbrace{[x_1 \ \ x_2]}_{x^T} \underbrace{\begin{bmatrix}\frac{1}{a_1^2} & \\ & \frac{1}{a_2^2}\end{bmatrix}}_{Q} \underbrace{\begin{bmatrix}x_1\\ x_2\end{bmatrix}}_{x} \le 1$$



c)



Claim: There exists an affine map A s.t.

$$A(B(0,1)) = E(\bar{x}, Q)$$

It follows from the proposition that $E(\bar{x}, Q)$ is convex