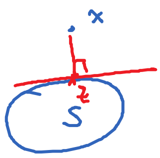
Q: Given a set $S \subseteq \mathbb{R}^n$, $S \neq \phi$, and a point $x \notin S$, we want to find a point $z \in S$ that is closest to x

↳ Euclidean distance

e.g.



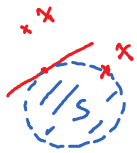
$z \in S$ is closest to x

Formally, $z = \arg\min_{y \in S} \|x-y\|_2$ is the projection of x onto S.

Notation: $z = \Pi_S(x)$

Q: Existence?

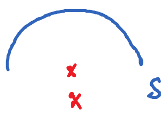
A: Not always



$x \notin S$

Q: Uniqueness?

A: Not always



Every point on S is a projection of x onto S.

Q: Conditions that can guarantee the existence and uniqueness of the projection?

Topological Properties

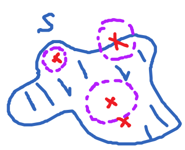
Let $S \subseteq \mathbb{R}^n$ be a set.

Definitions:

① We say that x is an interior point of S if

$$\exists \varepsilon > 0 : B^o(x, \varepsilon) = \{ y \in \mathbb{R}^n : \|x-y\|_2 < \varepsilon \} \subseteq S$$

e.g.



The collection of all interior points of S is called the interior of S denoted by $\text{int}(S)$.

$\text{int}(S)$

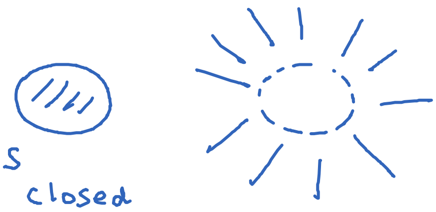
② We say that S is open if $S = \mathrm{int}(S)$.

e.g. 

not open

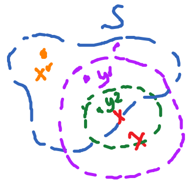
③ We say that S is closed if $\underbrace{\mathbb{R}^n \setminus S}_{S^c}$ is open

S^c complement of S

e.g. 

④ We say that x is a limit point of S if every open ball centered at x contains $y \in S$ s.t. $y \neq x$.

In particular, $\exists$ sequence $y^k \in S$ s.t. $y^k \to x$, $y^k \neq x$.

e.g. 

Note: A limit point need not be in S

e.g. $S = (0, 1]$ Is $x=0$ a limit point of S?

Yes: $y^k = \frac{1}{k} \Rightarrow y^k \in S$, $y^k \to x$, $y^k \neq x$

e.g. A finite point set has no limit point.

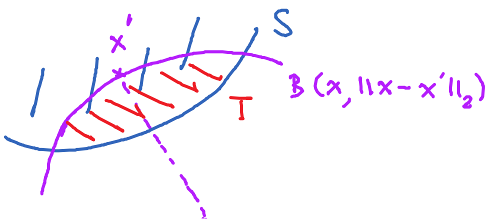
Fact: S is closed iff every limit point of S belongs to S.

Hence, to prove S is closed, take any sequence $\underbrace{y^k \in S}_{(i)}$ s.t. $\underbrace{y^k \to x}_{(ii)}$, $\underbrace{y^k \neq x}_{(iii)}$, then show $x \in S$.

Theorem: Let $S \subseteq \mathbb{R}^n$ be a non-empty, closed, convex set. Then, for every $x \in \mathbb{R}^n$, $\exists$ unique $z \in S$ s.t. $z = \Pi_S(x)$

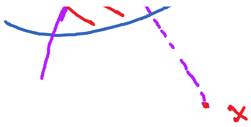
Proof: (Existence)

We may assume that $x \notin S$.



Consider any $x' \in S$ and define

We may assume that $x \notin S$,



Consider any $x' \in S$ and define

$$T = S \cap B(x, \|x - x'\|_2)$$

Observe:

① $\min\limits_{y \in S} \|x - y\|_2 = \min\limits_{y \in T} \|x - y\|_2$

② T is closed (it is the intersection of 2 closed sets)

and bounded (i.e., $\exists R > 0$ s.t. $T \subseteq B(0, R)$)

In other words, T is compact.

Fact (Weierstrass Theorem)

If f is continuous over a compact set T, then both

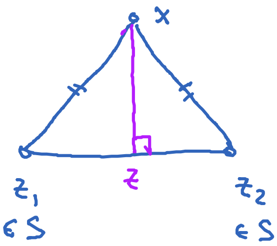
$\min\limits_{x \in T} f(x)$ and $\max\limits_{x \in T} f(x)$ have optimal solutions.

Note: The above argument does not need convexity.

(Uniqueness)

Suppose that $z_1, z_2 \in S$ s.t. $z_1 = \Pi_S(x)$, $z_2 = \Pi_S(x)$,

and $z_1 \neq z_2$.



① $\|x - z_1\|_2 = \|x - z_2\|_2$

② $z \in S$ by convexity of S

③ (Exercise)

$\|x - z\|_2 < \|x - z_1\|_2$

$\Rightarrow$ Contradiction

Q: Let S be as above, $x \notin S$. Suppose that $z \in S$ is given.

How to verify if $z = \Pi_S(x)$?

A: Theorem: Given $x \in \mathbb{R}^n$,

$$z = \Pi_S(x) \iff z \in S, \quad (y-z)^T(x-z) \leq 0 \quad \forall y \in S$$