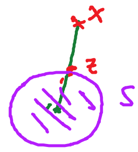
Projection (cont'd), Separation

2022年2月24日 15:33

Example: Consider $S = B(0,1)$. Let $x \notin S$.



Note that $\|x\|_2 > 1$. Then,

$$\Pi_S(x) = \frac{x}{\|x\|_2}$$

Indeed, $\forall y \in B(0,1)$,

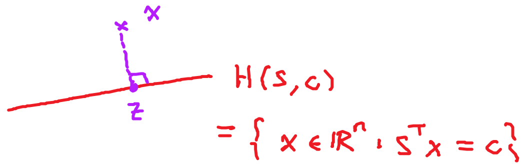
$$\left(y - \frac{x}{\|x\|_2}\right)^T \left(x - \frac{x}{\|x\|_2}\right) = x^T y - \frac{x^T y}{\|x\|_2} - \|x\|_2 + 1$$

$$= \underbrace{\left(1 - \frac{1}{\|x\|_2}\right)}_{>0} \underbrace{x^T y}_{\text{Cauchy-Schwarz}} - \|x\|_2 + 1$$

$$\leq \left(1 - \frac{1}{\|x\|_2}\right) \|x\|_2 \cdot \underbrace{\|y\|_2}_{\leq 1 \ (\because\ y \in B(0,1))} - \|x\|_2 + 1$$

$$\leq \left(1 - \frac{1}{\|x\|_2}\right) \|x\|_2 - \|x\|_2 + 1 = 0$$

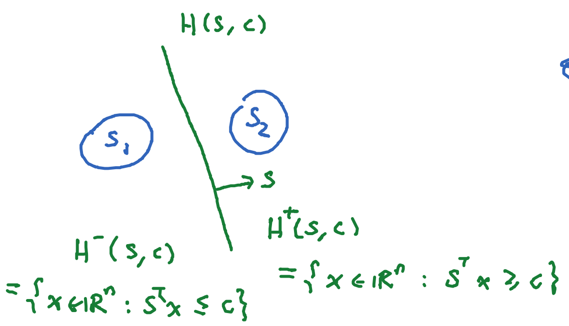
Exercise: Determine $\Pi_{H(s,c)}(x)$.



Separation

Motivation. Given $S_1, S_2 \subseteq \mathbb{R}^n$, how can we certify $S_1 \cap S_2 = \phi$?

Idea:

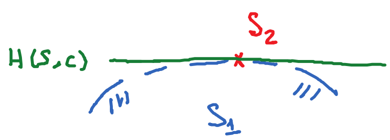


Observe:

$S_1 \subseteq H^-(s,c)$
$S_2 \subseteq H^+(s,c)$ $\longrightarrow$ use hyperplane to certify $S_1 \cap S_2 = \phi$

but

(1)



$S_1 \cap S_2 = \phi$, but a hyperplane cannot "properly" separate them

(2) 

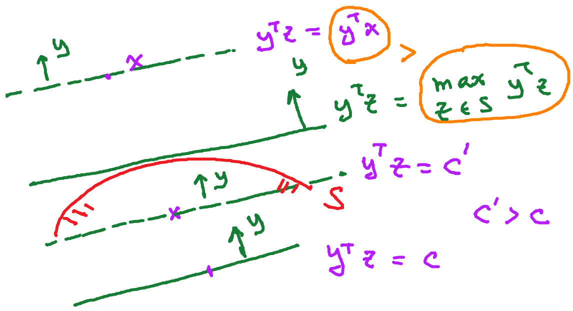
$S_1 \cap S_2 = \phi$, but no hyperplane can separate them

Theorem (Point-Set Separation)

Let $S \subseteq \mathbb{R}^n$ be non-empty, closed, convex and $\underbrace{x \notin S}_{\{x\} \cap S = \phi}$.

Then, $\exists\, y \in \mathbb{R}^n$ s.t.

$$\max_{z \in S} y^T z < y^T x$$



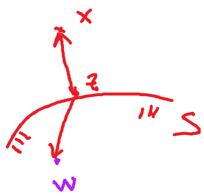
Proof: By the projection theorem,

$z = \Pi_S(x)$ exists and is unique.

Define $y = x - z$. Note that $y \neq 0$,

since $x \notin S$. Then,

$\forall w \in S$, $(w - z)^T y \leq 0$

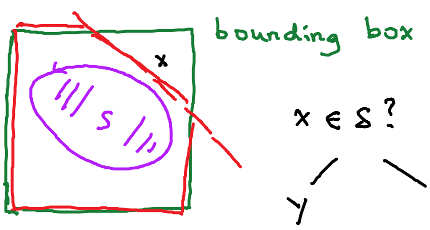


$$\Rightarrow \quad y^T w \leq y^T z = y^T x + y^T \underbrace{(z - x)}_{-y} = y^T x - \|y\|_2^2$$

$$\Rightarrow \quad \max_{w \in S} y^T w \leq y^T x - \underbrace{\|y\|_2^2}_{>0} < y^T x.$$

Application: Cutting-plane method

Find a point in the closed convex set S



$x \in S$?

Y: end

N: Update bounding box and repeat

Convex Functions

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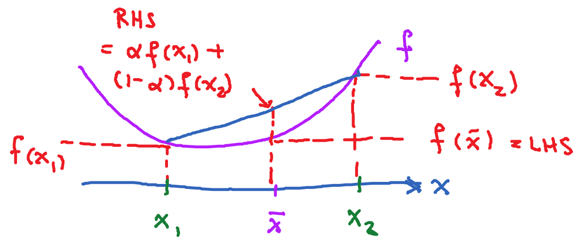
Definitions: Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function that is not identically $+\infty$.

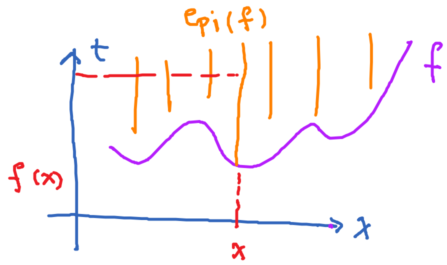
① We say that f is convex if $\forall\, x_1, x_2 \in \mathbb{R}^n$, $\alpha \in [0,1]$,

$$f(\underbrace{\alpha x_1 + (1-\alpha) x_2}_{\bar{x}}) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$$



② The epigraph of f is the set

$$\text{epi}(f) \triangleq \{ (x,t) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq t \} \subseteq \mathbb{R}^{n+1}$$



Fact: f is $\underbrace{\text{convex}}_{\text{as a function}}$ iff epi(f) is $\underbrace{\text{convex}}_{\text{as a set}}$

(Exercise)