Convex Functions (cont'd)

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Definitions: Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function that is not identically $+\infty$.

③ Let $S \subseteq \mathbb{R}^n$ be a set. The indicator of S is

$$\mathbb{1}_S(x) = \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{otherwise} \end{cases}$$

e.g.

$$\underbrace{\inf_{x \in S} f(x)}_{\text{constrained}} \iff \underbrace{\inf_{x \in \mathbb{R}^n} \{ f(x) + \mathbb{1}_S(x) \}}_{\text{unconstrained}}$$

Proposition: Let f, S be as before.

① f is convex $\iff$ $\text{epi}(f)$ is convex.

② S is convex $\iff$ $\mathbb{1}_S(x)$ is convex

(Exercise)

Corollary (Jensen's inequality)

Let f be as before. Then, f is convex iff

$$\forall\, x_1, \ldots, x_k \in \mathbb{R}^n, \quad \alpha_1, \ldots, \alpha_k \geq 0, \quad \sum_{j=1}^{k} \alpha_j = 1 \qquad f\left(\sum_{j=1}^{k} \alpha_j x_j\right) \leq \sum_{j=1}^{k} \alpha_j f(x_j)$$

Note. When $k = 2$, this is just the definition of a convex function.

Proof:

($\Leftarrow$) Obvious

($\Rightarrow$) Suppose that f is convex. Note that

$$(x_j, f(x_j)) \in \text{epi}(f), \quad \forall j \qquad \text{epi}(f) = \{(x,t) : f(x) \leq t\}$$

By the proposition, $\text{epi}(f)$ is convex.

$$\Rightarrow \quad \sum_{j=1}^{k} \alpha_j \underbrace{(x_j, f(x_j))}_{\in\, \text{epi}(f)} \in \text{epi}(f)$$

$$\Rightarrow \quad \sum_{j=1} \alpha_j \underbrace{(x_j, f(x_j))}_{\in \, epi(f)} \quad \in \; epi(f)$$

$$\Rightarrow \quad \Big(\underbrace{\sum_{j=1}^{k} \alpha_j x_j}_{x} \, , \; \underbrace{\sum_{j=1}^{k} \alpha_j f(x_j)}_{t} \Big) \in \; epi(f)$$

$$\Rightarrow \quad f\Big(\sum_{j=1}^{k} \alpha_j x_j \Big) \leq \sum_{j=1}^{k} \alpha_j f(x_j)$$

Operations preserving convexity

① (Non-negative combinations)

Let $f_1, \dots, f_k$ be convex functions; $\alpha_1, \dots, \alpha_k \geq 0$. Then,

$$f = \sum_{j=1}^{k} \alpha_j f_j \quad \text{is convex}$$

Proof: Take any $x_1, x_2 \in \mathbb{R}^n$; $\alpha \in [0,1]$.

$$f(\alpha x_1 + (1-\alpha) x_2) = \sum_{j=1}^{k} \underbrace{\alpha_j}_{\geq 0} \underbrace{f_j(\alpha x_1 + (1-\alpha) x_2)}_{\leq \, \alpha f_j(x_1) + (1-\alpha) f_j(x_2)}$$

$$\leq \sum_{j=1}^{k} \alpha_j \left[\alpha f_j(x_1) + (1-\alpha) f_j(x_2) \right]$$

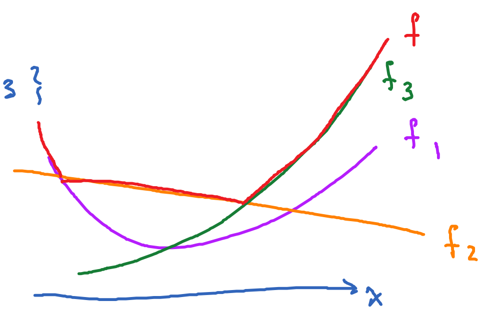
$$= \alpha f(x_1) + (1-\alpha) f(x_2).$$

② (Pointwise supremum)

Let I be index set and $\{f_i\}_{i \in I}$ be a family of convex functions. Then,

$$f = \sup_{i \in I} f_i \quad \text{is convex}$$

e.g. $I = \{1, 2, 3\}$



e.g. Let $\sigma : \mathbb{R}^{m \times n} \to \mathbb{R}_+$ be defined as

$$\sigma(X) = \text{largest singular value of } X = \sqrt{\text{largest eigenvalue of } \underbrace{XX^T}_{\text{psd}}}$$

Claim: $\sigma(\cdot)$ is convex

Proof: By the Courant-Fischer theorem,

$$\sigma(X) = \max_{\substack{\|u\|_2 = 1 \\ \|v\|_2 = 1}} u^T X v$$

Let $\underbrace{f_{u,v}}_{\text{index}}(X) = u^T X v$, $I = \{ (u,v) : \|u\|_2 = 1, \|v\|_2 = 1 \}$.

Then, $$\sigma(X) = \sup_{(u,v) \in I} f_{u,v}(X).$$

Observe that $X \mapsto f_{u,v}(X)$ is linear hence convex

$\hookrightarrow f_{u,v}(\alpha X + \beta Y) = \alpha f_{u,v}(X) + \beta f_{u,v}(Y)$

③ (Composition with increasing convex function)

Let $g : \mathbb{R}^n \to \mathbb{R}$ be convex, $h : \mathbb{R} \to \mathbb{R}$ be convex. Then,

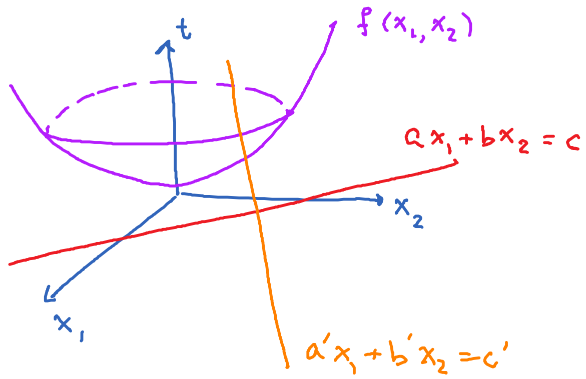
$f = h \circ g$ (i.e., $f(x) = h(g(x))$) is not necessarily convex

e.g. $g(x) = x^2$, $h(x) = -x \Rightarrow f(x) = -x^2$

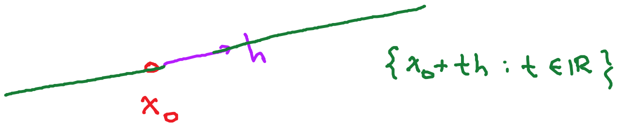
However, if we assume in addition that h is increasing,

then $f = h \circ g$ is convex.

④ (Restriction on Lines)



Definition: Given a point $x_0 \in \mathbb{R}^n$, a direction $h \in \mathbb{R}^n \setminus \{0\}$, we call the set $\{ x_0 + th : t \in \mathbb{R} \}$ a line through x_0 in the direction h.



Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and define

$$\mathbb{R} \ni t \mapsto \underbrace{\tilde{f}_{x_0,h}(t)}_{\text{single-variable}} = f(\underbrace{x_0 + th}_{\in \mathbb{R}^n})$$

to be the restriction of f on the line $\{x_0 + th : t \in \mathbb{R}\}$.

Then,

$$f \text{ is convex} \iff \forall x_0, h, \quad \tilde{f}_{x_0,h} \text{ is convex}$$