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A Duality Analysis on Stochastic Partial Differential Equations

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The duality equations of stochastic partial differential equations are solved in the Sobolev space H^m $(=W_2^m(R^d))$, and the H^m-norm estimates of the solutions are obtained. As an application, the H^m-norm estimates with negative m for the solutions of stochastic partial differential equations are derived.

1. INTRODUCTION

The duality analysis has proved effective and powerful in various fields of mathematics. In differential equation theory, the duality argument is usually applied through the so-called duality equation. Let us begin with the simplest case. Given $A \in L^\infty(0, 1; R^{d\times d})$, $f, F \in L^2(0, 1; R^d)$, and $x_0, \lambda_1 \in R^d$. Consider the following two ordinary differential equations (ODE):

$$\begin{cases} dx(t)/dt = A(t)\,x(t) + f(t), \\ x(0) = x_0, \end{cases} \tag{1.1}$$

$$\begin{cases} d\lambda(t)/dt = -A^T(t)\,\lambda(t) - F(t), \\ \lambda(1) = \lambda_1. \end{cases} \tag{1.2}$$

Using integration by parts, we have

$$\int_0^1 (x(t), F(t))\,dt + (x(1), \lambda_1) = \int_0^1 (\lambda(t), f(t))\,dt + (\lambda(0), x_0). \tag{1.3}$$

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Equation (1.2) is called the duality equation (or, as it is sometimes called, the adjoint equation) of (1.1), and (1.3) is called the duality (adjoint) equality. In general, a duality equation is a *backward* equation with the end state being given. In ODE cases, (1.2) can be easily obtained by inversing the time. But in stochastic problems the duality equations can not be obtained by simply inversing the time, since we must be careful with the adaptness. Bismut [2, 3] first solved the problem, for stochastic differential equations (SDE), by introducing a duality equation with an additional martingale term. In this paper, we will study the duality equations of the following stochastic partial differential equations (SPDE):

$$\begin{cases} dq(t) = [A(t)\,q(t) + f(t)]\,dt + \sum_{k=1}^{d'} [M^k(t)\,q(t) + g^k(t)]\,dW_k(t), \\ q(0) = q_0, \end{cases} \tag{1.4}$$

where $W := (W_1, W_2, ..., W_{d'})$ is a d'-dimensional Brownian motion with $W(0) = 0$, and the random operators $A(t, \omega)$, $M^k(t, \omega)$ are given as

$$\begin{aligned} A(t, \omega)\,\phi(x) :=\ & \partial_i(a^{ij}(t, x, \omega)\,\partial_j\phi(x)) \\ & + b^i(t, x, \omega)\,\partial_i\phi(x) + c(t, x, \omega)\,\phi(x), \end{aligned} \tag{1.5}$$

$$\begin{aligned} M^k(t, \omega)\,\phi(x) :=\ & \sigma^{ik}(t, x, \omega)\,\partial_i\phi(x) \\ & + h^k(t, x, \omega)\,\phi(x), \qquad (t, x, \omega) \in [0, 1] \times R^d \times \Omega, \end{aligned} \tag{1.6}$$

where a^{ij}, b^i, c, σ^{ik}, and h^k are real valued functions, for $i, j = 1, 2, ..., d$; $k = 1, 2, ..., d'$. Note $\partial_i := \partial/\partial x_i$, and the conventional repeated indices for summation are used.

When the operators M^k are of order zero (i.e., $\sigma^{ik} = 0$), Bensoussan [1] has derived the duality equation of (1.4) by a rather complicated method, and it seems that his method fails to work if $\sigma^{ik} \neq 0$, the case that is of importance in that (σ^{ik}) influences the behavior of the solution of (1.4) just as strongly as does (a^{ij}). Employing a finite dimensional approximation approach, the duality equation of (1.4) for the general case $\sigma^{ik} \neq 0$ has been derived in Zhou [11] as

$$\begin{cases} d\lambda(t) = -[A^*(t)\,\lambda(t) + \sum_{k=1}^{d'} M^{k*}(t)\,r^k(t) + F(t)]\,dt \\ \qquad\qquad + \sum_{k=1}^{d'} r^k(t)\,dW_k(t), \qquad t \in [0, 1], \\ \lambda(1) = G, \end{cases} \tag{1.7}$$

where $A^*(t)$, $M^{k*}(t)$ are the adjoints of $A(t)$, $M^k(t)$, respectively, given as (omitting to write ω)

$$A^*(t)\,\phi(x) := \partial_i(a^{ij}(t,x)\,\partial_j\phi(x)) - b^i(t,x)\,\partial_i\phi(x) + [c(t,x) - \partial_i b^i(t,x)]\,\phi(x), \tag{1.8}$$

$$M^{k*}(t)\,\phi(x) := -\sigma^{ik}(t,x)\,\partial_i\phi(x) + [h^k(t,x) - \partial_i\sigma^{ik}(t,x)]\,\phi(x), \qquad x \in R^d. \tag{1.9}$$

Moreover, the existence and uniqueness of the $W(t)$-*adapted* solution *pair* (λ, r) for (1.7) have been established in the space $L^2([0,1]\times\Omega; H^1)\times L^2([0,1]\times\Omega; H^0)^{d'}$ in [11], where $H^m :=$ the Sobolev space $W_2^m(R^d)$.

The purpose of this paper is to study further the analytic and qualitative properties of the solution pair of the duality equation of (1.7); more precisely, we hope to solve (1.7) in the space of Sovolev type $L^2([0,1]\times\Omega; H^{m+1})\times L^2([0,1]\times\Omega; H^m)^{d'}$ $(m \geqslant 0)$ and give the corresponding estimates of the Sobolev norms. Our method relies heavily on the delicate results of Krylov and Rozovskii [4–6] concerning the SPDE theory together with some a prior estimates of differential operators. As an application, the H^m-norm estimates with *negative* m for the solutions of SPDE (1.4) are obtained through the duality relationship. The application of our results to the optimal stochastic control theory and the Hamilton–Jacobi–Bellman equation in infinite dimensional spaces will be studied in some later papers.

2. Preliminaries

We denote by H^m the Sobolev space

$$H^m := \{\phi : D^\alpha\phi \in L^2(R^d), \text{ for any } \alpha := (\alpha_1, ..., \alpha_d) \text{ with } |\alpha| := |\alpha_1| + \cdots + |\alpha_d| \leqslant m\}, \qquad m = 0, 1, 2, ...,$$

with the Sobolev norm

$$\|\phi\|_m := \left\{ \sum_{|\alpha| \leqslant m} \int_{R^d} |D^\alpha\phi(x)|^2\, dx \right\}^{1/2}, \qquad \text{for} \quad \phi \in H^m.$$

Denote by $H^{-m} := (H^m)^*$, the dual of H^m for $m = 1, 2, ...$, under that H^0 is identified with its dual.

Denote by $\langle \cdot, \cdot \rangle_m$ the duality pairing between H^{m-1} and H^{m+1} under $(H^m)^* = H^m$, and by $(\cdot, \cdot)_m$ the inner product in H^m.

For any second-order differential operator L which has the same form as (1.5), when we write $\langle L\phi, \psi \rangle_m$, then L is understood to be an operator from H^{m+1} to H^{m-1} by using formally Green's formula. For example, for the operator $A(t)$ defined as in (1.5), we have

$$\langle A(t)\phi, \psi\rangle_m := -(a^{ij}(t,\cdot)\,\partial_j\phi, \partial_i\psi)_m + (b^i(t,\cdot)\,\partial_i\phi, \psi)_m + (c(t,\cdot)\phi, \psi)_m. \tag{2.1}$$

Remark 2.1. Let $A(t)$, $M^k(t)$ be given by (1.5), (1.6) and their formal adjoints $A^*(t)$, $M^{k*}(t)$ given by (1.8), (1.9). We have obviously $\langle A(t)\phi, \psi\rangle_0 = \langle \phi, A^*(t)\psi\rangle_0$, $(M^k(t)\phi, \psi)_0 = (\phi, M^{k*}(t)\psi)_0$ for $\phi, \psi \in H^1$. However, neither $\langle A(t)\phi, \psi\rangle_m = \langle \phi, A^*(t)\psi\rangle_m$ nor $(M^k(t)\phi, \psi)_m = (\phi, M^{k*}(t)\psi)_m$ holds if $m \geqslant 1$.

For $\alpha, \beta \in (-\infty, +\infty)$ with $\alpha < \beta$, we are given a filtered probability space $(\Omega, \mathscr{F}, P, \mathscr{F}_t \colon \alpha \leqslant t \leqslant \beta)$ and a Hilbert space X. For $p \in [1, +\infty]$, define

$$L^p_{\mathscr{F}}(\alpha, \beta; X) := \{\phi \colon \phi \text{ is an } X\text{-valued } \mathscr{F}_t\text{-adapted process on } [\alpha, \beta], \text{ and } \phi \in L^p([\alpha, \beta] \times \Omega; X)\}.$$

We identify ϕ and ϕ' in $L^p_{\mathscr{F}}(\alpha, \beta; X)$ if $E\int_\alpha^\beta \|\phi(t) - \phi'(t)\|^p\, dt = 0$. In particular, $L^2_{\mathscr{F}}(\alpha, \beta; X)$ is a Hilbert space as a subspace of the Hilbert space $L^2([\alpha, \beta] \times \Omega; X)$.

Throughout this paper we fix a standard probability space $(\Omega, \mathscr{F}, P)$ with a d'-dimensional Brownian motion $\{W(t) \colon 0 \leqslant t \leqslant 1\}$ and the filtration $\mathscr{F}_t := \sigma\{W(s) \colon 0 \leqslant s \leqslant t\}$. Let us fix an integer $m \geqslant 0$ and positive constants K, δ. We introduce the following conditions on the coefficients of $A(t)$, $M^k(t)$:

(A1)$_m$ a^{ij}, b^i, c, σ^{ik}, and h^k are measurable in (t, ω) for each x and are adapted to $\{\mathscr{F}_t\}$; the functions a^{ij}, b^i, c, σ^{ik}, h^k, $\partial_j b^i$, $\partial_j \sigma^{ik}$, and $\partial_j h^k$ and their derivatives in x up to the order m do not exceed K in absolute value.

(A2) $a^{ij} = a^{ji}$, $i, j = 1, 2, \ldots, d$; the matrix $(A^{ij}) := (a^{ij} - 1/2\sum_{k=1}^{d'} \sigma^{ik}\sigma^{jk}) \geqslant 0$ for all (t, x, ω).

(A2)$'$ $a^{ij} = a^{ji}$, $i, j = 1, 2, \ldots, d$; the matrix (A^{ij}) is uniformly positive definite: $A^{ij}\xi_i\xi_j \geqslant \delta|\xi|^2$, for any (t, x, ω), and any $\xi \in R^d$.

The SPDE theory has been studied deeply by Krylov and Rozovskii [4–6], including the H^m-norm estimates of the solutions of SPDE for nonnegative m. We shall state some of their results in a way which is convenient to our later discussion. First, the following a priori estimates will play an important role in this paper.

LEMMA 2.1. (a) *Assume* (A1)$_m$, (A2)$'$ *for some* $m \geqslant 0$. *Then there exists a constant* N_1, *depending only on* K, m, δ, *such that*

$$2[\langle A(t)\phi, \phi\rangle_{\bar m} + \langle f, \phi\rangle_{\bar m}] + \sum_{k=1}^{d'} \|M^k(t)\phi + g^k\|_{\bar m}^2$$

$$\leqslant -\delta\|\phi\|_{\bar m+1}^2 + N_1\left(\|\phi\|_{\bar m}^2 + \|f\|_{\bar m-1}^2 + \sum_{k=1}^{d'} \|g^k\|_{\bar m}^2\right),$$

for any $\phi \in H^{\bar m+1}$, $f \in H^{\bar m-1}$, $g \in H^{\bar m}$; $\bar m = 0, 1, ..., m.$ (2.2)

(b) *Assume* $(\mathrm{A1})_m$, (A2) *with* $\sigma^{ik} = 0$ *for some* $m \geqslant 0$. *Then there exists a constant* N_2, *depending only on* K, m, *such that*

$$2[\langle A(t)\phi, \phi\rangle_{\bar m} + (f, \phi)_{\bar m}] + \sum_{k=1}^{d'} \|M^k(t)\phi + g^k\|_{\bar m}^2$$

$$\leqslant N_2\left(\|\phi\|_{\bar m}^2 + \|f\|_{\bar m}^2 + \sum_{k=1}^{d'} \|g^k\|_{\bar m}^2\right),$$

for any $\phi \in H^{\bar m+1}$, $f, g \in H^{\bar m}$; $\bar m = 0, 1, ..., m.$ (2.3)

Proof. This is an easy consequence of [5, Lemma 2.1] (see also [11]). ∎

Remark 2.2. The estimate (2.2) (resp. (2.3)) holds for *any* second- and first-order differential operators which have the forms of (1.5) and (1.6) whose coefficients satisfy $(\mathrm{A1})_m$ and (A2)′ (resp. (A2)).

PROPOSITION 2.1 (Krylov and Rozovskii [4–6]). (a) *Assume* $(\mathrm{A1})_m$, (A2)′ *for some* $m \geqslant 0$, *and assume* $f \in L^2_{\mathscr{F}}(0, 1; H^{m-1})$, $g^k \in L^2_{\mathscr{F}}(0, 1; H^m)$, $q_0 \in L^2(\Omega, \mathscr{F}_0; H^m)$. *Then* (1.4) *has a unique solution* $q \in L^2_{\mathscr{F}}(0, 1; H^{m+1}) \cap L^2(\Omega; C(0, 1; H^m))$ *and there exists a constant* N_3, *depending only on* K, m, δ, *such that*

$$\sup_{0 \leqslant t \leqslant 1} E\|q(t)\|_{\bar m}^2 + E\int_0^1 \|q(t)\|_{\bar m+1}^2\, dt$$

$$\leqslant N_3 E\left\{\|q_0\|_{\bar m}^2 + \int_0^1 \left[\|f(t)\|_{\bar m-1}^2 + \sum_{k=1}^{d'} \|g^k(t)\|_{\bar m}^2\right] dt\right\},$$

$\bar m = 0, 1, ..., m.$ (2.4)

(b) *Assume* $(\mathrm{A1})_m$, (A2) *with* $\sigma^{ik} = 0$ *for some* $m \geqslant 1$, *and assume* $f, g^k \in L^2_{\mathscr{F}}(0, 1; H^m)$, $q_0 \in L^2(\Omega, \mathscr{F}_0; H^m)$. *Then* (1.4) *has a unique solution* $q \in L^2_{\mathscr{F}}(0, 1; H^m) \cap L^2(\Omega; C(0, 1; H^{m-1}))$ *and there exists a constant* N_4, *depending only on* K, m, *such that*

$$\sup_{0\leq t\leq 1} E\|q(t)\|_{\bar{m}}^2$$
$$\leq N_4 E\left\{\|q_0\|_{\bar{m}}^2+\int_0^1\left[\|f(t)\|_{\bar{m}}^2+\sum_{k=1}^{d'}\|g^k(t)\|_{\bar{m}}^2\right]dt\right\},$$
$$\bar{m}=0, 1, ..., m. \tag{2.5}$$

We conclude this section with the following useful remark.

Remark 2.3. Define $\Lambda := 1-\Delta$, where Δ is the Laplacian on R^d. The operator Λ maps H^1 into H^{-1} and has an inverse Λ^{-1}. It is easy to check that $\Lambda^{-1}H^m = H^{m+2}$. Moreover, if m is a nonnegative integer, then

$$(\phi, \Lambda^{-m}\psi)_m = (\phi, \psi)_0, \tag{2.6}$$

for any $\phi \in H^m$, $\psi \in H^0$ (see, for example, [4]).

3. Duality Equation: Nondegenerate Case

Throughout this section, we assume $(A1)_m$ and $(A2)'$ for some $m \geq 0$.

Lemma 3.1. *Let the operators $M^k(t)$, $M^{k*}(t)$ be defined by* (1.6), (1.9), *respectively. Then, there exists a constant N_5 which depends only on K and m, such that*

$$|(M^k(t)\phi, \psi)_{\bar{m}} - (\phi, M^{k*}(t)\psi)_{\bar{m}}| \leq N_5 \|\phi\|_{\bar{m}} \|\psi\|_{\bar{m}}, \tag{3.1}$$

for any $\phi, \psi \in H^{m+1}$; $\bar{m}=0, 1, ..., m$, $k=1, 2, ..., d'$.

Proof. Fix $\bar{m}$. We will denote by $I_i(\phi, \psi)$ $(i=1, 2, ...)$ some finite sums of terms of the form $\sum_{|\beta|\leq \bar{m}, |\gamma|\leq \bar{m}} (f^\beta D^\beta\phi, g^\gamma D^\gamma\psi)_0$, where f^β, g^γ are bounded measurable functions.

Then it is easy to compute that (we omit to write the variables t, x)

$$(M^k\phi, \psi)_{\bar{m}} - (\phi, M^{k*}\psi)_{\bar{m}}$$
$$= \sum_{|\alpha|=\bar{m}} \{(D^\alpha(\sigma^{ik}\partial_i\phi), D^\alpha\psi)_0 - (D^\alpha\phi, D^\alpha(-\sigma^{ik}\partial_i\psi))_0\} + I_1(\phi, \psi)$$
$$= \sum_{|\alpha|=\bar{m}} \{(\sigma^{ik} D^\alpha(\partial_i\phi), D^\alpha\psi)_0 + (D^\alpha\phi, \sigma^{ik} D^\alpha(\partial_i\psi))_0\} + I_2(\phi, \psi)$$
$$= \sum_{|\alpha|=\bar{m}} \{-(D^\alpha\phi, \partial_i(\sigma^{ik} D^\alpha\psi))_0 + (D^\alpha\phi, \sigma^{ik} D^\alpha(\partial_i\phi))_0\} + I_2(\phi, \psi)$$
$$= -\sum_{|\alpha|=\bar{m}} (D^\alpha\phi, (\partial_i\sigma^{ik})\cdot D^\alpha\psi)_0 + I_2(\phi, \psi) = I_3(\phi, \psi).$$

This yields the desired result. ∎

Define a linear operator $M^{k\Delta}(t)$ mapping H^{m+1} into H^m in the following way. For $\phi \in H^{m+1}$,

$$(M^{k\Delta}(t)\phi, \psi)_m := \lim_{\substack{\psi_n \in H^{m+1} \\ \psi_n \to \psi \text{ in } H^m}} (\phi, M^{k*}(t)\psi_n), \qquad \text{for any} \quad \psi \in H^m. \tag{3.2}$$

LEMMA 3.2. *The above $M^{k\Delta}(t)$ is well-defined. Further, we have*

$$\|M^{k\Delta}(t)\phi - M^k(t)\phi\|_m \leq N_5 \|\phi\|_m, \qquad \textit{for any} \quad \phi \in H^{m+1}. \tag{3.3}$$

Proof. Let $\phi \in H^{m+1}$ and $\{\psi_n\} \subset H^{m+1}$ be Cauchy in H^m. Then Lemma 3.1 yields

$$\begin{aligned} &|(\phi, M^{k*}(t)(\psi_n - \psi_{n'}))_m| \\ &\quad \leq N_5 \|\phi\|_m \|\psi_n - \psi_{n'}\|_m + |(M^k(t)\phi, \psi_n - \psi_{n'})_m| \to 0, \qquad \text{as} \quad n, n' \to \infty, \end{aligned}$$

hence $(\phi, M^{k*}(t)\psi_n)_m$ converges as $n \to \infty$, and the limit is independent of the choice of the sequence $\{\psi_n\}$ that converges to a fixed $\psi \in H^m$ in H^m-topology. So (3.2) is well-defined. Moreover, for any $\phi \in H^{m+1}$ and $\psi \in H^m$,

$$\begin{aligned} &|(M^{k\Delta}(t)\phi, \psi)_m - (M^k(t)\phi, \psi)_m| \\ &\quad = \lim_{\substack{\psi_n \in H^{m+1} \\ \psi_n \to \psi \text{ in } H^m}} |(\phi, M^{k*}(t)\psi_n)_m - (M^k(t)\phi, \psi_n)_m| \\ &\quad \leq \lim_n N_5 \|\phi\|_m \|\psi_n\|_m = N_5 \|\phi\|_m \|\psi\|_m, \end{aligned} \tag{3.4}$$

thus (3.3) follows. ∎

Before studying the duality equation (1.7), we give the definition of the solution.

DEFINITION 3.1. A *pair* $(\lambda, r) \in L^2_{\mathcal{F}}(0, 1; H^1) \times L^2_{\mathcal{F}}(0, 1; H^0)^{d'}$ is said to be a solution of the equation (1.7), if for any $\eta \in C_0^\infty(R^d)$ (=smooth function on R^d with compact support) and almost all $(t, \omega) \in [0, 1] \times \Omega$,

$$\begin{aligned} &(\lambda(t), \eta)_0 \\ &\quad = (G, \eta)_0 + \int_t^1 \left[\langle \lambda(s), A(s)\eta \rangle_0 + \sum_{k=1}^{d'} (r^k(s), M^k(s)\eta)_0 + \langle F(s), \eta \rangle_0 \right] ds \\ &\qquad - \sum_{k=1}^{d'} \int_t^1 (r^k(s), \eta)_0 \, dW_k(s). \end{aligned} \tag{3.5}$$

THEOREM 3.1. *Assume that* $F \in L^2_{\mathscr{F}}(0, 1; H^{m-1})$, $G \in L^2(\Omega, \mathscr{F}_1; H^m)$. *Then* (1.7) *admits a unique solution* $(\lambda, r) \in L^2_{\mathscr{F}}(0, 1; H^{m+1}) \times L^2_{\mathscr{F}}(0, 1; H^m)^{d'}$, *where* $r := (r^1, r^2, ..., r^{d'})$. *Moreover, there exists a constant* N_6, *depending only on K, m, and* δ, *such that*

$$\sup_{0 \leq t \leq 1} \|\lambda(t)\|^2_{\bar{m}} + E\int_0^1 \left[\|\lambda(t)\|^2_{\bar{m}+1} + \sum_{k=1}^{d'} \|r^k(t)\|^2_{\bar{m}} \right] dt$$

$$\leq N_6 E\left[\int_0^1 \|F(t)\|^2_{\bar{m}-1}\, dt + \|G\|^2_{\bar{m}} \right], \qquad \bar{m} = 0, 1, ..., m. \tag{3.6}$$

Proof. To avoid notational complexity, we will prove the theorem for $d' = 1$ (there is no essential difficulty when $d' > 1$). Thus the index k will be dropped throughout the proof.

Uniqueness. Suppose $(\lambda, r) \in L^2_{\mathscr{F}}(0, 1; H^1) \times L^2_{\mathscr{F}}(0, 1; H^0)$ satisfies

$$\begin{cases} d\lambda(t) = -[A^*(t)\, \lambda(t) + M^*(t)\, r(t)]\, dt + r(t)\, dW(t), & t \in [0, 1], \\ \lambda(1) = 0, \end{cases}$$

then Ito's formula yields

$$\begin{aligned} E\|\lambda(t)\|^2_0 &= 2E\int_t^1 [\langle \lambda(s), A(s)\, \lambda(s) \rangle_0 + (r(s), M(s)\, \lambda(s))_0]\, ds \\ &\quad - E\int_t^1 \|r(s)\|^2_0\, ds \\ &\leq 2E\int_t^1 [\langle \lambda(s), A(s)\, \lambda(s) \rangle_0 + 1/2\, \|M(s)\, \lambda(s)\|^2_0] \\ &\leq 2N_1 E\int_t^1 \|\lambda(s)\|^2_0\, ds, \end{aligned}$$

hence $\lambda(t) \equiv 0$ by virtue of Gronwall's inequality. By Definition 3.1, for any $\phi \in H^1$,

$$\int_0^t (r(s), M(s)\phi)_0\, ds - \int_0^t (r(s), \phi)_0\, dW(s) \equiv 0.$$

The uniqueness of decompositions of the semimartingale leads to $(r(t), \phi)_0 \equiv 0$, hence $r(t) \equiv 0$. This proves the uniqueness.

Existence. Consider the triplet (H^{m-1}, H^m, H^{m+1}) with $(H^m)^* = H^m$. Let $e_1, e_2, ..., e_n, ...$ be a Hilbert basis of H^{m+1}, which is orthonormal as a basis of H^m.

Fix n, by Bismut [3], there exists uniquely $\tilde{\lambda}_n := (\lambda_{n1}, \lambda_{n2}, ..., \lambda_{nn})^T \in L^2_{\mathscr{F}}(0, 1; R^n)$ and $\tilde{r}_n := (r_{n1}, r_{n2}, ..., r_{nn})^T \in L^2_{\mathscr{F}}(0, 1; R^n)$ such that

$$\begin{cases} d\lambda_{ni}(t) = -\Big[\sum_{j=1}^n \langle A^*(t)e_j, e_i\rangle_m \lambda_{nj}(t) + \sum_{j=1}^n (e_j, M^{\Delta}(t)e_i)_m r_{nj}(t) \\ \qquad + \langle F(t), e_i\rangle_m\Big] dt + r_{ni}(t)\, dW(t), \qquad t \in [0, 1], \\ \lambda_{ni}(1) = G_{ni}, \qquad i = 1, 2, ..., n, \end{cases} \tag{3.7}$$

where $M^{\Delta}(t)$ is defined by (3.2), $G_{ni} \in L^2(\Omega, \mathscr{F}_1; R^1)$, and $\sum_{i=1}^n G_{ni} e_i \equiv G_n \to G$ in $L^2(\Omega; H^m)$ as $n \to \infty$. Define $\lambda_n := \sum_{i=1}^n \lambda_{ni} e_i \in L^2_{\mathscr{F}}(0, 1; H^{m+1})$, $r_n := \sum_{i=1}^n r_{ni} e_i \in L^2_{\mathscr{F}}(0, 1; H^{m+1})$. Then Ito's formula implies

$$\begin{aligned} E\|\lambda_n(t)\|_m^2 &= E\|G_n\|_m^2 + 2E\int_t^1 [\langle A^*(s)\lambda_n(s), \lambda_n(s)\rangle_m \\ &\quad + (r_n(s), M^{\Delta}(s)\lambda_n(s))_m + \langle F(s), \lambda_n(s)\rangle_m - 1/2\|r_n(s)\|_m^2]\, ds \\ &\leqslant E\|G_n\|_m^2 + 2E\int_t^1 [\langle A^*(s)\lambda_n(s), \lambda_n(s)\rangle_m \\ &\quad + (M(s)\lambda_n(s), r_n(s))_m + N_5\|\lambda_n(s)\|_m \|r_n(s)\|_m \\ &\quad + \langle F(s), \lambda_n(s)\rangle_m - 1/2\|r_n(s)\|_m^2]\, ds \qquad \text{(by Lemma 3.2)} \\ &\leqslant E\|G_n\|_m^2 + E\int_t^1 [2\langle A^*(s)\lambda_n(s), \lambda_n(s)\rangle_m \\ &\quad + \|M(s)\lambda_n(s)\|_m^2 + N_5\|\lambda_n(s)\|_m^2 + N_5\|r_n(s)\|_m^2 \\ &\quad + 2/\delta \cdot \|F(s)\|_{m-1}^2 + \delta/2 \cdot \|\lambda_n(s)\|_{m+1}^2]\, ds \\ &\leqslant E\|G_n\|_m^2 + E\int_t^1 [-\delta\|\lambda_n(s)\|_{m+1}^2 + (N_1 + N_5)\|\lambda_n(s)\|_m^2 \\ &\quad + N_5\|r_n(s)\|_m^2 + 2/\delta \cdot \|F(s)\|_{m-1}^2 + \delta/2 \cdot \|\lambda_n(s)\|_{m+1}^2]\, ds, \end{aligned}$$

hence Gronwall's inequality yields

$$\begin{aligned} &\sup_{0 \leqslant t \leqslant 1} E\|\lambda_n(t)\|_m^2 + \delta/2 \cdot E\int_0^1 \|\lambda_n(t)\|_{m+1}^2\, dt \\ &\qquad \leqslant N_7 E\left[\int_0^1 (\|F(t)\|_{m-1}^2 + \|r_n(t)\|_m^2)\, dt + \|G_n\|_m^2\right], \end{aligned} \tag{3.8}$$

where N_7 depends only on K, m, δ.

Now let $\tilde{\rho}_n := (\rho_{n1}, \rho_{n2}, ..., \rho_{nn})^T \in L^2_{\mathscr{F}}(0, 1; R^n)$ be the solution of the following SDE in R^n:

$$\begin{cases} d\rho_{ni}(t) = \sum_{j=1}^{n} \langle e_j, A^*(t)e_i \rangle_m \rho_{nj}(t)\, dt \\ \qquad\qquad + \left[\sum_{j=1}^{n} (M^A(t)e_j, e_i)_m \rho_{nj}(t) + r_{ni}(t) \right] dW(t), \\ \rho_{ni}(0) = 0, \qquad i = 1, 2, ..., n. \end{cases} \tag{3.9}$$

Define $\rho_n := \sum_{i=1}^n \rho_{ni} e_i \in L^2_{\mathscr{F}}(0, 1; H^{m+1})$. Choose $\varepsilon > 0$ such that

$$2\varepsilon \|M(s)\, \rho_n(s)\|^2_m \leqslant \delta/2\, \|\rho_n(s)\|^2_{m+1} \qquad \text{(noting } (\mathrm{A1})_m).$$

Using the inequality $(a+b)^2 \leqslant (1+\varepsilon)a^2 + (1+1/\varepsilon)b^2$, we have by Lemma 3.2,

$$\begin{aligned} &2\langle A^*(s)\, \rho_n(s), \rho_n(s) \rangle_m + \|M^A(s)\, \rho_n(s) + r_n(s)\|^2_m \\ &\leqslant 2\langle A^*(s)\, \rho_n(s), \rho_n(s) \rangle_m + (1+\varepsilon)\, \|M(s)\, \rho_n(s) + r_n(s)\|^2_m \\ &\quad + (1 + 1/\varepsilon) N_5^2\, \|\rho_n(s)\|^2_m \\ &\leqslant 2\langle A^*(s)\, \rho_n(s), \rho_n(s) \rangle_m + \|M(s)\, \rho_n(s) + r_n(s)\|^2_m \\ &\quad + 2\varepsilon \|M(s)\, \rho_n(s)\|^2_m + 2\varepsilon \|r_n(s)\|^2_m + (1 + 1/\varepsilon) N_5^2\, \|\rho_n(s)\|^2_m \\ &\leqslant -\delta \|\rho_n(s)\|^2_{m+1} + N_1 \|r_n(s)\|^2_m + \delta/2\, \|\rho_n(s)\|^2_{m+1} \\ &\quad + 2\varepsilon \|r_n(s)\|^2_m + (1 + 1/\varepsilon) N_5^2\, \|\rho_n(s)\|^2_m \\ &\leqslant -\delta/2\, \|\rho_n(s)\|^2_{m+1} + N_8(\|\rho_n(s)\|^2_m + \|r_n(s)\|^2_m). \end{aligned} \tag{3.10}$$

Now it follows by (3.9),

$$\begin{aligned} E\|\rho_n(t)\|^2_m &= E\int_0^t [2\langle A^*(s)\, \rho_n(s), \rho_n(s) \rangle_m \\ &\qquad + \|M^A(s)\, \rho_n(s) + r_n(s)\|^2_m]\, ds \\ &\leqslant -\delta/2 E \int_0^t \|\rho_n(s)\|^2_{m+1}\, ds \\ &\qquad + N_8 E \int_0^t [\|\rho_n(s)\|^2_m + \|r_n(s)\|^2_m]\, ds. \end{aligned}$$

Applying Gronwall's inequality, we obtain

$$\begin{aligned} &\sup_{0 \leqslant t \leqslant 1} E\|\rho_n(t)\|^2_m + \delta/2 E \int_0^1 \|\rho_n(t)\|^2_{m+1}\, dt \\ &\qquad \leqslant N_8 \exp(N_8) E \int_0^1 \|r_n(t)\|^2_m\, dt. \end{aligned} \tag{3.11}$$

Appealing to Ito's formula, we get

$$\begin{aligned}
d \sum_{i=1}^{n} \lambda_{ni}(t)\, \rho_{ni}(t)
&= \sum_{i=1}^{n} \{ -[\langle A^*(t)\, \lambda_n(t), e_i \rangle_m + (r_n(t), M^{\Delta}(t) e_i)_m \\
&\quad + \langle F(t), e_i \rangle_m]\, \rho_{ni}(t) + \lambda_{ni}(t) \langle \rho_n(t), A^*(t) e_i \rangle_m \\
&\quad + r_{ni}(t) [(M^{\Delta}(t)\, \rho_n(t), e_i)_m + r_{ni}(t)] \}\, dt + \{ \cdots \}\, dW(t) \\
&= [\|r_n(t)\|_m^2 - \langle F(t), \rho_n(t) \rangle_m]\, dt + \{ \cdots \}\, dW(t).
\end{aligned}$$

Integrating from 0 to 1 and taking expectation, we find out

$$\begin{aligned}
E \int_0^1 \|r_n(t)\|_m^2\, dt
&= E \left[\int_0^1 \langle F(t), \rho_n(t) \rangle_m\, dt + (G_n, \rho_n(1))_m \right] \\
&\leqslant \left(E \int_0^1 \|F(t)\|_{m-1}^2\, dt \right)^{1/2} \left(E \int_0^1 \|\rho_n(t)\|_{m+1}^2 \right)^{1/2} \\
&\quad + (E \|G_n\|_m^2)^{1/2} (E \|\rho_n(1)\|_m^2)^{1/2} \\
&\leqslant \left(N_9 E \int_0^1 \|r_n(t)\|_m^2\, dt \right)^{1/2} \\
&\quad \times \left\{ \left(E \int_0^1 \|F(t)\|_{m-1}^2\, dt \right)^{1/2} + (E \|G_n\|_m^2)^{1/2} \right\},
\end{aligned}$$

where $N_9 := \max\{N_8 \exp(N_8), 2/\delta \cdot N_8 \exp(N_8)\}$, hence

$$E \int_0^1 \|r_n(t)\|_m^2\, dt \leqslant 2 N_9 E \left[\int_0^1 \|F(t)\|_{m-1}^2\, dt + \|G_n\|_m^2 \right]. \tag{3.12}$$

Combining (3.8) and (3.12), we know that there exist subsequence $\{n'\}$ of $\{n\}$ and $(\lambda, r) \in L^2_{\mathscr{F}}(0, 1; H^{m+1}) \times L^2_{\mathscr{F}}(0, 1; H^m)$ such that

$$\lambda_{n'} \to \lambda \quad \text{weakly in } L^2([0,1] \times \Omega; H^{m+1}) \tag{3.13}$$

$$r_{n'} \to r \quad \text{weakly in } L^2([0,1] \times \Omega; H^m), \quad \text{as } n' \to \infty. \tag{3.14}$$

Let us now show (λ, r) satisfies (1.7). Let γ be an absolutely continuous function from $[0, 1]$ to R^1 with $\dot{\gamma} := d\gamma/dt \in L^2[0, 1]$ and $\gamma(0) = 0$. Define $\gamma_i(t) := \gamma(t) e_i$. Multiplying (3.6) by $\gamma_i(t)$ and using Ito's formula, we have

$$\int_0^1 (\lambda_n(t), \dot{\gamma}_i(t))_m \, dt + \int_0^1 (r_n'(t), \gamma_i(t))_m \, dW(t)$$

$$= (G_n, \gamma_i(1))_m + \int_0^1 [\langle A^*(t)\lambda_n(t), \gamma_i(t)\rangle_m$$

$$+ (r_n(t), M^{\Delta}(t)\gamma_i(t))_m + \langle F(t), \gamma_i(t)\rangle_m] \, dt.$$

By virtue of (3.13) and (3.14), letting $n' \to \infty$, we have, for any $\phi \in H^{m+1}$,

$$\int_0^1 (\lambda(t), \phi)_m \dot{\gamma}(t) \, dt + \int_0^1 (r(t), \phi)_m \gamma(t) \, dW(t)$$

$$= (G, \phi)_m \gamma(1) + \int_0^1 [\langle A^*(t)\lambda(t), \phi\rangle_m$$

$$+ (r(t), M^{\Delta}(t)\phi)_m + \langle F(t), \phi\rangle_m] \gamma(t) \, dt. \qquad (3.15)$$

Appealing to Remark 2.3, for $\psi \in H^1$, we take $\phi := \Lambda^{-m}\psi$ in (3.15) and note that

$$(r(t), M^{\Delta}(t)\Lambda^{-m}\psi)_m = \lim_{\substack{c_n \in H^{m+1} \\ c_n \to r(t) \text{ in } H^m}} (M^*(t)c_n, \Lambda^{-m}\psi)_m = \lim_n (M^*(t)c_n, \psi)_0$$

$$= \lim_n (c_n, M(t)\psi)_0 = (r(t), M(t)\psi)_0,$$

then (3.15) reduces, for any $\psi \in H^1$,

$$\int_0^1 (\lambda(t), \psi)_0 \dot{\gamma}(t) \, dt + \int_0^1 (r(t), \psi)_0 \gamma(t) \, dW(t)$$

$$= (G, \psi)_0 \gamma(1) + \int_0^1 [\langle \lambda(t), A(t)\psi\rangle_0$$

$$+ (r(t), M(t)\psi)_0 + \langle F(t), \psi\rangle_0] \gamma(t) \, dt. \qquad (3.16)$$

For any $t \in (0, 1)$, we take γ_ε defined by

$$\gamma_\varepsilon(s) := \begin{cases} 0, & \text{if } s \leqslant t - \varepsilon/2, \\ 1/\varepsilon \cdot (s - t + \varepsilon/2), & \text{if } t - \varepsilon/2 < s < t + \varepsilon/2, \\ 1, & \text{if } s \geqslant t + \varepsilon/2. \end{cases}$$

Substituting (3.16) with γ_ε and letting $\varepsilon \to 0$, we arrive at

$$(\lambda(t), \psi)_0 + \int_t^1 (r(s), \psi)_0 \, dW(s)$$

$$= (G, \psi)_0 + \int_t^1 [\langle \lambda(s), A(s)\psi \rangle_0 + (r(s), M(s)\psi)_0$$

$$+ \langle F(s), \psi \rangle_0] \, ds, \qquad \text{for any} \quad \psi \in H^1, \text{ a.e. } t \in [0, 1]. \tag{3.17}$$

This means (λ, r) satisfies (3.5).

Now let us show (3.6). By virtue of (3.8), we know that for each fixed t, there exists a subsequence $\{n''\}$ of $\{n\}$ and $\tilde{\lambda}(t) \in L^2(\Omega; H^m)$ such that $\lambda_{n''}(t) \to \tilde{\lambda}(t)$ weakly in $L^2(\Omega; H^m)$. But

$$(\lambda_n(t), e_i)_m + \int_t^1 (r_n(s), e_i)_m \, dW(s)$$

$$= (G_n, e_i)_m + \int_t^1 [\langle A^*(s) \lambda_n(s), e_i \rangle_m + (r_n(s), M^{\Delta}(s) e_i)_m + \langle F(s), e_i \rangle_m] \, ds,$$

letting $n'' \to \infty$, we have $\tilde{\lambda}(t) = \lambda(t)$ for almost $[0, 1] \times \Omega$, observing (3.17). Hence combining (3.8), (3.12), (3.13), and (3.14), we get (3.6) for $\bar{m} = m$. As for $\bar{m} = 0, 1, ..., m-1$, the argument is totally the same if we note the uniqueness of the solution. The proof is now completed. ▮

Now let us give the duality equality.

COROLLARY 3.1. *Let the same assumptions as in Theorem* 3.1 *be satisfied with* $m = 0$. *Given* $f \in L^2_{\mathscr{F}}(0, 1; H^{-1})$, $g^k \in L^2_{\mathscr{F}}(0, 1; H^0)$, $k = 1, 2, ..., d'$, *and* $q_0 \in L^2(\Omega, \mathscr{F}_0; H^0)$. *Suppose* $q \in L^2_{\mathscr{F}}(0, 1; H^1) \cap L^2(\Omega; C(0, 1; H^0))$ *is the solution of* (1.4) *and* (λ, r) *is the solution of* (1.7), *then for any* $[\alpha, \beta] \subset [0, 1]$,

$$E\left[\int_\alpha^\beta \langle F(t), q(t) \rangle_0 \, dt + (\lambda(\beta), q(\beta))_0\right]$$

$$= E\left\{\int_\alpha^\beta \left[\langle \lambda(t), f(t) \rangle_0 + \sum_{k=1}^{d'} (r^k(t), g^k(t))_0\right] dt + (\lambda(\alpha), q(\alpha))_0\right\}.$$

Proof. Applying Ito's formula to $(\lambda(t), q(t))_0$, we easily get the result. ▮

Remark 3.1. If we check the proof of Theorem 3.1, we will find that when $m = 0$, Theorem 3.1 (and therefore Corollary 3.1) still remains valid even if all the coefficients a^{ij}, etc., are only bounded measurable.

4. Duality Equation: Degenerate Case

The argument in the previous section fails to work in general if SPDE (1.4) is degenerate. In this section, we shall treat a special case of the degenerate equations, i.e., the first-order derivatives in the diffusion term of (1.4) vanish.

Throughout this section, we assume $(A1)_m$, (A2) for some $m \geq 1$ and $\sigma^{ik} = 0$.

THEOREM 4.1. *Assume that* $F \in L^2_{\mathscr{F}}(0, 1; H^m)$, $G \in L^2(\Omega, \mathscr{F}_1; H^m)$. *Then the duality equation* (1.7) *admits a unique solution* $(\lambda, r) \in L^2_{\mathscr{F}}(0, 1; H^m) \times L^2_{\mathscr{F}}(0, 1; H^m)^{d'}$. *Moreover, there exists a constant* N_{10} *which depends only on* K *and* m, *such that*

$$\sup_{0 \leq t \leq 1} E\|\lambda(t)\|^2_{\bar{m}} + E\int_0^1 \sum_{k=1}^{d'} \|r^k(t)\|^2_{\bar{m}}\, dt \leq N_{10} E\left[\int_0^1 \|F(t)\|^2_{\bar{m}}\, dt + \|G\|^2_{\bar{m}}\right], \qquad \bar{m} = 0, 1, 2, ..., m. \tag{4.1}$$

Proof. We assume $d' = 1$ and omit to write the index k. Uniqueness can be proved by exactly the same way as in the proof of Theorem 3.1. We only show the existence.

We define $A_\varepsilon(t)$ and $A_\varepsilon^*(t)$ by (1.5) and (1.8), respectively, with a^{ij} replaced by $a^{ij} + \varepsilon\,\delta^{ij}$, where $\delta^{ij} = 1$ as $i = j$; $\delta^{ij} = 0$ as $i \neq j$, and $\varepsilon \in (0, 1)$.

By Theorem 3.1, there exists uniquely $(\lambda_\varepsilon, r_\varepsilon) \in L^2_{\mathscr{F}}(0, 1; H^{m+1}) \times L^2_{\mathscr{F}}(0, 1; H^m)$ such that

$$\begin{cases} d\lambda_\varepsilon(t) = -[A_\varepsilon^*(t)\,\lambda_\varepsilon(t) + M^*(t)\,r_\varepsilon(t) + F(t)]\,dt + r_\varepsilon(t)\,dW(t), \\ \lambda_\varepsilon(1) = G. \end{cases} \tag{4.2}$$

Since $|\langle \phi, A_\varepsilon^*(t)\,\psi \rangle_m| \leq \text{const.}\ \|\phi\|_{m+1}\,\|\psi\|_{m+1}$, for any $\phi, \psi \in H^{m+1}$, so the formula

$$\langle A_\varepsilon^A(t)\phi, \psi \rangle_m := \langle \phi, A_\varepsilon^*(t)\psi \rangle_m, \qquad \text{for} \quad \phi, \psi \in H^{m+1},$$

defines a linear operator $A_\varepsilon^A(t)$ mapping from H^{m+1} into H^{m-1}. Similarly, the formula

$$(M^A(t)\phi, \psi)_m := (\phi, M^*(t)\psi)_m, \qquad \text{for} \quad \phi, \psi \in H^m,$$

defines a linear operator from H^m into itself. Note since $\sigma^{ik} = 0$, $M^A(t)$ is a bounded operator with the bound independent of t.

By Lemma 2.1(b), there exists a constant N_{11} which is independent of ε, such that

$$2[\langle A_\varepsilon^A(t)\phi, \phi\rangle_m + (f, \phi)_m] + \|M^A(t)\phi + g\|_m^2 \leqslant -2\varepsilon\|\phi\|_{m+1}^2 + N_{11}(\|\phi\|_m^2 + \|f\|_m^2 + \|g\|_m^2), \quad \text{for any} \quad \phi \in H^{m+1}, f \in H^m, g \in H^m. \tag{4.3}$$

The above estimate together with a routine finite dimensional approximation approach (cf. [9]) yields that there exists uniquely $\rho_\varepsilon \in L^2_{\mathscr{F}}(0, 1; H^{m+1})$ satisfying the SPDE

$$\begin{cases} d\rho_\varepsilon(t) = [A_\varepsilon^A(t)\,\rho_\varepsilon(t) + \lambda_\varepsilon(t)]\,dt + [M^A(t)\,\rho_\varepsilon(t) + r_\varepsilon(t)]\,dW(t), \\ \rho_\varepsilon(0) = 0. \end{cases} \tag{4.4}$$

Further,

$$\begin{aligned} E\|\rho_\varepsilon(t)\|_m^2 &= E\int_0^t \{2[\langle A^A(s)\,\rho_\varepsilon(s), \rho_\varepsilon(s)\rangle_m + (\lambda_\varepsilon(s), \rho_\varepsilon(s))_m] \\ &\qquad + \|M^A(s)\,\rho_\varepsilon(s) + r_\varepsilon(s)\|_m^2\}\,ds \\ &\leqslant N_{11}E\int_0^t [\|\rho_\varepsilon(s)\|_m^2 + \|\lambda_\varepsilon(s)\|_m^2 + \|r_\varepsilon(s)\|_m^2]\,ds, \end{aligned}$$

hence

$$\sup_{0\leqslant t\leqslant 1} E\|\rho_\varepsilon(t)\|_m^2 \leqslant N_{11}\exp(N_{11})E\int_0^1 [\|\lambda_\varepsilon(t)\|_m^2 + \|r_\varepsilon(t)\|_m^2]\,dt. \tag{4.5}$$

Applying Ito's formula for $(\lambda_\varepsilon(t), \rho_\varepsilon(t))_m$, we easily get

$$\begin{aligned} &E\int_0^1 [\|\lambda_\varepsilon(t)\|_m^2 + \|r_\varepsilon(t)\|_m^2]\,dt \\ &\quad = E\left[\int_0^1 (F(t), \rho_\varepsilon(t))_m\,dt + (G, \rho_\varepsilon(1))_m\right] \\ &\quad \leqslant \left\{N_{11}\exp(N_{11})E\int_0^1 [\|\lambda_\varepsilon(t)\|_m^2 + \|r_\varepsilon(t)\|_m^2]\,dt\right\}^{1/2} \\ &\qquad \times \left\{\left(E\int_0^1 \|F(t)\|_m^2\,dt\right)^{1/2} + (E\|G\|_m^2)^{1/2}\right\}, \end{aligned}$$

so

$$E\int_0^1 [\|\lambda_\varepsilon(t)\|_m^2 + \|r_\varepsilon(t)\|_m^2]\,dt \leqslant 2N_{11}\exp(N_{11})E\left[\int_0^1 \|F(t)\|_m^2\,dt + \|G\|_m^2\right]. \tag{4.6}$$

Hence there exist subsequence $\{\varepsilon_n\}$ and $(\lambda, r) \in L^2_{\mathscr{F}}(0, 1; H^m) \times L^2_{\mathscr{F}}(0, 1; H^m)$ such that

$$\lambda_{\varepsilon_n} \to \lambda \quad \text{weakly in } L^2([0, 1] \times \Omega; H^m),$$

$$r_{\varepsilon_n} \to r \quad \text{weakly in } L^2([0, 1] \times \Omega; H^m), \text{ as } \varepsilon_n \to 0.$$

By (4.2), for any absolutely continuous function γ with $\dot{\gamma} \in L^2[0, 1]$ and $\gamma(0) = 0$,

$$\int_0^1 (\lambda_\varepsilon(t), \phi)_0 \dot{\gamma}(t)\, dt + \int_0^1 (r_\varepsilon(t), \phi)_0 \gamma(t)\, dW(t)$$

$$= (G, \phi)_0 \gamma(1) + \int_0^1 [\langle \lambda_\varepsilon(t), A(t)\phi \rangle_0 + (r_\varepsilon(t), M(t)\phi)_0$$

$$+ (F(t), \phi)_0] \gamma(t)\, dt, \qquad \text{for any} \quad \phi \in H^1.$$

It follows that (λ, r) satisfies (3.5), by letting $\varepsilon_n \to 0$.

Finally let us give the estimate for $\sup_{0 \leqslant t \leqslant 1} E\|\lambda(t)\|_m^2$. Indeed, noting that $M^*(t)$ is a bounded operator mapping H^m into itself, we have by (4.2)

$$E\|\lambda_\varepsilon(t)\|_m^2 = E\|G\|_m^2 + 2E\int_0^1 [\langle A^*(s)\lambda_\varepsilon(s), \lambda_\varepsilon(s)\rangle_m + (M^*(s) r_\varepsilon(s), \lambda_\varepsilon(s))_m$$

$$+ (F(s), \lambda_\varepsilon(s))_m - 1/2 \cdot \|r_\varepsilon(s)\|_m^2]\, ds$$

$$\leqslant E\|G\|_m^2 + 2E\int_0^1 [1/2 \cdot N_2 \|\lambda_\varepsilon(s)\|_m^2 + N_{12}\|r_\varepsilon(s)\|_m \|\lambda_\varepsilon(s)\|_m$$

$$+ \|F(s)\|_m \|\lambda_\varepsilon(s)\|_m - 1/2 \cdot \|r_\varepsilon(s)\|_m^2]\, ds,$$

thus

$$\sup_{0 \leqslant t \leqslant 1} E\|\lambda_\varepsilon(t)\|_m^2 \leqslant N_{13} E\left[\int_0^1 \|F(t)\|_m^2\, dt + \|G\|_m^2\right],$$

where the constant N_{13} is independent of ε. Now by the same argument as in the proof of Theorem 3.1, (λ, r) satisfies (4.1). ∎

Remark 4.1. By Sobolev's well-known embedding theorem (see, for example, [7]), if $m - n > d/2$ for a nonnegative n, then the solutions (λ, r) obtained in Theorems 3.1 and 4.1 belong to $C^n(R^d)$ for almost all (t, ω), and the H^m-norms in the left-hand sides of (3.6) and (4.1) can be replaced by $C^n(R^d)$-norms (i.e., the summation of the sup-norms up to nth order). In particular, if $m > d/2 + 1$ in Theorem 3.1 (resp. $m > d/2 + 2$ in Theorem 4.1), then the respective solutions of (1.7) are classical.

5. Application

The application of the duality equation (1.7) to the necessary conditions of optimal stochastic controls has been given in [11]. In this section, we will present an application to the study of SPDE (1.4).

Theorem 5.1. *In Proposition* 2.1(a), *the estimate* (2.4) *holds for* $\bar{m}=0, \pm 1, ..., \pm m$.

Proof. It suffices to prove the result for $d'=1$ and $\bar{m}=-m$.

Let q be the solution of (1.4) and $\bar{t}\in(0,1]$ be fixed. Consider the triplet $(L^2(\Omega,\mathscr{F}_{\bar{t}};H^{-m}), L^2(\Omega,\mathscr{F}_{\bar{t}};H^0), L^2(\Omega,\mathscr{F}_{\bar{t}};H^m))$ under $(L^2(\Omega,\mathscr{F}_{\bar{t}};H^0))^*=L^2(\Omega,\mathscr{F}_{\bar{t}};H^0)$ (cf. [7, p. 47]). By the Hahn–Banach theorem (see, for example, [10]), there exists $G\in L^2(\Omega,\mathscr{F}_{\bar{t}};H^m)=(L^2(\Omega,\mathscr{F}_{\bar{t}};H^{-m}))^*$, such that

$$E(G,q(\bar{t}))_0=E\|G\|_m^2=E\|q(\bar{t})\|_{-m}^2. \tag{5.1}$$

Similarly, there exists $F\in L^2_{\mathscr{F}}(0,\bar{t};H^{m-1})$ such that

$$E\int_0^{\bar{t}}(F(t),q(t))_0\,dt=E\int_0^{\bar{t}}\|F(t)\|_{m-1}^2\,dt=E\int_0^{\bar{t}}\|q(t)\|_{-m+1}^2\,dt. \tag{5.2}$$

Appealing to Theorem 3.1, there exists uniquely $(\lambda,r)\in L^2_{\mathscr{F}}(0,\bar{t};H^{m+1})\times L^2_{\mathscr{F}}(0,\bar{t};H^m)$ satisfying

$$\begin{cases} d\lambda(t)=-[A^*(t)\,\lambda(t)+M^*(t)\,r(t)+F(t)]\,dt+r(t)\,dW(t),\\ \lambda(\bar{t})=G. \end{cases} \tag{5.3}$$

The duality equality (Corollary 3.1) yields

$$\begin{aligned} &E\left[\int_0^{\bar{t}}(F(t),q(t))_0\,dt+(G,q(\bar{t}))_0\right]\\ &\quad=E\left\{\int_0^{\bar{t}}[(\lambda(t),f(t))_0+(r(t),g(t))_0]\,dt+(\lambda(0),q_0)_0\right\}\\ &\quad\leqslant\left(E\int_0^{\bar{t}}\|\lambda(t)\|_{m+1}^2\,dt\right)^{1/2}\left(E\int_0^{\bar{t}}\|f(t)\|_{-m-1}^2\,dt\right)^{1/2}\\ &\qquad+\left(E\int_0^{\bar{t}}\|r(t)\|_m^2\,dt\right)^{1/2}\left(E\int_0^{\bar{t}}\|g(t)\|_{-m}^2\,dt\right)^{1/2}\\ &\qquad+(E\|\lambda(0)\|_m^2)^{1/2}\,(E\|q_0\|_{-m}^2)^{1/2}\\ &\quad\leqslant\left\{N_6E\left[\int_0^{\bar{t}}\|F(t)\|_{m-1}^2\,dt+\|G\|_m^2\right]\right\}^{1/2}\left\{\left(E\int_0^{\bar{t}}\|f(t)\|_{-m-1}^2\,dt\right)^{1/2}\right.\\ &\qquad\left.+\left(E\int_0^{\bar{t}}\|g(t)\|_{-m}^2\,dt\right)^{1/2}+(E\|q_0\|_{-m}^2)^{1/2}\right\}. \end{aligned}$$

Noting (5.1) and (5.2), we get immediately

$$E\|q(\bar{t})\|^2_{-m}+E\int_0^{\bar{t}}\|q(t)\|^2_{-m+1}\,dt$$
$$\leqslant 3N_6E\left\{\|q_0\|^2_{-m}+\int_0^{\bar{t}}[\|f(t)\|^2_{-m-1}+\|g(t)\|^2_{-m}]\,dt\right\}.$$

Since $\bar{t}$ is arbitrary, the above inequality is just what we want. ∎

In the complete same fashion, we have

THEOREM 5.2. *In Proposition* 2.1(b), *the estimate* (2.5) *holds for* $m=0, \pm 1, ..., \pm m$.

Observing the duality relationship between (1.4) and (1.7), by a similar method as above, we can prove the following

THEOREM 5.3. *In Theorems* 3.1 *and* 4.1, *the estimates* (3.6) *and* (4.1) *hold for* $m=0, \pm 1, ..., \pm m$.

In the above results, the higher regularity conditions on f, g, q_0 in Theorems 5.1, 5.2 (resp. F, G in Theorem 5.3) are posed. But if we are only concerned with the H^m-norm estimates with the negative m, these conditions can be considerably relaxed.

COROLLARY 5.1. (a) *Assume* $(A1)_m$, $(A2)'$ *for some* $m\geqslant 0$, *and assume* $f, F\in L^2_{\mathscr{F}}(0,1;H^{-1})$, $g\in L^2_{\mathscr{F}}(0,1;H^0)$, $q_0\in L^2(\Omega,\mathscr{F}_0;H^0)$, $G\in L^2(\Omega,\mathscr{F}_1;H^0)$. *Then the estimates* (2.4) *and* (3.6) *hold for* $\bar{m}=-1, ..., -m$.

(b) *Assume* $(A1)_m$, (A2) *for some* $m\geqslant 1$ *with* $\sigma^{ik}=0$, *and assume that* $f, g, F\in L^2_{\mathscr{F}}(0,1;H^1)$, $q_0\in L^2(\Omega,\mathscr{F}_0;H^1)$, $G\in L^2(\Omega,\mathscr{F}_1;H^1)$. *Then the estimates* (2.5) *and* (4.1) *hold for* $\bar{m}=-1, ..., -m$.

Proof. Proving the above claims is a simple approximation procedure. We only show that of (2.4) for example. We assume $d'=1$ for simplicity. Choose $\{f_n\}\subset L^2_{\mathscr{F}}(0,1;H^{m-1})$, $\{g_n\}\subset L^2_{\mathscr{F}}(0,1;H^m)$, and $\{q_{0n}\}\subset L^2(\Omega,\mathscr{F}_0;H^m)$ such that

$$f_n\to f \quad \text{in} \quad L^2([0,1]\times\Omega;H^{-1}),$$
$$g_n\to g \quad \text{in} \quad L^2([0,1]\times\Omega;H^0),$$
$$q_{0n}\to q_0 \quad \text{in} \quad L^2(\Omega,\mathscr{F}_0;H^0), \quad \text{as} \quad n\to\infty.$$

Let q_n be the solution of (1.4) with f, g, q_0 replaced respectively by f_n, g_n, q_{0n}. Then (2.4) with $m=0$ yields

$$\sup_{0\leqslant t\leqslant 1} E\|q_n(t)-q(t)\|_0^2+E\int_0^1\|q_n(t)-q(t)\|_1^2\,dt\to 0, \qquad \text{as} \quad n\to\infty.$$

On the other hand, by Theorem 5.1, q_n satisfies (2.4) for $\bar{m} = -1, ..., -m$ with the constant therein *independent of n*. Hence the desired result follows by letting $n \to \infty$. ▌

Remark 5.1. The $H^{\bar{m}}$-norm estimates of the solution of SPDE with negative m have been found useful in the study of the Hamilton–Jacobi–Bellman equation in infinite dimensional spaces. For example, in order to obtain the uniqueness of viscosity solutions of a special class of H-J-B equations, Lions [8] has proved the estimate (2.5) for $\bar{m} = -2$ by using a specific method. The duality analysis of the present paper may be useful in treating the optimal control problem of much more general SPDE and the corresponding H-J-B equation. We hope to study this subject in some future papers.

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