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A class of semilinear stochastic partial differential equations and their controls: Existence results*

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This paper concerns a class of similinear stochastic partial differential equations, of which the drift term is a second-order differential operator plus a nonlinearity, and the diffusion term is a first-order differential operator. When the nonlinearity is only continuous in the state, it is shown that there exist solutions of the equation provided that the Wiener process involved is one-dimensional. The existence of optimal relaxed controls for this class of equations is also proved. Our method is based on a group analysis of the first-order differential operator and a time change technique.

semilinear stochastic partial differential equations * group of operators * time change * compact embedding * optimal relaxed controls

1. Introduction

The linear stochastic partial differential equations (SPDE in short) have been studied extensively by many authors (cf. Pardoux, 1975, 1979; Kunita, 1982; Walsh, 1986), especially by Krylov and Rozovskii (1977, 1982a, 1982b). For nonlinear SPDEs, however, even the existence and uniqueness of solutions are not clear in general. In this paper, we will consider the following kind of nonlinear (semilinear) SPDE:

$$\begin{cases} dq(t,x)=[\partial_i(a^{ij}(x)\partial_j q(t,x)+f^i(x,q(t,x)))]\,dt \\ \qquad\qquad +[\sigma^i(x)\partial_i q(t,x)+h(x)q(t,x)+g(x)]\,dW(t), \quad x\in\mathbb{R}^d, \quad t\in[0,1], \\ q(0,x)=q_0(x), \quad x\in\mathbb{R}^d, \end{cases} \tag{1.1}$$

where W is a one-dimensional Wiener process with $W(0)=0$, and $\partial_i:=\partial/\partial x_i$. Note here and in the following we always use the conventional repeated indices for summation.

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SPDE (1.1) describes intuitively a physical object governed by a semilinear partial differential equation perturbed by some random forces. We emphasize that the diffusion term of (1.1) is a first-order differential operator. Roughly speaking, as we will see at the later stage, the appearance of σ^i means that the random perturbation influences the behavior of the solutions of (1.1) so strongly as does the drift term containing the second-order operator. This will also result in one of the main difficulties in this paper.

The objective of this paper is twofold. First we will be concerned with the existence of solutions (the precise meaning of 'solution' will be given later on) of (1.1) with the continuous nonlinearity. It should be noted that when $f^i(x, \cdot)$ is Lipschitz continuous, the existence and uniqueness of solutions can be proved by a standard Picard's method. One may refer to Pardoux (1975), Walsh (1986) and Tudor (1989) for this method (though for slightly different forms of nonlinear SPDEs than (1.1)). When $f^i(x, \cdot)$ is only continuous, however, Picard's method is not effective. On the other hand, one may recall that, in stochastic differential equation (SDE) theory, a typical method of proving the existence with the continuous nonlinearities is to employ the Ascoli–Arzela theorem and Skorohod theorem (cf. Ikeda and Watanabe, 1989). But we are now treating the SPDE, the state space of which is infinite dimension, where there is no A–A theorem available to us. In this paper, we will employ a similar argument to that in Zhou (1991) to overcome the difficulty. The main idea is to use a time change technique, based on an analysis of the group generated by the first-order differential operator, to turn the equation (1.1) into a P-a.s. *deterministic* equation. This allows us to apply a compact embedding lemma (Lemma 2.3 below) and establish the desired existence theorem.

Second, we will study the optimal control problem for SPDEs like (1.1). Most of the existing results on this aspect are for linear SPDEs, the reason perhaps being that Zakai's equations for partially observed diffusions are linear SPDEs (cf. Bensoussan, 1983; Nagase and Nisio, 1990; Bensoussan and Nisio, 1990; Zhou, 1991). But the study on the nonlinear SPDEs as (1.1) is of its own interest in both theory and application. Using the same method mentioned above, we are able to show that there exists an optimal relaxed control.

The main restriction of this paper is that the Wiener process W is required to be one-dimensional. As for multi-dimensional cases, our method applies only to some special cases (for example, the diffusion operators are commutative) which in particular include those that the diffusion operators are of order zero.

It should be noted that the 'time change' technique, sometimes also called 'reduction to robust equation,' has been employed before in the literature by Da Prato and Tubaro (1985), Cannarsa and Vespri (1987), etc. They reduced the nonlinear SPDEs to P-a.s. deterministic PDEs (robust equations) and then solved them by semigroup theory. However, they assumed some Lipschitz continuity and/or monotonicity of the nonlinearities in order to guarantee the existence of solutions to the robust equations. In the present paper, we can handle such SPDEs with only continuous nonlinearities by applying a compact embedding lemma and Skorohod theorem to the approximating solutions of the robust equations. Furthermore, our approach can allow us to treat SPDEs with *degenerate* second-order differential operators in the drift, for which the semigroup theory can hardly apply.

The paper is organized as follows: In Section 2 we will give some basic notations as well as some preliminary lemmas which will play essential roles in this paper. In Section 3 we will prove the existence of solutions of (1.1). In Section 4 we study a variant, where the existence theorem is obtained for a more 'abstract' equation. Section 5 is devoted to the existence of optimal relaxed controls.

2. Preliminaries

Let us define operators A and M by

$$A\phi(x) := \partial_i(a^{ij}(x)\partial_j\phi(x)), \tag{2.1}$$

$$M\phi(x) := \sigma^i(x)\partial_i\phi(x) + h(x)\phi(x) \quad \text{for } x \in \mathbb{R}^d. \tag{2.2}$$

We may then rewrite the SPDE (1.1) as follows:

$$\begin{cases} \mathrm{d}q(t) = [Aq(t) + \partial_i f^i(\cdot, q(t,\cdot))]\,\mathrm{d}t + [Mq(t) + g]\,\mathrm{d}W(t), \quad t \in [0,1], \\ q(0) = q_0. \end{cases} \tag{2.3}$$

In this paper, we shall consider the triplet $H^1 \hookrightarrow H^0 \hookrightarrow H^{-1}$, where H^k denotes the Sobolev space $W_2^k(\mathbb{R}^d)$ with the Sobolev norm $\|\cdot\|_k$ $(k=-1,0,1)$. We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between H^{-1} and H^1 under $(H^0)^* = H^0$, and by $(\cdot,\cdot)$ the inner product in H^0.

For the second-order differential operator A, when we write $\langle A\phi, \psi\rangle$, then A is understood to be an operator from H^1 to H^{-1} in the following way:

$$\langle A\phi, \psi\rangle := -(a^{ij}(\cdot)\partial_j\phi, \partial_i\psi) \quad \text{for } \phi, \psi \in H^1.$$

Given a filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, a number p with $1 \le p \le +\infty$, and a Hilbert space X with the norm $\|\cdot\|_X$. Define

$$L^p_{\mathcal{F}}(0,1;X) := \{\phi\colon \phi \text{ is an } X\text{-valued } \mathcal{F}_t\text{-adapted measurable process on } [0,1], \text{ and } \phi \in L^p([0,1]\times\Omega; X)\}.$$

We identify ϕ and ϕ' in $L^p_{\mathcal{F}}(0,1;X)$ if $E\int_0^1 \|\phi(t)-\phi'(t)\|_X^p\,\mathrm{d}t = 0$.

Now let us clarify the meaning of a solution of (2.3).

Definition 2.1. By a (weak) solution of the eq. (2.3), we mean an H^1-valued process $q = \{q(t)\colon 0 \le t \le 1\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\colon 0 \le t \le 1\}$ such that

(i) there exists a one-dimensional $\mathscr{F}_t$-Wiener process $\{W(t): 0 \leq t \leq 1\}$ with $W(0) = 0$;

(ii) $q \in L^2_{\mathscr{F}}(0, 1; H^1)$;

(iii) for any $\eta \in C_0^\infty(\mathbb{R}^d)$ (smooth function on $\mathbb{R}^d$ with compact support) and almost all $(t, \omega) \in [0, 1] \times \Omega$,

$$(q(t), \eta) = (q_0, \eta) + \int_0^t [\langle Aq(s), \eta\rangle - (f^i(\cdot, q(s, \cdot)), \partial_i \eta)] \, ds + \int_0^t (Mq(s) + g, \eta) \, dW(s). \tag{2.4}$$

To emphasize the particular role of the Wiener process W, sometimes we call (q, W) a solution of (2.3).

Remark 2.1. According to Da Prato and Tubaro (1985) and Tudor (1989), one can also define so-called mild solutions of (2.3) as follows: suppose A generates a C_0-semigroup $T(t)$ on H^0 and H^{-1}, then a mild solution is a solution of the following integral equation:

$$q(t) = T(t)q_0 + \int_0^t T(t-s)\partial_i f^i(\cdot, q(s, \cdot)) \, ds + \int_0^t T(t-s)(Mq(s) + g) \, dW(s).$$

Note the mild solutions, which satisfy (2.4) on any *given* probability space with any *given* Wiener process, is analogous to those in PDE theory (cf. Ahmed and Teo, 1981, and Pazy, 1983). In Definition 2.1, on the other hand, probability spaces together with Wiener processes are also a part of the solutions. In this sense, the solutions considered in this paper are weaker than the mild solutions. Moreover, it is difficult, if not impossible, to extend the concept of mild solutions to degenerate A which no longer generates a C_0-semigroup, while in our definition, it does not matter whether A is degenerate or nondegenerate (see also Section 4 below).

Let us fix two positive constants K and δ. We introduce the following assumptions on the functions appearing in (2.3):

(A1) a^{ij}, σ^i, $h: \mathbb{R}^d \to \mathbb{R}^1$ are bounded measurable functions; the derivatives of σ^i up to second order and those of h up to first order do not exceed K in absolute value.

(A2) $a^{ij} = a^{ji}$, $i, j = 1, 2, \ldots, d$, and $(a^{ij} - \frac{1}{2}\sigma^i\sigma^j)_{ij} \geq \delta I$, where I is the identity matrix.

(A3) $f^i: \mathbb{R}^d \times \mathbb{R}^1 \to \mathbb{R}^1$ is jointly measurable, continuous in the second argument, and there exists $\lambda \in H^0$ such that

$$|f^i(x, r)| \leq K(\lambda(x) + |r|), \quad i = 1, 2, \ldots, d.$$

On the other hand, $g \in H^1$.

(A4) $q_0 \in H^0$.

Remark 2.2. In this paper, the operator A is considered to be a purely second-order operator without lower terms. But it does not lose any generality. Indeed, if $A\phi(x) = \partial_i(a^{ij}(x)\partial_j\phi(x) + b^i(x)\phi(x) + c(x))$, then the lower order terms $b^i(x)\phi(x) + c(x)$ can be included to $f^i(x, \phi(x))$, and (A3) is satisfied if b^i and c are uniformly bounded.

The following result concerning the solutions of linear SPDEs is an easy variant of Krylov and Rozovskii (1977).

Lemma 2.1. *Given a filtered probability space* $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ *with a one-dimensional Wiener process* W, *consider the following linear SPDE*

$$\begin{cases} \mathrm{d}q(t) = [Aq(t) + \partial_i F^i(t)]\,\mathrm{d}t + [Mq(t) + g]\,\mathrm{d}W(t), & t \in [0, 1], \\ q(0) = q_0. \end{cases} \tag{2.5}$$

We assume that (A1), (A2) *and* (A4) *are satisfied and that* $F^i \in L^2_{\mathcal{F}}(0, 1; H^0)$, $g \in H^0$. *Then,* (2.5) *has a unique solution* $q \in L^2_{\mathcal{F}}(0, 1; H^1) \cap L^2(\Omega; C(0, 1; H^0))$ *and there exists a constant* C, *depending only on* K *and* δ, *such that*

$$E\|q(t)\|_0^2 + E\int_0^t \|q(t)\|_1^2\,\mathrm{d}t \leq CE\left\{\|q_0\|_0^2 + \int_0^t\left[\sum_{i=1}^d \|F^i(s)\|_0^2 + \|g\|_0^2\right]\mathrm{d}s\right\}. \tag{2.6}$$

Moreover, for any $p \geq 2$, if $F^i \in L^{2p}_{\mathcal{F}}(0, 1; H^0)$, then there is a constant $C(p)$ such that

$$E\|q(t)\|_0^{2p} + E\left(\int_0^t \|q(s)\|_1^2\,\mathrm{d}s\right)^p \leq C(p)E\left\{\|q_0\|_0^{2p} + \int_0^t\left[\sum_{i=1}^d \|F^i(s)\|_0^{2p} + \|g\|_0^{2p}\right]\mathrm{d}s\right\}. \quad \square \tag{2.7}$$

Now let us introduce two lemmas which play the essential roles in this paper.

Lemma 2.2. *Assume* (A1). *On the Hilbert space* H^0, *define an operator* M *by* (2.2) *with the domain* $D(M) := H^1$, *then*

(i) *M can be extended to a closed operator (still denoted by M) which generates a strongly continuous group* $\{e^{Mt}: -\infty < t < +\infty\}$ *on* H^0. *Moreover,* H^1 *is an invariant subspace of the operator* e^{Mt} *for each* t. *Further, there exists a positive constant* N *such that*

$$\|e^{Mt}\|_{L(H^k \to H^k)} \leq e^{N|t|} \quad \text{for any } t \in (-\infty, +\infty), \qquad k = 0, 1; \tag{2.8}$$

(ii) *Denote by* M^* *the adjoint operator of* M *on* H^0, *then* $H^1 \subset D(M^*)$ *and* M^* *also generates a strongly continuous group* $\{e^{M^*t} = (e^{Mt})^*: -\infty < t < +\infty\}$ *on* H^0. *Moreover, with the same constant* N, *we have*

$$\|e^{M^*t}\|_{L(H^k \to H^k)} \leq e^{N|t|} \quad \text{for any } t \in (-\infty, +\infty), \qquad k = 0, 1. \tag{2.9}$$

(iii) *Define two operators* M^2, M^{*2} *from* H^1 *to* H^{-1} *by the following formula:*

$$\langle M^2\phi, \psi\rangle = (M\phi, M^*\psi) = \langle \phi, M^{*2}\psi\rangle \quad \text{for } \phi, \psi \in H^1,$$

then M^2 *and* M^{*2} *are bounded linear operators from* H^1 *to* H^{-1}.

Proof. See Appendix of Zhou (1991). □

Remark 2.3. Intuitively speaking, Lemma 2.1 simply says that the first-order differential operator corresponds to a *reversible* evolution process.

From now on, when we write M, M^*, M^2 and M^{*2}, it is always understood to be in the sense of that in Lemma 2.2.

Lemma 2.3. *Let* D *be a set in* $\mathbb{R}^d$ *such that*

$$D \text{ is bounded, open and with smooth boundary.} \tag{2.10}$$

Define $W_D[0,1] := \{\phi\colon \phi \in L^2(0,1; H^1(D)),\ \mathrm{d}\phi/\mathrm{d}t \in L^2(0,1; H^{-1}(D))\}$ *with the norm*

$$\|\phi\|_{W_D[0,1]} := \left(\int_0^1 \|\phi(t)\|_{1,D}^2\,\mathrm{d}t + \int_0^1 \|\mathrm{d}\phi(t)/\mathrm{d}t\|_{-1,D}^2\,\mathrm{d}t\right)^{1/2},$$

where $H^k(D)$ *is the Sobolev space* $W_2^k(D)$ *with the Sobolev norm* $\|\cdot\|_{k,D}$. *Then the embedding*: $W_D[0,1] \to L^2(0,1; H^0(D))$ *is compact.*

Proof. See, for example, Lions (1969). □

3. Existence of solutions

Theorem 3.1. *Under* (A1)–(A4), *there exists at least one solution of the equation* (2.3). *Moreover, there is a positive constant* N_1, *depending only on* K *and* δ, *such that for any solution* q,

$$\sup_{0\le t\le 1} E\|q(t)\|_0^2 + E\int_0^1 \|q(t)\|_1^2\,\mathrm{d}t \le N_1(\|q_0\|_0^2 + \|\lambda\|_0^2 + \|g\|_0^2). \tag{3.1}$$

Proof. Throughout the proof, N_i $(i = 1, 2, \ldots)$ will denote some constants depending only on K and δ.

Fix a standard probability space $(\Omega, \mathcal{F}, P)$ and a one-dimensional Wiener process W with $W(0) = 0$. Let $\mathcal{F}_t := \sigma\{W(s)\colon 0 \le s \le t\}$. Define a sequence $\{q_n\}_{n=1}^{\infty} \subset L^2_{\mathcal{F}}(0,1; H^1)$ as follows: $q_0(t) \equiv q_0$; once q_{n-1} is defined, then let $q_n \in L^2_{\mathcal{F}}(0,1; H^1) \cap L^2(\Omega; C(0,1; H^0))$ be the (unique) solution of the following linear SPDE

$$\begin{cases} \mathrm{d}q_n(t) = [Aq_n(t) + \partial_i f^i(\cdot, q_{n-1}(t,\cdot))]\,\mathrm{d}t + (Mq_n(t) + g)\,\mathrm{d}W(t), & t \in [0,1], \\ q_n(0) = q_0, \quad n = 1, 2, \ldots. \end{cases} \tag{3.2}$$

Note the existence and uniqueness of solutions of (3.2) follow by Lemma 2.1 and the fact that $E\int_0^1 \|f^i(\cdot, q_{n-1}(t,\cdot))\|_0^2\,\mathrm{d}t \le 2K^2 E\int_0^1 [\|\lambda\|_0^2 + \|q_{n-1}(t)\|_0^2]\,\mathrm{d}t$.

Due to (2.6), we have, for $n=1,2,\dots$,

$$\begin{aligned}
&E\|q_n(t)\|_0^2+E\int_0^t\|q_n(t)\|_1^2\,\mathrm{d}t\\
&\leq CE\left\{\|q_0\|_0^2+\|g\|_0^2+\int_0^t 2\,\mathrm{d}K^2[\|\lambda\|_0^2+\|q_{n-1}(t)\|_0^2]\,\mathrm{d}t\right\}\\
&\leq N_2a\left(1+\int_0^t E\|q_{n-1}(t)\|_0^2\,\mathrm{d}t\right)\\
&\leq N_3a\left(1+\int_0^t E\|q_{n-1}(t)\|_0^2\,\mathrm{d}t\right),
\end{aligned}\tag{3.3}$$

where $a:=1+\|q_0\|_0^2+\|g\|_0^2$, $N_2:=C\cdot\max\{1+2\,\mathrm{d}K^2\|\lambda\|_0^2, 2\,\mathrm{d}K^2\}$, $N_3:=N_2a$. In particular, we have

$$E\|q_1(t)\|_0^2+E\int_0^t\|q_1(t)\|_1^2\,\mathrm{d}t\leq N_2a(1+at)\leq N_3a(1+t).\tag{3.4}$$

By (3.3) and (3.4), it is not difficult to obtain by induction that

$$\begin{aligned}
&E\|q_n(t)\|_0^2+E\int_0^t\|q_n(t)\|_1^2\,\mathrm{d}t\\
&\leq N_3a\left(\sum_{k=0}^{n-1}\frac{1}{k!}(N_3a)^kt^k+\frac{1}{n!}(N_3a)^{n-1}t^n\right)\\
&\leq N_3a\left(\exp(N_3at)+\frac{1}{n!}(N_3a)^{n-1}t^n\right),
\end{aligned}$$

hence

$$\sup_n E\int_0^1\|q_n(t)\|_1^2\,\mathrm{d}t<+\infty.\tag{3.5}$$

Note for $p=2,4$, we have

$$E\int_0^1\|f^i(\cdot,q_{n-1}(t,\cdot))\|_0^{2p}\,\mathrm{d}t\leq 2^{2p-1}K^{2p}E\int_0^1[\|\lambda\|_0^{2p}+\|q_{n-1}(t)\|_0^{2p}]\,\mathrm{d}t,$$

hence appealing to (2.7), a totally similar argument to above yields

$$\sup_n\sup_{0\leq t\leq 1}E(\|q_n(t)\|_0^4+\|q_n(t)\|_0^8)<+\infty.\tag{3.6}$$

Therefore, it follows from (2.7) that

$$\begin{aligned}
&\sup_n E\left(\int_0^1\|q_n(t)\|_1^2\,\mathrm{d}t\right)^2\\
&\leq\sup_n CE\left\{\|q_0\|_0^4+\int_0^1\left[\sum_{i=1}^d\|f^i(\cdot,q_{n-1}(s,\cdot))\|_0^4+\|g\|_0^4\right]\mathrm{d}s\right.\\
&\leq\sup_n CE\left\{\|q_0\|_0^4+8\,\mathrm{d}K^4\int_0^1[\|\lambda\|_0^4+\|q_{n-1}(s)\|_0^4]\,\mathrm{d}s+\|g\|_0^4\right\}<+\infty.
\end{aligned}\tag{3.7}$$

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Define $p_n(t) := \mathrm{e}^{-MW(t)} q_n(t)$. It will be seen in the sequel that p_n satisfies an SPDE with a constant diffusion term $p_{n,1}(t) := \int_0^t \mathrm{e}^{-MW(s)} g \, \mathrm{d}W(s)$. So we define $p_{n,2}(t) := p_n(t) - p_{n,1}(t)$. Then by Lemma 2.2 and (3.7), we have

$$\begin{aligned} &\sup_n E \int_0^1 \|p_n(t)\|_1^2 \, \mathrm{d}t \\ &\leq \sup_n E \int_0^1 \mathrm{e}^{2N|W(t)|} \|q_n(t)\|_1^2 \, \mathrm{d}t \\ &\leq \sup_n E\left(\sup_{0\leq t\leq 1} \mathrm{e}^{2N|W(t)|} \int_0^1 \|q_n(t)\|_1^2 \, \mathrm{d}t \right) \\ &\leq \sup_n \left(E \sup_{0\leq t\leq 1} \mathrm{e}^{4N|W(t)|} \right)^{1/2} \left[E\left(\int_0^1 \|q_n(t)\|_1^2 \, \mathrm{d}t \right)^2 \right]^{1/2} < +\infty. \end{aligned} \tag{3.8}$$

Similarly, we have

$$\sup_n E \int_0^1 \|p_{n,1}(t)\|_1^2 \, \mathrm{d}t \leq \sup_n E \int_0^1 \int_0^t \mathrm{e}^{2N|W(s)|} \|g\|_1^2 \, \mathrm{d}s \, \mathrm{d}t < +\infty.$$

It follows that

$$\sup_n E \int_0^1 \|p_{n,2}(t)\|_1^2 \, \mathrm{d}t < +\infty. \tag{3.9}$$

On the other hand, for any $\phi \in H^1$, we have the following formula in H^{-1} appealing to Lemma 2.2 and Ito's formula:

$$\mathrm{d}(\mathrm{e}^{-M^*W(t)}\phi) = \tfrac{1}{2} M^{*2} \mathrm{e}^{-M^*W(t)} \phi \, \mathrm{d}t - M^* \mathrm{e}^{-M^*W(t)} \phi \, \mathrm{d}W(t).$$

Therefore again by Ito's formula

$$\begin{aligned} \mathrm{d}(p_{n,2}(t), \phi) &= \mathrm{d}(q_n(t), \mathrm{e}^{-M^*W(t)}\phi) - (\mathrm{e}^{-MW(t)} g, \phi) \, \mathrm{d}W(t) \\ &= \langle (Aq_n(t) + \partial_i f^i(\cdot, q_{n-1}(t,\cdot)), \mathrm{e}^{-M^*W(t)}\phi \rangle \, \mathrm{d}t \\ &\quad + (Mq_n(t) + g, \mathrm{e}^{-M^*W(t)}\phi) \, \mathrm{d}W(t) \\ &\quad + \langle q_n(t), \tfrac{1}{2} M^{*2} \mathrm{e}^{-M^*W(t)}\phi \rangle \, \mathrm{d}t \\ &\quad - (q_n(t), M^* \mathrm{e}^{-M^*W(t)}\phi) \, \mathrm{d}W(t) \\ &\quad - (Mq_n(t) + g, M^* \mathrm{e}^{-M^*W(t)}\phi) \, \mathrm{d}t \\ &\quad - (\mathrm{e}^{-MW(t)} g, \phi) \, \mathrm{d}W(t) \\ &= \{ \langle (A - \tfrac{1}{2}M^2) q_n(t) - Mg, \mathrm{e}^{-M^*W(t)}\phi \rangle \\ &\quad - (f^i(\cdot, q_{n-1}(t,\cdot)), \partial_i(\mathrm{e}^{-M^*W(t)}\phi)) \} \, \mathrm{d}t. \end{aligned} \tag{3.10}$$

Hence

$$\begin{aligned} |\langle \mathrm{d}p_{n,2}(t)/\mathrm{d}t, \phi \rangle| &\leq \left[\|(A - \tfrac{1}{2}M^2) q_n(t)\|_{-1} + \|Mg\|_0 \right. \\ &\quad \left. + \sum_{i=1}^d \|f^i(\cdot, q_{n-1}(t,\cdot))\|_0 \right] \|\mathrm{e}^{-M^*W(t)}\phi\|_1 \\ &\leq N_4 \mathrm{e}^{N|W(t)|} (\|q_n(t)\|_1 + \|q_{n-1}(t)\|_0 + \|g\|_1) \|\phi\|_1. \end{aligned}$$

This yields

$$\sup_n E\int_0^1 \|\mathrm{d}p_{n,2}(t)/\mathrm{d}t\|_{-1}^2\,\mathrm{d}t$$

$$\leq \text{const}\cdot\sup_n E\int_0^1 e^{2N|W(t)|}(\|q_n(t)\|_1^2+\|q_{n-1}(t)\|_0^2+1)\,\mathrm{d}t<+\infty. \tag{3.11}$$

(3.9) and (3.11) imply that there exists a constant N_5 which is independent of n such that, for any $D\subset\mathbb{R}^d$ with the property (2.11),

$$E\|p_{n,2}\|^2_{W_D[0,1]}\leq N_5. \tag{3.12}$$

Let $D_k:=\{x\in\mathbb{R}^d: |x|<k\}$ for $k=1,2,\ldots$. Define a metric $\bar{d}$ on $L^2(0,1;H^0)$ by

$$\bar{d}(\phi,\psi):=\sum_{k=1}^{\infty}\frac{1}{2^k}\min\left\{1,\left(\int_0^1\|\phi(t)-\psi(t)\|^2_{0,D_k}\,\mathrm{d}t\right)^{1/2}\right\}.$$

We denote by $\bar{L}^2(0,1;H^0)$ the completion of $L^2(0,1;H^0)$ by $\bar{d}$, namely, for any $\phi\in\bar{L}^2(0,1;H^0)$, there is $\{\phi_n\}\subset L^2(0,1;H^0)$ such that $\int_0^1\|\phi_n-\phi\|^2_{0,D}\,\mathrm{d}t\to 0$ for any compact D. It should be noted that any $\phi\in\bar{L}^2(0,1;H^0)$ is still a function of (t,x,ω) instead of abstract object, since ϕ is pointwisely a limit of functions in the space $L^2(0,1;H^0)$.

For $\rho>0$, $B_\rho:=\{\phi\in\bar{L}^2(0,1;H^0): \|\phi\|_{W_{D_k}[0,1]}\leq(2^k\rho)^{1/2}, k=1,2,\ldots\}$ is compact in $\bar{L}^2(0,1;H^0)$ due to Lemma 2.3. Now (3.12) yields

$$P(p_{n,2}\notin B_\rho)\leq\sum_{k=1}^{\infty}\frac{1}{2^k\rho}N_5\leq N_5/\rho,\quad\text{for any }\rho>0,$$

hence $\{p_{n,2}\}$ is tight as a sequence of $\bar{L}^2(0,1;H^0)$-r.v. (cf. Ikeda and Watanabe, 1989). Thus by the Skorohod theorem, we can choose a subsequence (still denoted by $\{n\}$) and have $C(0,1;\mathbb{R}^1)\times\bar{L}^2(0,1;H^0)$-random variables $(\hat{W}_n,\hat{p}_{n,2})$, $(\hat{W},\hat{p}_2)$ on a suitable probability space $(\hat{\Omega},\hat{\mathcal{F}},\hat{P})$, such that

$$\text{law of }(\hat{W}_n,\hat{p}_{n,2})=\text{law of }(W,p_{n,2}), \tag{3.13}$$

and $\hat{P}$-a.s.

$$\hat{W}_n\to\hat{W}\quad\text{in }C(0,1;\mathbb{R}^1), \tag{3.14}$$

$$\hat{p}_{n,2}\to\hat{p}_2\quad\text{in }\bar{L}^2(0,1;H^0),\quad\text{as }n\to+\infty. \tag{3.15}$$

Define

$$\hat{q}_n(t):=e^{M\hat{W}_n(t)}\left(\int_0^t e^{-M\hat{W}_n(s)}g\,\mathrm{d}\hat{W}_n(s)+\hat{p}_{n,2}(t)\right),$$

$$q(t):=e^{M\hat{W}(t)}\left(\int_0^t e^{-M\hat{W}(s)}g\,\mathrm{d}W(s)+\hat{p}_2(t)\right).$$

By virtue of (3.13), we have

$$\text{law of }(\hat{W}_n,\hat{q}_n)=\text{law of }(W,q_n). \tag{3.16}$$

Further, observing (3.14) and (3.15), it is not difficult to derive that (cf. Zhou, 1991, Theor 3.1) there is a subsequence (still denoted by $\{n\}$) satisfying

$$\hat{E}\|\hat{q}_n-\hat{q}\|_{L^2(0,1;H^0)}\to 0, \quad \text{as } n\to+\infty. \tag{3.17}$$

On the other hand, in view of (3.7), there is a subsequence of $\{\hat{q}_n\}$ weakly converging in $L^2([0,1]\times\hat{\Omega};H^1)$, and the limit is necessarily $\hat{q}$. This implies $\hat{q}\in L^2([0,1]\times\hat{\Omega};H^1)$.

By (3.16), $(\hat{q}_n,\hat{W}_n)$ satisfies eq. (3.2). Let ψ be an absolutely continuous function from $[0,1]$ into $\mathbb{R}^1$, with $\dot{\psi}\equiv d\psi/dt\in L^2(0,1)$, $\psi(1)=0$ and $\eta\in C_0^\infty(\mathbb{R}^d)$. Ito's formula yields

$$\begin{aligned}0=(q_0,\eta)\psi(0)&+\int_0^1[\langle A\hat{q}_n(t),\eta\rangle-(f^i(\cdot,\hat{q}_{n-1}(t,\cdot)),\partial_i\eta)]\psi(t)\,dt\\&+\int_0^1(M\hat{q}_n(t)+g,\eta)\psi(t)\,d\hat{W}_n(t)+\int_0^1(\hat{q}_n(t),\eta)\dot{\psi}(t)\,dt,\quad P\text{-a.s.}\end{aligned} \tag{3.18}$$

Noting (3.17), there is a subsequence (still denoted by $\{n\}$) such that

$$\hat{q}_n(t,x,\omega)\to\hat{q}(t,x,\omega)\quad\text{a.e. in }[0,1]\times\operatorname{supp}\eta\times\hat{\Omega},\quad\text{as } n\to+\infty.$$

So the assumption (A3) and the dominated convergence theorem gives

$$\hat{E}\int_0^1\int_{\operatorname{supp}\eta}|f^i(x,\hat{q}_n(t,x))-f^i(x,\hat{q}(t,x))|^2\,dx\,dt\to 0\quad\text{as } n\to+\infty. \tag{3.19}$$

Now sending n to $+\infty$ in (3.18), we get

$$\begin{aligned}0=(q_0,\eta)\psi(0)&+\int_0^1[\langle A\hat{q}(t),\eta\rangle-(f^i(\cdot,\hat{q}(t,\cdot)),\partial_i\eta)]\psi(t)\,dt\\&+\int_0^1(M\hat{q}(t)+g,\eta)\psi(t)\,d\hat{W}(t)+\int_0^1(\hat{q}(t),\eta)\dot{\psi}(t)\,dt.\end{aligned} \tag{3.20}$$

In the above, the convergence of other terms rather than (3.19) can be proved by a routine argument as in linear SPDE cases (cf. Pardoux, 1979). Now (3.20) means that $(\hat{q},\hat{W})$ is a solution of (2.3) (cf. Pardoux, 1979). Finally, (3.1) follows easily from the estimate (2.6) and Gronwall's inequality. □

Remark 3.1. The main idea in proving Theorem 3.1 is to construct the transformation $p_n(t)=e^{-MW(t)}q_n(t)$, such that p_n satisfies an 'almost' deterministic equation in the sense that p_n satisfies an SPDE whose diffusion term is a constant stochastic process $(=\int_0^t e^{-MW(t)}g\,dW(t)$; see (3.10)). This method may be viewed as a *time change* technique, which, however, fails to be effective in general for the equation as follows:

$$\begin{cases}dq(t)=[Aq(t)+\partial_i f^i(\cdot,q(t,\cdot))]\,dt+\sum_{k=1}^{d'}[M_kq(t)+g_k]\,dW^k(t),\\ q(0)=q_0,\end{cases} \tag{3.21}$$

where $W:=(W^1,\dots,W^{d'})$ is a d'-dimensional Wiener process, and A, M_k have the same forms as (2.1), (2.2). But in some special cases, for example, if $\{M_k\}$ are commutative (i.e., $M_kM_j=M_jM_k$ for $k\neq j$), we can still obtain the existence of solutions of (3.21) by a similar argument to the one-dimensional Wiener process cases (cf. Zhou, 1991, Theor 5.1). Note the commutative cases include those that $\{M_k\}$ are of order zero or the coefficients of $\{M_k\}$ are constants. Now we have seen that the main difficulty of treating our problem comes from the unboundedness of the operators in diffusion term.

Remark 3.2. Take the transformation $p(t):=\mathrm{e}^{-MW(t)}q(t)$ in (2.3), then a similar calculation to that of (3.10) yields that the eq. (2.3) corresponds to (note the above transformation is reversible!) a deterministic evolution equation (called robust equation), the dynamics of which being $A-\frac{1}{2}M^2$, perturbed by a constant stochastic process. This justifies the observation in the Introduction that diffusion containing the first-order operator influences the behavior of the solutions of (2.3) as strongly as does the drift containing the second-order operator.

Remark 3.3. The uniqueness (in law sense) of solutions of (2.3), when $f'(x,\cdot)$ is merely continuous, seems to be a very difficult problem and remains open according to the author's knowledge. The difficulty arises from the dimensions infinity: in SDE cases, the corresponding uniqueness has been proved using some estimates of the differential operators (Stroock and Varadhan, 1979). However, when the differential operators concerned are on infinite dimensions, none of those estimates is known.

Let us conclude this section by an example.

Example 3.1. Consider a heat flow in a random medium with a temperature dependent source. The field of temperature q is governed by the following SPDE

$$\begin{cases} \mathrm{d}q(t,x)=(\Delta q(t,x)+\nabla f(q(t,x))\,\mathrm{d}t+g(x)\,\mathrm{d}W(t), & t\in[0,1], \quad x\in\mathbb{R}^1,\\ q(0,x)=q_0(x), \end{cases}$$

where Δ is the Laplacian, ∇ is the gradient in x, and W is a one-dimensional Wiener process. A deterministic version and a linear version of the above system have been discussed in Pazy (1983) and Nagase and Nisio (1990), respectively. By Theorem 3.1, there exists at least one solution of the equation provided that $q_0\in H^0$, $g\in H^1$, and f satisfies continuity and linear growth conditions.

4. A variant

In this section, we shall consider the following type of equations:

$$\begin{cases} \mathrm{d}q(t)=[Aq(t)+F(q(t))]\,\mathrm{d}t+[Mq(t)+g]\,\mathrm{d}W(t), & t\in[0,1],\\ q(0)=q_0, \end{cases} \tag{4.1}$$

where A, M are defined as in (2.1), (2.2), and F maps H^0 into H^{-1}. This type of equations is of a more general and abstract feature than (2.3), and has been studied by many authors (Pardoux, 1975; Walsh, 1986; Nagase, 1990; ...). The existence of its solutions can be solved by the same method as that for (2.3), except for a technical problem, which will be explained below.

First let us introduce the following function space. For a positive r, $L_r^2 := \{\phi\colon \phi$ is a real valued Borel function on $\mathbb{R}^d$, and $(1+|\cdot|^2)^{r/2}\phi(\cdot) \in H^0\}$ with the norm $\|\phi\|_{0,r} := (\int_{\mathbb{R}^d} |(1+|x|^2)^{r/2}\phi(x)|^2\,\mathrm{d}x)^{1/2}$. L_r^2 thus defined is a Hilbert space which is a subspace of H^0.

Let $H_r^1 := \{\phi\colon \phi, \partial_i\phi \in L_r^2\}$ with the norm

$$\|\phi\|_{1,r} := \left(\|\phi\|_{0,r}^2 + \sum_{i=1}^{d} \|\partial_i\phi\|_{0,r}^2\right)^{1/2}.$$

It also becomes a Hilbert space.

Theorem 4.1. *In addition to the assumptions* (A1), (A2), *we assume that*

(A3)′ $F\colon H^0 \to H^{-1}$ *is continuous, maps* L_r^2 *into* L_r^2, *and*

$$\|F(\phi)\|_{-1} \le K(1+\|\phi\|_0) \quad \text{for } \phi \in H^0,$$

$$\|F(\psi)\|_{0,r} \le K(1+\|\psi\|_{0,r}) \quad \text{for } \psi \in L_r^2,$$

$$g \in H^1 \cap L_r^2.$$

(A4)′ $q_0 \in L_r^2$.

Then, there exists at least one solution of (4.1).

Proof. We construct $\{q_n\}$ in a similar fashion to (3.2). By virtue of (A3)′, we can obtain that $\hat{q}_n$ satisfies (3.17) using the entirely same argument as in the proof of Theorem 3.1. But this is not enough, since (3.17) only means, roughly speaking, that $\hat{q}_n$ converges to $\hat{q}$ in $H^0(D)$ for every bounded D. In the present case, we must show that the convergence is also in H^0. To this end, we make use of the result of Krylov and Rozovskii (1982b) concerning the L_r^2-norm estimates of the solution of linear SPDE, to obtain that (noting (A3)′)

$$\sup_n \sup_{0\le t\le 1} \hat{E}\|\hat{q}_n(t)\|_{0,r}^2 \le \text{const}(\|q_0\|_{0,r}^2 + \|g\|_{0,r}^2) < +\infty.$$

Then,

$$\hat{E}\int_0^1 \int_{|x|>\rho} |\hat{q}(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$= \lim_{k\to\infty} \lim_{n\to\infty} \hat{E}\int_0^1 \int_{\rho<|x|<k} |\hat{q}_n(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$\le \text{const}/(1+\rho^2)^r \to 0, \quad \text{as } \rho \to +\infty.$$

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Hence $\hat{q} \in L^2([0,1]\times\hat{\Omega}; H^0)$, and

$$\hat{E}\int_0^1 \|\hat{q}_n(t)-\hat{q}(t)\|_0^2\,\mathrm{d}t \to 0, \quad \text{as } n\to+\infty. \tag{4.2}$$

Due to the continuity of F, we can complete the proof by the same argument as in the proof of Theorem 3.1. □

In the foregoing, the existence results have been obtained under the assumption (A2), i.e., the equations considered are nondegenerate. Now we will show that we can allow the equation to be degenerate at the cost of posing more regularity conditions on the coefficients and initial state.

Let us introduce the following conditions:

(B1) $a^{ij}, \sigma^i, h:\mathbb{R}^d\to\mathbb{R}^1$ are measurable; these functions and their derivatives up to second-order do not exceed K in absolute value.

(B2) $a^{ij}=a^{ji}$, $i,j=1,2,\ldots,d$, and $(a^{ij}-\frac{1}{2}\sigma^i\sigma^j)_{ij}\geq 0$.

(B3) $F: H^0\to H^0$ is continuous, maps L^2_r and H^1 into themselves, respectively, and

$$\|F(\phi)\|_k \leq K(1+\|\phi\|_k) \quad \text{for } \phi\in H^k, \qquad k=0,1,$$

$$\|F(\psi)\|_{0,r} \leq K(1+\|\psi\|_{0,r}) \quad \text{for } \psi\in L^2_r,$$

$$g\in H^2\cap H^1_r.$$

(B4) $q_0\in H^1\cap L^2_r$.

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Lemma 4.1 (Krylov and Rozovskii, 1982a, b). *Given a filtered probability space $(\Omega,\mathcal{F},P,\mathcal{F}_t)$ and a one-dimensional Wiener process W. Assume* (B1), (B2), (B4) *and that $\tilde{F}\in L^2_{\mathcal{F}}(0,1;H^1)\cap L^2_{\mathcal{F}}(0,1;L^2_r)$, $\tilde{G}\in H^2\cap H^1_r$. Then the following equation*

$$\begin{cases} \mathrm{d}q(t)=[Aq(t)+\tilde{F}(t)]\,\mathrm{d}t+[Mq(t)+\tilde{G}]\,\mathrm{d}W(t), & t\in[0,1],\\ q(0)=q_0, \end{cases} \tag{4.3}$$

has a unique solution $q\in L^2_{\mathcal{F}}(0,1;H^1)\cap L^2_{\mathcal{F}}(0,1;L^2_r)$ and there exists a constant C', depending only on K and r, such that

$$E\|q(t)\|^2_{0,r}\leq C'E\left\{\|q_0\|^2_{0,r}+\int_0^t [\|\tilde{F}(s)\|^2_{0,r}+\|\tilde{G}\|^2_{1,r}]\,\mathrm{d}s\right\}. \tag{4.4}$$

Moreover, if $\tilde{F}\in L^p_{\mathcal{F}}(0,1;H^1)$ for $p\geq 2$, then there is a constant $C'(p)$ such that

$$E\|q(t)\|^p_k\leq C'(p)E\left\{\|q_0\|^p_k+\int_0^t [\|\tilde{F}(s)\|^p_k+\|\tilde{G}\|^p_{k+1}]\,\mathrm{d}s\right\}, \quad k=0,1. \qquad \square \tag{4.5}$$

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Theorem 4.2. *Under assumptions* (B1)–(B4), *there exists at least one solution of* (4.1).

Proof. By virtue of Lemma 4.1, the result can be proved by the same argument as in the proof of Theorem 4.1. □

5. Optimal relaxed controls

In this section we shall study the existence of optimal relaxed controls for systems governed by semilinear SPDE like (2.3). First let us introduce the definition of relaxed control.

Let Γ be a given compact set in $\mathbb{R}^m$. By Λ we denote the set of all measures μ on $[0,1]\times\mathbb{R}^d\times\Gamma$ such that

$$\mu(S\times\Gamma)=m(S), \quad \text{for any Lebesque set } S \text{ in } [0,1]\times\mathbb{R}^d, \tag{5.1}$$

where m is the Lebesque measure on $[0,1]\times\mathbb{R}^d$.

We define by Λ_k the limitation of Λ on $V_k:=[0,1]\times[-k,k]^d\times\Gamma$, i.e., $\Lambda_k:=\{\mu|_{[0,1]\times[-k,k]^d\times\Gamma}: \mu\in\Lambda\}$, $k=1,2,\ldots$. Denoting by d_k the Prohorov metric on Λ_k, we define a metric on Λ as follows:

$$d(\mu,\mu'):=\sum_{k=1}^{\infty}\frac{1}{2^k}\min\{1, d_k(\mu|_{V_k},\mu'|_{V_k})\}. \tag{5.2}$$

Lemma 5.1. (i) $d(\mu_n,\mu)\to 0$ *iff* $\int f\,\mathrm{d}\mu_n\to\int f\,\mathrm{d}\mu$ *for any bounded continuous function* f *with compact support on* $[0,1]\times\mathbb{R}^d\times\Gamma$, *as* $n\to+\infty$.

(ii) Λ *is compact under the metric* d.

Proof. (i) It is clear.

(ii) Each Λ_k is compact under d_k since V_k is compact (cf. Ikeda and Watanabe, 1989), thus the desired result follows from a standard diagonal argument. □

Set $\sigma_t(\Lambda):=$ the σ-field generated by $\{\mu: \mu([0,s]\times S)\in\mathcal{B}(R^+),\ s\le t,\ S\in\mathcal{B}(\mathbb{R}^d\times\Gamma)\}$ and $\sigma(\Lambda):=\sigma_1(\Lambda)$. Let $\mathcal{P}:=\mathcal{P}(\Lambda)$ be the space of probabilities on $(\Lambda,\sigma(\Lambda))$, then Lemma 5.1(ii) yields that $\mathcal{P}$ is a compact metric space under the Prohorov metric.

By (5.1), μ is represented by $\mu(\mathrm{d}t,\mathrm{d}x,\mathrm{d}u)=\mu'(t,x,\mathrm{d}u)\,\mathrm{d}t\,\mathrm{d}x$, where $\mu'(t,x,\cdot)$ is a probability on Γ for almost all (t,x) and determined uniquely expect (t,x)-null set.

Now we introduce the relaxed system.

Definition 5.1. $\mathcal{R}=(\Omega,\mathcal{F},P,\mathcal{F}_t,W,\mu)$ is called a relaxed system, if

(i) $(\Omega,\mathcal{F},P,\mathcal{F}_t)$ is a standard probability space with filteration $\{\mathcal{F}_t: 0\le t\le 1\}$;

(ii) W is an $\mathcal{F}_t$-adapted one-dimensional Wiener process with $W(0)=0$;

(iii) μ is an $\mathcal{F}_t$-adapted Λ-valued random variable (Λ-r.v. in short), i.e., $\mu(B_1\times B_2)$ is $\mathcal{F}_t$-measurable whenever $B_1\in\mathcal{B}([0,t])$ and $B_2\in\mathcal{B}(\mathbb{R}^d\times\Gamma)$.

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For simplicity, we put $\mathscr{R} = (W, \mu)$ if no confusion arises.

R denotes the totality of relaxed controls. For $\mathscr{R} = (W, \mu)$, $\pi(\mathscr{R})$ denotes the image measure of (W, μ) on $C(0, T; \mathbb{R}^1) \times \Lambda$. Again by endowing the space $\Pi := \{\pi(\mathscr{R}) \colon \mathscr{R} \in R\}$ with the Prohorov metric, we have the following fact (Nagase and Nisio, 1990):

Lemma 5.2. *Π is a compact metric space.* □

Given $\mathscr{R} = (\Omega, \mathscr{F}, P, \mathscr{F}_t, W, \mu)$, consider the following SPDE

$$\begin{cases} \mathrm{d}q(t) = [Aq(t) + \partial_i \bar{f}^i(t, \cdot, q(t, \cdot), \mu)]\,\mathrm{d}t \\ \qquad\qquad + [Mq(t) + g]\,\mathrm{d}W(t), \quad t \in [0, 1], \\ q(0) = q_0, \end{cases} \tag{5.3}$$

where A, M are defined by (2.1), (2.2), and

$$\bar{f}^i(t, x, r, \mu) := \int_\Gamma f^i(t, x, r, u)\mu^t(t, x, \mathrm{d}u), \tag{5.4}$$

for given functions f^i on $[0, 1] \times \mathbb{R}^d \times \mathbb{R}^1 \times \Gamma$, $i = 1, 2, \ldots, d$.

Remark 5.1. Under some assumptions which will be specified below, for each $\pi \in \Pi$, there are $\mathscr{R} \in R$ and q such that $\pi(\mathscr{R}) = \pi$ and q is a solution of (5.3) for $\mathscr{R}$. This can be proved by the same method as in proving Theorem 3.1. It is in this sense that we will call either an $\mathscr{R} \in R$ or a $\pi \in \Pi$ a relaxed control. Since we do not know the uniqueness of solutions, let us denote by $S(\pi)$ the totality of solutions of (5.3) corresponding to $\pi \in \Pi$.

Remark 5.2. The controlled system (5.3) is the relaxed one of the following system:

$$\begin{cases} \mathrm{d}q(t) = [Aq(t) + \partial_i f^i(t, \cdot, q(t, \cdot), u(t, \cdot))]\,\mathrm{d}t \\ \qquad\qquad + [Mq(t) + g]\,\mathrm{d}W(t), \quad t \in [0, 1], \\ q(0) = q_0, \end{cases} \tag{5.5}$$

where the admissible control $u : [0, 1] \times \mathbb{R}^d \times \Omega \to \Gamma$ is measurable and $\mathscr{F}_t$-adapted. Indeed, take $\mu^t(t, x, \mathrm{d}u) = \delta_{u(t,x)}(\mathrm{d}u)$, where $\delta_a(\mathrm{d}u)$ is the Dirac measure on Γ, then (5.3) reduces to (5.5). Note in the most of existing results concerning the optimal control of SPDE the controls were taken to be independent of the space variable x (Bensoussan and Nisio, 1990; Nagase and Nisio, 1990; Zhou, 1991; ...), the reason being that their motivation was to study the partially observed diffusion where the controls were indeed space-independent. In this paper we allow the controls to be space-dependent; it is natural to do so since we are concerned with the control problem for SPDE itself. It is also worth noting that in the literature of control problems for deterministic PDE, the controls always took the form of $u(t, x)$ (cf. Ahmed and Teo, 1981).

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For each $\pi \in \Pi$ and $q \in S(\pi)$, we are given a cost functional

$$J(\pi, q) := E\{F(q(\cdot)) + G(q(1))\}, \tag{5.6}$$

where F and G are given functionals on $L^2(0, 1; H^1)$ and H^0 respectively. The optimal relaxed control problem is to find $\pi^* \in \Pi$ and $q^* \in S(\pi^*)$ such that $J(\pi^*, q^*) = \min\{J(\pi, q): \pi \in \Pi, q \in S(\pi)\}$.

Let us introduce some conditions:

(A5) The mapping $(t, x, r, u) \in [0, 1] \times \mathbb{R}^d \times \mathbb{R}^1 \times \Gamma \to f^i(t, x, r, u)$ is measurable. It is continuous in r, uniformly in u, and continuous in u. There are $\lambda \in H^0$ and $K > 0$ such that

$$|f^i(t, x, r, u)| \leq K(\lambda(x) + |r|).$$

On the other hand, $g \in H^1$.

(A6) F and G are weakly continuous mappings from $L^2(0, 1; H^1)$ and H^0 to $\mathbb{R}^1$ respectively, and

$$|F(\phi)| \leq K(1 + \|\phi\|_{L^2(0,1;H^1)}) \quad \text{for } \phi \in L^2(0, 1; H^1),$$

$$|G(\psi)| \leq K(1 + \|\psi\|_0), \quad \text{for } \psi \in H^0.$$

Theorem 5.1. *Under* (A1), (A2), (A4)–(A6), *there exists at least one optimal relaxed control for the system* (5.3) *with the cost functional* (5.6).

Proof. First observing Theorem 3.1 (especially (3.1)) and (A6), $J(\pi, q)$ is bounded below. Hence there is a sequence of $\pi_n \in \Pi$ and $q_n \in S(\pi_n)$ such that

$$J(\pi_n, q_n) \to \inf\{J(\pi, q): \pi \in \Pi, q \in S(\pi)\}. \tag{5.7}$$

By Lemma 5.2, there is a subsequence (still denoted by $\{n\}$) of $\{\pi_n\}$ and $\pi^* \in \Pi$ such that $\pi_n \to \pi^*$ in Prohorov metric. Suppose $\pi_n = \pi(\mathcal{R}_n) = \pi((W_n, \mu_n))$, $\pi^* = \pi(\mathcal{R}^*) = \pi((W^*, \mu^*))$.

Noting the compactness of Λ, by the entirely same argument as in proving Theorem 3.1 (cf. (3.16), (3.17)), we can choose a subsequence (still denoted by $\{n\}$) and have $(\hat{W}_n, \hat{\mu}_n, \hat{q}_n)$, $(\hat{W}^*, \hat{\mu}^*, \hat{q}^*)$ on a suitable space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$, such that

$$\text{law of } (\hat{W}_n, \hat{\mu}_n, \hat{q}_n) = \text{law of } (W_n, \mu_n, q_n) \tag{5.8}$$

as $C(0, 1; \mathbb{R}^1) \times \Lambda \times \bar{L}^2(0, 1; H^0)$-r.v., and $\hat{P}$-a.s.:

$$\hat{W}_n \to \hat{W}^* \quad \text{in } C(0, 1; \mathbb{R}^1), \tag{5.9}$$

$$f(\hat{\mu}_n, \hat{\mu}^*) \to 0, \tag{5.10}$$

$$\hat{q}_n \to \hat{q}^* \quad \text{in } \bar{L}^2(0, 1; H^0), \quad \text{as } n \to +\infty. \tag{5.11}$$

By a similar calculation to that of (3.18), we have

$$\begin{aligned}(q_0, \eta)\psi(0) &+ \int_0^1 [\langle A\hat{q}_n(t), \eta\rangle - (\tilde{f}^i(t, \cdot, \hat{q}_n(t, \cdot), \hat{\mu}_n), \partial_i\eta)]\psi(t)\, dt \\ &+ \int_0^1 (M\hat{q}_n(t) + g, \eta)\psi(t)\, d\hat{W}_n(t) + \int_0^1 (\hat{q}_n(t), \eta)\dot{\psi}(t)\, dt = 0,\end{aligned} \tag{5.12}$$

where ψ, η are specified in the paragraph followed by (3.18). Now we can write

$$\begin{aligned}
&\int_0^1 (\tilde{f}^i(t,\cdot,\hat{q}_n(t,\cdot),\hat{\mu}_n)-\tilde{f}^i(t,\cdot,\hat{q}^*(t,\cdot),\hat{\mu}^*),\partial_i\eta)\psi(t)\,\mathrm{d}t\\
&\quad=\int_0^1\int_{\operatorname{supp}\eta}\int_\Gamma [f^i(t,x,\hat{q}_n(t,x),u)-f^i(t,x,\hat{q}^*(t,x),u)]\\
&\qquad\times\hat{\mu}'_n(t,x,\mathrm{d}u)\partial_i\eta(x)\psi(t)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_\Gamma\int_{\operatorname{supp}\eta}\int_0^1 f^i(t,x,\hat{q}^*(t,x),u)\partial_i\eta(x)\psi(t)\\
&\qquad\times[\hat{\mu}_n(\mathrm{d}t,\mathrm{d}x,\mathrm{d}u)-\hat{\mu}^*(\mathrm{d}t,\mathrm{d}x,\mathrm{d}u)]\\
&\quad\to 0,\quad \text{as } n\to+\infty.
\end{aligned}$$

Sending n to $+\infty$ in (5.12), we arrive at

$$\begin{aligned}
&(q_0,\eta)\psi(0)+\int_0^1[\langle A\hat{q}^*(t),\eta\rangle-(\tilde{f}^i(t,\cdot,\hat{q}^*(t,\cdot),\hat{\mu}^*),\partial_i\eta)]\psi(t)\,\mathrm{d}t\\
&\quad+\int_0^1(M\hat{q}^*(t)+g,\eta)\psi(t)\,\mathrm{d}W^*(t)+\int_0^1(\hat{q}^*(t),\eta)\dot{\psi}(t)\,\mathrm{d}t=0.
\end{aligned}$$

This means $\hat{q}^*\in S(\pi(\hat{W}^*,\hat{\mu}^*))=S(\pi(W^*,\mu^*))=S(\pi^*)$, noting (5.8)-(5.11).

On the other hand, observing the compactness of the embeddings $L^2(0,1;H^1)\to$ w-$L^2(0,1;H^1)$ and $H^0\to$ w-H^0 (w-X means the Banach space endowed with the weak-topology), we can show, by the same argument as above, that $\hat{q}_n\to\hat{q}^*$ weakly in $L^2(0,1;H^1)$, $\hat{q}_n(1)\to\hat{q}^*(1)$ weakly in H^0 as $n\to+\infty$, $\hat{P}$-a.s. So $J(\pi_n,\hat{q}_n)\to J(\pi^*,\hat{q}^*)$ as $n\to+\infty$ by virtue of (A6). Now (5.7) implies that $(\pi^*,\hat{q}^*)$ is an optimal one. □

Remark 5.1. The relaxed controlled system (5.3) reduces to the (usual) controlled system (5.5) when assuming some convex conditions (Roxin's condition) on $f^i(t,x,r,\Gamma)$ (cf., for example, Nagase and Nisio, 1990). In particular, the existence result holds for the controlled systems governed by deterministic semilinear PDE. Note the existence of optimal controls for linear PDE with Roxin's condition has been known for a long time (cf. Ahmed and Teo, 1981).

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