

# Continuous-Time Mean–Variance Portfolio Selection with Bankruptcy Prohibition

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A continuous-time mean–variance portfolio selection problem is studied where all the market coefficients are random and the wealth process under any admissible trading strategy is not allowed to be below zero at any time. The trading strategy under consideration is defined in terms of the dollar amounts, rather than the proportions of wealth, allocated in individual stocks. The problem is completely solved using a decomposition approach. Specifically, a (constrained) variance minimizing problem is formulated and its feasibility is

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characterized. Then, after having solved a system of equations for two Lagrange multipliers, variance minimizing portfolios are derived as the replicating portfolios of some contingent claims, and the variance minimizing frontier is obtained. Finally, the efficient frontier is identified as an appropriate portion of the variance minimizing frontier after the monotonicity of the minimum variance on the expected terminal wealth over this portion is proved, and all the efficient portfolios are found. In the special case where the market coefficients are deterministic, efficient portfolios are explicitly expressed as feedback of the current wealth, and the efficient frontier is represented by parameterized equations. Our results indicate that the efficient policy for a mean–variance investor is simply to purchase a European put option that is chosen, according to his or her risk preferences, from a particular class of options.

**KEY WORDS:** Mean–variance portfolio selection, Lagrange multiplier, backward stochastic differential equation, contingent claim, Black–Scholes equation, continuous time

## 1 INTRODUCTION

Mean–variance portfolio selection is concerned with the allocation of wealth among a variety of securities so as to achieve the optimal trade-off between the expected return of the investment and its risk over a fixed planning horizon. The model was first proposed and solved more than fifty years ago in the single-period setting by Markowitz in his Nobel-Prize winning work Markowitz (1952), Markowitz (1959). With the risk of a portfolio measured by the variance of its return, Markowitz showed how to formulate the problem of minimizing a portfolio’s variance subject to the constraint that its expected return equals a prescribed level as a quadratic program. Such an optimal portfolio is said to be *variance minimizing*,

and if it also achieves the maximum expected return among all portfolios having the same variance of return, then it is said to be *efficient*. The set of all points in the two-dimensional plane of variance (or standard deviation) and expected return that are produced by efficient portfolios is called the *efficient frontier*. Hence investors should focus on the efficient frontier, with different investors selecting different efficient portfolios, depending upon their risk preferences.

Not only have this model and its single period variations (e.g., there might be constraints on the investments in individual assets) seen widespread use in the financial industry, but also the basic concepts underlying this model have become the cornerstone of classical financial theory. For example, in Markowitz's world (i.e., the world where all the investors act in accordance with the single period, mean–variance theory), one of the important consequences is the so-called mutual fund theorem, which asserts that two mutual funds, both of which are efficient portfolios, can be established so that all investors will be content to divide their assets between these two funds. Moreover, if a risk-free asset (such as a bank account) is available, then it can serve as one of the two mutual funds. A logical consequence of this is that the other mutual fund, which itself is efficient, must correspond to the “market.” This, in turn, leads to the elegant *capital asset pricing model* (CAPM), see Sharp (1964), Lintner (1965), Mossin (1966).

Meanwhile, in subsequent years there has been considerable development of multiperiod and, pioneered by the famous work Merton (1971;1973), continuous-time models for portfolio management. In all this work, however, the approach is considerably different, as expected utility criteria are employed. For example, for the problem of maximizing the expected utility of the investor's wealth at a fixed planning horizon, Merton (1971;1973) used dynamic

programming and partial differential equation theory to derive and analyze the relevant Hamilton–Jacobi–Bellman (HJB) equation. Recent books Karatzas and Shreve (1998) and Korn (1997) summarize much of this continuous time, portfolio management theory.

Multiperiod, discrete-time mean–variance portfolio selection has been studied by Samuelson (1986), Hakansson (1971), Grauer and Hakansson (1993), and Pliska (1997). But somewhat surprisingly, the *exact, faithful* continuous-time versions of the mean–variance problem have not been developed until very recently. This is surprising because the mean–variance portfolio problem is known to be very similar to the problem of maximizing the expected *quadratic* utility of terminal wealth. Solving the expected quadratic utility problem can produce a point on the mean–variance efficient frontier, although *a priori* it is often unclear what the portfolio’s expected return will turn out to be. So while it is straightforward to formulate a continuous-time version of the mean–variance problem as a dynamic programming problem, researchers have been slow to produce significant results.

A more modern approach to continuous-time portfolio management, first introduced by Pliska (1982), Pliska (1986), avoids dynamic programming by using the risk neutral (martingale) probability measure; but this has not been much helpful either. This risk neutral computational approach decomposes the problem into two sub-problems, where first one uses convex optimization theory to find the random variable representing the optimal terminal wealth, and then one solves the sub-problem of finding the trading strategy that replicates the terminal wealth. The solution for the mean–variance problem of the first sub-problem is known for the unconstrained case,<sup>1</sup> but apparently nobody has successfully solved

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<sup>1</sup>See, for example, Pliska (1997); the treatment there was for the single-period situation, but the basic result easily generalizes to very similar results for the multiperiod and continuous-time situations.

for continuous time applications the second sub-problem, which is essentially a martingale representation problem.

A breakthrough of sorts was provided in a recent paper Li and Ng (2000), who studied the discrete-time, multiperiod, mean–variance problem using the framework of multi-objective optimization, where the variance of the terminal wealth and its expectation are viewed as competing objectives. They are combined in a particular way to give a single-objective “cost” for the problem. An important feature of this paper is an embedding technique, introduced because dynamic programming could not be directly used to deal with their particular cost functional. Their embedding technique was used to transform their problem to one where dynamic programming was used to obtain explicit, optimal solutions.

Zhou and Li (2000) used the embedding technique and linear–quadratic (LQ) optimal control theory to solve the continuous-time, mean–variance problem with assets having deterministic diffusion coefficients. In their LQ formulation, the dollar amounts, rather than the proportions of wealth, in individual assets are used to define the trading strategy. This leads to a dynamic system that is linear in both the state (i.e., the level of wealth) and the control (i.e., the trading strategy) variables. Together with the quadratic form of the objective function, this formulation falls naturally into the realm of stochastic LQ control. Moreover, since there is no running cost in the objective function, the resulting problem is inherently an *indefinite stochastic LQ control problem*, the theory of which has been developed only very recently (see, e.g., Chapter 6 in Yong and Zhou (1999)).

Exploiting the stochastic LQ control theory, Zhou and his colleagues have considerably extended the initial continuous-time, mean–variance results obtained by Zhou and Li (2000). Lim and Zhou (2002) allowed for stocks which are modeled by processes having random drift

and diffusion coefficients, Zhou and Yin (2003) featured assets in a regime switching market, and Li, Zhou, and Lim (2001) introduced a constraint on short selling. Kohlmann and Zhou (2000) went in a slightly different direction, studying the problem of mean–variance hedging of a given contingent claim. In all these papers, explicit forms of efficient/optimal portfolios and efficient frontiers were presented. While many results in the continuous-time Markowitz world are analogous to their single-period counterparts, there are some results that are strikingly different. Most of these results are summarized by Zhou (2003) who also provided a number of examples that illustrate the similarities as well as differences between the continuous-time and single-period settings.

In view of all this recent work on the continuous-time, mean–variance problem, what is left to be done? The answer is that it is desirable to address a significant shortcoming of the preceding models, for their resulting optimal trading strategies can cause bankruptcy for the investor. Moreover, these models assume a bankrupt investor can keep on trading, borrowing money even though his or her wealth is negative. In most of the portfolio optimization literature the trading strategies are expressed as the proportions of wealth in the individual assets, so with technical assumptions (such as finiteness of the integration of a portfolio) about these strategies the portfolio’s monetary value will automatically be *strictly* positive. But with strategies described by the money invested in individual assets, as dictated by the stochastic LQ control theory approach, a *larger* set of trading strategies is available, including ones which allow the portfolio’s value to reach zero or to become and remain strictly negative (e.g., borrow from the bank, buy stock on margin, and watch the stock’s price go into the tank). The ability to continue trading even though the value of an investor’s portfolio is strictly negative is highly unrealistic, so that brings us to the subject of this paper: the

study of the continuous-time, mean–variance problem with the additional restriction that bankruptcy is prohibited<sup>2</sup>.

In this paper we use an extension of the risk neutral approach rather than making heavy use of stochastic LQ control theory. However, we retain the specification of trading strategies in terms of the monetary amounts invested in individual assets, and we add the explicit constraint that feasible strategies must be such that the corresponding monetary value of the portfolio is nonnegative (rather than strictly positive) at every point in time with probability one. The resulting continuous time, mean–variance portfolio selection problem is straightforward to formulate, as will be seen in the following section. Our model of the securities market is complete, although we allow the asset drift and diffusion coefficients, as well as the interest rate for the bank account, to be random. Once again, we emphasize that the set of trading strategies we consider is larger than that of the proportional strategies, and we will show that the efficient strategies we obtain are in general *not* obtainable by the proportional ones. In Section 2 we also demonstrate that the original nonnegativity constraint can be replaced by the constraint which simply requires the terminal monetary value of the portfolio to be nonnegative. This leads to the first sub-problem in the risk neutral computational approach: find the nonnegative random variable having minimum variance and satisfying two constraints, one calling for the expectation of this random variable under the original probability measure to equal a specified value, and the other calling for the expectation of the *discounted* value of this random variable under the risk neutral measure to equal the

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<sup>2</sup>Here the bankruptcy is defined as the wealth being *strictly* negative. A zero wealth is *not* regarded as in bankruptcy. In fact, as will be seen in the sequel the wealth process associated with an efficient portfolio may indeed “touch” zero with a positive probability.

initial wealth.

In Section 3 we study the feasibility of our problem, an issue that has never been addressed by other authors to the best of our knowledge. There we provide two nonnegative numbers with the property that the variance minimizing problem has a unique, optimal solution if and only if the ratio of the initial wealth to the desired expected wealth falls between these two numbers. In Section 4 we solve the first sub-problem by introducing two Lagrange multipliers that enable the problem to be transformed to one where the only constraint is that the random variable, i.e., the terminal wealth, must be nonnegative. This leads to an explicit expression for the optimal random variable, an expression that is in terms of the two Lagrange multipliers which must, in turn, satisfy a system of two equations. In Section 5 we show this system has a unique solution, and we establish simple conditions for what the signs of the Lagrange multipliers will be. A consequence here is the observation that the optimal terminal wealth can be interpreted as the payoff of either, depending on the signs of the Lagrange multipliers, a European put or a call on a fictitious security.

In Section 6 we turn to the second sub-problem, showing that the optimal trading strategy of the variance minimizing problem can be expressed in terms of the solution of a backward stochastic differential equation. We also provide an explicit characterization of the mean-variance efficient frontier, which is a proper portion of the variance minimizing frontier. Unlike the situation where bankruptcy is allowed, the expected wealth on the efficient frontier is not necessarily a linear function of the standard deviation of the wealth. In Section 7 we consider the special case where the interest rate and the risk premium are deterministic functions of time (if not constants). Here we provide explicit expressions for the Lagrange multipliers, the optimal trading strategies, and the efficient frontier. We conclude in Section



8 with some remarks.

Somewhat related to our work are the continuous-time studies of mean–variance hedging by Duffie and Richardson (1991), and Schweizer (1992). More pertinent is the study of continuous-time, mean–variance portfolio selection in Richardson (1989), a study where the portfolio’s monetary value was allowed to become strictly negative. Also in the working paper of Zhao and Ziemba a mean–variance portfolio selection problem with deterministic market coefficients and with bankruptcy allowed is solved using a martingale approach. Closely connected to our research is the work by Korn and Trautmann (1995) and Korn (1997). They considered the continuous-time mean–variance portfolio selection problem with nonnegativity constraints on the terminal wealth for the case of the Black–Scholes market where there is a single risky asset that is modeled as simple geometric Brownian motion and where the bank account has a constant interest rate. They provided expressions for the optimal terminal wealth as well as the optimal trading strategy using a duality method. Their first sub-problem fixes a single Lagrange multiplier and then solves an unconstrained convex optimization problem for the optimal proportional strategy; their second sub-problem is to find the “correct” value of their Lagrange multiplier. Actually, they do not have an explicit constraint for nonnegative wealth, but by using strategies that are in terms of proportions of wealth, a strictly positive wealth is automatically achieved. In our paper we include strategies that allow the wealth to become zero at intermediate dates, so apparently our set of feasible strategies is larger. Our results are considerably more general, for we allow stochastic interest rates, an arbitrary number of assets, and asset drift and diffusion coefficients that are random. And we provide characterizations of efficient frontiers, necessary and sufficient conditions for existence of solutions, and several other kinds of results that Korn and Trautmann (1995)

did not address at all.

## 2 PROBLEM FORMULATION

In this paper  $T$  is a fixed terminal time and  $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \geq 0})$  is a fixed filtered complete probability space on which is defined a standard  $\mathcal{F}_t$ -adapted  $m$ -dimensional Brownian motion  $W(t) \equiv (W^1(t), \dots, W^m(t))'$  with  $W(0) = 0$ . It is assumed that  $\mathcal{F}_t = \sigma\{W(s) : s \leq t\}$ . We denote by  $L^2_{\mathcal{F}}(0, T; \mathbb{R}^d)$  the set of all  $\mathbb{R}^d$ -valued, progressively measurable stochastic processes  $f(\cdot) = \{f(t) : 0 \leq t \leq T\}$  adapted to  $\mathcal{F}_t$  such that  $E \int_0^T |f(t)|^2 dt < +\infty$ , and by  $L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^d)$  the set of all  $\mathbb{R}^d$ -valued,  $\mathcal{F}_T$ -measurable random variables  $\eta$  such that  $E|\eta|^2 < +\infty$ . Throughout this paper, a.s. signifies that the corresponding statement holds true with probability 1 (with respect to  $P$ ).

**Notation.** We use the following additional notation:

$M'$  : the transpose of any vector or matrix  $M$ ;

$|M|$  :  $= \sqrt{\sum_{i,j} m_{ij}^2}$  for any matrix or vector  $M = (m_{ij})$ ;

$\alpha^+$  :  $= \max\{\alpha, 0\}$  for any real number  $\alpha$ ;

$\mathbf{1}_A$  : the indicator function of any set  $A$ .

Suppose there is a market in which  $m + 1$  assets (or securities) are traded continuously. One of the assets is the bank account whose price process  $S_0(t)$  is subject to the following

(stochastic) ordinary differential equation:

$$(2.1) \quad \begin{cases} dS_0(t) = r(t)S_0(t)dt, & t \in [0, T], \\ S_0(0) = s_0 > 0, \end{cases}$$

where the interest rate  $r(t)$  is a uniformly bounded,  $\mathcal{F}_t$ -adapted, scalar-valued stochastic process. Note that normally one would assume that  $r(t) \geq 0$ ; yet this assumption is not necessary in our subsequent analysis. The other  $m$  assets are stocks whose price processes  $S_i(t)$ ,  $i = 1, \dots, m$ , satisfy the following stochastic differential equation (SDE):

$$(2.2) \quad \begin{cases} dS_i(t) = S_i(t)[b_i(t)dt + \sum_{j=1}^m \sigma_{ij}(t)dW^j(t)], & t \in [0, T], \\ S_i(0) = s_i > 0, \end{cases}$$

where  $b_i(t)$  and  $\sigma_{ij}(t)$ , the appreciation and dispersion (or volatility) rates, respectively, are scalar-valued,  $\mathcal{F}_t$ -adapted, uniformly bounded stochastic processes.

Define the volatility matrix  $\sigma(t) := (\sigma_{ij}(t))_{m \times m}$ . A basic assumption throughout this paper is that the *covariance* matrix

$$(2.3) \quad \sigma(t)\sigma(t)' \geq \delta I_m, \quad \forall t \in [0, T], \quad \text{a.s.},$$

for some  $\delta > 0$ , where  $I_m$  is the  $m \times m$  identity matrix. Consequently, we have a complete model of a securities market. In particular, there exists a unique risk-neutral (martingale) probability measure that we shall denote by  $Q$ .

Consider an agent whose total wealth at time  $t \geq 0$  is denoted by  $x(t)$ . Assume that the trading of shares takes place continuously in a self-financing fashion (i.e., there is no

consumption or income) and there are no transaction costs. Then  $x(\cdot)$  satisfies (see, e.g., Karatzas and Shreve (1998) and Elliott and Kopp (1999) )

$$(2.4) \quad \left\{ \begin{array}{l} dx(t) = \left\{ r(t)x(t) + \sum_{i=1}^m [b_i(t) - r(t)]\pi_i(t) \right\} dt \\ \quad \quad \quad + \sum_{j=1}^m \sum_{i=1}^m \sigma_{ij}(t)\pi_i(t)dW^j(t), \\ x(0) = x_0 \geq 0, \end{array} \right.$$

where  $\pi_i(t)$ ,  $i = 0, 1, 2, \dots, m$ , denotes the total market value of the agent's wealth in the  $i$ -th asset. Hence  $N_i(t) := \pi_i(t)/S_i(t)$  is the number of shares of the  $i$ -th asset held by the agent at times  $t$ . Of course we have that  $\pi_0(t) + \pi_1(t) + \dots + \pi_m(t) = x(t)$ , where  $\pi_0(t)$  is the time- $t$  value of the bank account. We call  $\pi(\cdot) \equiv (\pi_1(\cdot), \dots, \pi_m(\cdot))'$  the *portfolio* of the agent.

Set

$$(2.5) \quad B(t) := (b_1(t) - r(t), \dots, b_m(t) - r(t)),$$

and define the *risk premium process*

$$(2.6) \quad \theta(t) \equiv (\theta_1(t), \dots, \theta_m(t)) := B(t)(\sigma(t)')^{-1}.$$

With this notation, equation (2.4) becomes

$$(2.7) \quad \left\{ \begin{array}{l} dx(t) = [r(t)x(t) + B(t)\pi(t)]dt + \pi(t)'\sigma(t)dW(t), \\ x(0) = x_0. \end{array} \right.$$

We of course allow only for portfolios  $\pi(\cdot)$  for which the wealth equation (2.7) admits a unique, strong solution  $x(\cdot)$ . Observe, however, that *a priori* the wealth process  $x(\cdot)$  that is

the solution to (2.7) might not be a nonnegative process. This is sometimes unacceptable for practical purposes, because normally investors cannot buy assets on margin when their wealth is negative. Thus an important restriction that we shall now make and impose throughout the balance of this paper is the prohibition of bankruptcy of the agent. That is, we shall limit our considerations to investment strategies  $\pi(\cdot)$  for which the corresponding wealth processes are such that  $x(t) \geq 0$ , a.s.,  $\forall t \in [0, T]$ . Observe that in our set-up there is at least one no-bankruptcy policy which is to put all the money in the bank account.

Before we formulate our continuous time mean–variance portfolio selection model, we specify the “allowable” investment policies with

DEFINITION 2.1. A portfolio  $\pi(\cdot)$  is said to be *admissible* if  $\pi(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ .

Observe that by standard SDE theory a unique strong solution exists for the wealth equation (2.7) for any admissible portfolio  $\pi(\cdot)$ . We would like to emphasize an important point concerning the way we specify our trading strategies. Most papers in the research literature define a trading strategy or portfolio, say  $u(\cdot)$ , as the (vector of) proportions or fractions of wealth allocated to different assets, perhaps with some other “technical” constraints such as  $\int_0^T |u(t)|^2 dt < \infty$ , a.s., being specified (see, e.g., Cvitanic and Karatzas (1992) and Karatzas and Shreve (1998)). With this definition, and if additionally the self-financing property is postulated, then the wealth at any time  $t \geq 0$  can be shown to be proportional to the wealth at time  $t = 0$ , in the sense that  $x(t) = x_0 \tilde{x}(t)$ , where  $\tilde{x}(t)$  is an (almost surely) strictly positive process. In fact, with a proportional, self-financing strategy  $u(\cdot)$  satisfying the above condition, it can be shown that the wealth process is a unique

strong solution of the following equation

$$(2.8) \quad \begin{cases} dx(t) = x(t)[r(t) + B(t)u(t)]dt + x(t)u(t)'\sigma(t)dW(t), \\ x(0) = x_0. \end{cases}$$

Thus,  $x(t) = x_0\tilde{x}(t)$ , where

$$\tilde{x}(t) = \exp \left\{ \int_0^t \left( [r(s) + B(s)u(s)]^2 - \frac{1}{2}|u(s)'\sigma(s)|^2 \right) ds + \int_0^t u(s)'\sigma(s)dW(s) \right\}.$$

Consequently, with proportional, self-financing strategies satisfying the above condition, the wealth process is *strictly* positive if the initial wealth  $x_0$  is strictly positive. In fact, in this case the value  $x = 0$  becomes a natural barrier of the wealth process.

However, in our model, with the portfolio defined to be the amounts of money allocated to different assets, the wealth process can take zero or negative values, and we require the nonnegativity of the wealth as an *additional constraint* rather than as a by-product of the “proportions of wealth” approach. Clearly the class of admissible, proportional, self-financing strategies is a proper sub-class of our set of admissible self-financing strategies. In fact, any admissible strategy  $\pi(\cdot)$  which produces a (strictly) positive wealth process  $x(t) > 0$  gives rise to a proportional strategy, defined as  $u(t) := \frac{\pi(t)}{x(t)}$ . On the other hand, any proportional strategy  $u(\cdot)$  gives rise to a “monetary amount” strategy  $\pi(\cdot)$  defined as  $\pi(t) = u(t)x(t)$ . We will see later that our final solutions involve strategies that *cannot* be expressed as proportional ones. Thus our model is *fundamentally different* from approaches based upon (2.8).

Our first result makes the simplifying observation that the wealth process  $x(\cdot)$  is nonnegative if and only if the terminal wealth  $x(T)$  is nonnegative. From the economic standpoint,

this is a consequence of the fact that there exists a risk neutral probability measure under which the *discounted* wealth process is a martingale. Hence if the terminal wealth is non-negative, then so are the discounted wealth process and thus  $x(\cdot)$ . We prove this by taking a mathematical approach, however.

**PROPOSITION 2.1.** *Let  $x(\cdot)$  be a wealth process under an admissible portfolio  $\pi(\cdot)$ . If  $x(T) \geq 0$ , a.s., then  $x(t) \geq 0$ , a.s.  $\forall t \in [0, T]$ .*

*Proof.* Let us fix an admissible portfolio  $\pi(\cdot)$  and let  $x(\cdot)$  be the unique wealth process that solves (2.7), with  $x(T) \geq 0$ , a.s.. Note that  $\xi := x(T)$  is a positive square-integrable  $\mathcal{F}_T$ -random variable; hence  $(x(\cdot), z(\cdot)) := (x(\cdot), \sigma(\cdot)' \pi(\cdot))$  satisfies the following backward stochastic differential equation (BSDE):

$$(2.9) \quad \begin{cases} dx(t) = [r(t)x(t) + \theta(t)z(t)]dt + z(t)'dW(t), \\ x(T) = \xi. \end{cases}$$

Applying Proposition 2.2 (p. 22) in El Karoui, Peng and Quenez (1997), we obtain the following representation

$$(2.10) \quad x(t) = \rho(t)^{-1} E(\rho(T)x(T) | \mathcal{F}_t), \quad \forall t \in [0, T], \quad \text{a.s.},$$

where  $\rho(\cdot)$  satisfies

$$(2.11) \quad \begin{cases} d\rho(t) = \rho(t)[-r(t)dt - \theta(t)dW(t)], \\ \rho(0) = 1, \end{cases}$$

or, equivalently,

$$(2.12) \quad \rho(t) = \exp \left\{ - \int_0^t [r(s) + \frac{1}{2} |\theta(s)|^2] ds - \int_0^t \theta(s) dW(s) \right\}.$$

It follows from (2.10) then that  $x(t) \geq 0$ , a.s.,  $\forall t \in [0, T]$ .  $\square$

Observe that the above process  $\rho(\cdot)$  in (2.12) is nothing else but what financial economists call the *deflator* process. Since for our market there exists a unique equivalent martingale measure  $Q$ , it must satisfy

$$\frac{dQ}{dP} \Big|_{\mathcal{F}_t} = \eta(t), \quad \text{a.s.},$$

where  $\eta(t) := \frac{S_0(t)}{s_0} \rho(t)$ . Thus representation (2.10) can be rewritten as the *risk-neutral* valuation formula

$$x(t) = S_0(t) E_Q [S_0(T)^{-1} x(T) | \mathcal{F}_t], \quad \forall t \in [0, T], \quad \text{a.s.},$$

where we denoted by  $E_Q$  the expectation with respect to the probability  $Q$ .

The importance of Proposition 2.1 is that it enables us to replace the pointwise (in time  $t$ ) constraint  $x(t) \geq 0$  by the terminal constraint  $x(T) \geq 0$ , thereby greatly simplifying our problem, which we formulate as follows.

**DEFINITION 2.2.** Consider the following optimization problem parameterized by  $z \in \mathbb{R}$ :



$$\begin{aligned}
& \text{Minimize} && \text{Var } x(T) \equiv Ex(T)^2 - z^2, \\
(2.13) & && \left\{ \begin{array}{l} Ex(T) = z, \\ x(T) \geq 0, \text{ a.s.}, \\ \pi(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m), \\ (x(\cdot), \pi(\cdot)) \text{ satisfies equation (2.7)}. \end{array} \right. \\
& \text{subject to} &&
\end{aligned}$$

The optimal portfolio for this problem (corresponding to a fixed  $z$ ) is called a *variance minimizing* portfolio, and the set of all points  $(\text{Var } x^*(T), z)$ , where  $\text{Var } x^*(T)$  denotes the optimal value of (2.13) corresponding to  $z$  and  $z$  runs over  $\mathbb{R}$ , is called the *variance minimizing frontier*.

The efficient frontier, to be defined in Section 6, is a portion of the minimizing variance frontier. Once the minimizing variance frontier is identified, the efficient frontier can be easily obtained as an appropriate subset of the former<sup>3</sup>; see Section 6. Hence in this paper we shall focus on problem (2.13).

If the initial wealth  $x_0$  of the agent is zero and if the constraint  $x(T) \geq 0$  is in force, then it follows from (2.10) that  $x(t) \equiv 0$  under all admissible  $\pi(\cdot)$ . On the other hand, if  $z$  is set to be 0, then the constraints of (2.13) yield  $x(T) = 0$ , a.s., which in turn leads to  $x(t) \equiv 0$  by (2.10). Hence to eliminate these trivial cases from consideration we assume from now on

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<sup>3</sup>In some of the literature, problem (2.13) itself is defined as the mean–variance portfolio selection problem, with  $z$  required to be in a certain range.

that

$$(2.14) \quad x_0 > 0, \quad z > 0.$$

To solve problem (2.13) we use an extension of the risk-neutral computational approach that was first introduced by Pliska (1982), Pliska (1986). The idea is to decompose the problem into two sub-problems, the first of which is find the optimal attainable wealth  $X^*$ , that is, the random variable that is the optimal value of all possible  $x(T)$  obtainable by admissible portfolios. The second sub-problem is to find the trading strategy  $\pi(\cdot)$  that replicates  $X^*$ , which is essentially a martingale representation problem.

To be specific, the first sub-problem is

$$(2.15) \quad \begin{array}{l} \text{Minimize} \\ \text{subject to} \end{array} \quad \left\{ \begin{array}{l} EX^2 - z^2, \\ EX = z, \\ E[\rho(T)X] = x_0, \\ X \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}), \quad X \geq 0, \text{ a.s..} \end{array} \right.$$

Assuming a solution  $X^*$  exists for this problem, consider the following terminal-valued equation:

$$(2.16) \quad \left\{ \begin{array}{l} dx(t) = [r(t)x(t) + B(t)\pi(t)]dt + \pi(t)'\sigma(t)dW(t), \\ x(T) = X^*. \end{array} \right.$$

The following result verifies that problems (2.15) and (2.16) indeed lead to a solution of our original problem.

THEOREM 2.1. *If  $(x^*(\cdot), \pi^*(\cdot))$  is optimal for problem (2.13), then  $x^*(T)$  is optimal for problem (2.15) and  $(x^*(\cdot), \pi^*(\cdot))$  satisfies (2.16). Conversely, if  $X^*$  is optimal for problem (2.15), then (2.16) must have a solution  $(x^*(\cdot), \pi^*(\cdot))$  which is an optimal solution for (2.13).*

*Proof.* Suppose  $(x^*(\cdot), \pi^*(\cdot))$  is optimal for problem (2.13). First of all, by virtue of (2.10) we have  $E[\rho(T)x^*(T)] = x_0$ . Hence  $x^*(T)$  is feasible for problem (2.15). Assume there is another feasible solution, denoted by  $Y$ , of (2.15) with

$$(2.17) \quad EY^2 < Ex^*(T)^2.$$

The following linear BSDE

$$(2.18) \quad \begin{cases} dx(t) = [r(t)x(t) + \theta(t)z(t)]dt + z(t)'dW(t) \\ x(T) = Y \end{cases}$$

admits a unique square-integrable,  $\mathcal{F}_t$ -adapted solution  $(x(\cdot), z(\cdot))$  since the coefficients of (2.18) are uniformly bounded due to the underlying assumptions. Write  $\pi(t) = (\sigma(t)')^{-1}z(t)$ , which is square integrable due to the uniform boundedness of  $(\sigma(t)')^{-1}$ . Hence  $\pi(\cdot)$  is an admissible portfolio, and  $(x(\cdot), \pi(\cdot))$  satisfies the same dynamics of (2.7). Moreover, it follows from (2.10) that

$$x(0) = E[\rho(T)Y] = x_0,$$

where the second equality is due to the feasibility of  $Y$  to (2.15). This implies  $(x(\cdot), \pi(\cdot))$  is a feasible solution to (2.13). However, (2.17) yields  $Ex(T)^2 = EY^2 < Ex^*(T)^2$ , contradicting the optimality of  $(x^*(\cdot), \pi^*(\cdot))$ .

Conversely, let  $X^*$  be optimal for problem (2.15). Then by a similar argument to that above, and using the BSDE (2.18) with terminal condition  $x(T) = X^*$ , one sees that one

can construct a feasible solution  $(x^*(\cdot), \pi^*(\cdot))$  to (2.13). Moreover, if there is another feasible solution  $(x(\cdot), \pi(\cdot))$  to (2.13) that is better than  $(x^*(\cdot), \pi^*(\cdot))$ , then  $x(T)$  would be better than  $X^*$  for problem (2.15), leading to a contradiction.  $\square$

REMARK 2.1. By virtue of the above theorem, solving the variance minimizing problem boils down to solving the optimization problem (2.15). Once (2.15) is solved, the solution to (2.16) can be obtained via standard BSDE theory.  $\square$

### 3 FEASIBILITY

Since problem (2.13) involves several constraints, the first issue is its feasibility, which is the subject of this section.

PROPOSITION 3.1. *Problem (2.13) either has no feasible solution or it admits a unique optimal solution.*

*Proof.* In view of Remark 2.1 it suffices to investigate the feasibility of (2.15). Now (2.15) can be regarded as an optimization problem on the Hilbert space  $L^2_{\mathcal{F}_T}(\Omega; R)$ , with the constraint set

$$D := \{Y \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}) : EY = z, E[\rho(T)Y] = x_0, Y \geq 0\}.$$

If  $D$  is nonempty, say with  $Y_0 \in D$ , then an optimal solution of (2.15), if any, must be in the set  $D' := D \cap \{EY^2 \leq EY_0^2\}$ . In this case, clearly  $D'$  is a nonempty, bounded, closed convex set in  $L^2_{\mathcal{F}_T}(\Omega; \mathbb{R})$ . Moreover, the cost functional of (2.15) is strictly convex on  $D'$  with a lower bound  $-z^2$ . Hence (2.15) must admit a unique optimal solution.  $\square$

Define

$$(3.1) \quad a := \inf_{Y \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}), Y \geq 0, EY > 0} \frac{E[\rho(T)Y]}{EY},$$

$$b := \sup_{Y \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}), Y \geq 0, EY > 0} \frac{E[\rho(T)Y]}{EY}.$$

As will be evident from the sequel, the values  $a$  and  $b$  are critical. The following representations of  $a$  and  $b$  are useful.

PROPOSITION 3.2. *We have the following representation*

$$(3.2) \quad a = \inf\{\eta \in \mathbb{R} : P(\rho(T) < \eta) > 0\},$$

$$b = \sup\{\eta \in \mathbb{R} : P(\rho(T) > \eta) > 0\}.$$

*Proof.* Denote  $\bar{a} := \inf\{\eta \in \mathbb{R} : P(\rho(T) < \eta) > 0\}$ . For any  $\eta$  satisfying  $P(\rho(T) < \eta) > 0$ , take  $Y := \mathbf{1}_{\rho(T) < \eta}$ . Then

$$Y \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}), \quad Y \geq 0, \quad EY > 0, \quad \text{and} \quad \frac{E[\rho(T)Y]}{EY} < \eta.$$

As a result, by the definition of  $a$ , we have  $a \leq \frac{E[\rho(T)Y]}{EY} < \eta$ . Hence  $a \leq \bar{a}$ . Conversely, by the definition of  $\bar{a}$  we must have  $P(\rho(T) < \bar{a} - \epsilon) = 0$  for any  $\epsilon > 0$ , namely,  $\rho(T) \geq \bar{a} - \epsilon$ , a.s.. Hence for any  $Y \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R})$  with  $Y \geq 0, EY > 0$ , we have  $\frac{E[\rho(T)Y]}{EY} \geq \bar{a} - \epsilon$ . This implies  $a \geq \bar{a} - \epsilon$  for any  $\epsilon > 0$ ; thus  $a \geq \bar{a}$ .

We have now proved the first equality of (3.2). The second one can be proved in a similar fashion.  $\square$

REMARK 3.1. When the risk premium process  $\theta(\cdot)$  is deterministic, and when  $\int_0^T |\theta(t)|^2 dt > 0$ , the exponent in (2.12) at  $t = T$  is the sum of a bounded random variable and a normal random variable with a strictly positive variance; hence  $a = 0, b = +\infty$  by

Proposition 3.2. But when  $\theta(\cdot)$  is a stochastic process, both  $a > 0$  and  $b < +\infty$  are possible even if  $\int_0^T |\theta(t)|^2 dt > 0$ , a.s.. To show this, by (2.12) it suffices to construct an example where  $\int_0^T \theta(t) dW(t)$  is uniformly bounded. Indeed, consider a market with one bank account and one stock with the corresponding one-dimensional standard Brownian Motion  $W(t)$ . For a given real number  $K > 0$ , define

$$(3.3) \quad \tau := \begin{cases} \inf\{t \geq 0 : |W(t)| > K\}, & \text{if } \sup_{0 \leq t \leq T} |W(t)| > K, \\ T, & \text{if } \sup_{0 \leq t \leq T} |W(t)| \leq K. \end{cases}$$

Take  $r(t) = 0.1$ ,  $b(t) = 0.1 + \mathbf{1}_{t \leq \tau}$  and  $\sigma(t) = 1$ . Thus  $\theta(t) = \mathbf{1}_{t \leq \tau}$ . Then  $\int_0^T \theta(t) dW(t) = W(\tau)$ , which is uniformly bounded by  $K$ .  $\square$

The next result is very important, for it specifies an interval such that our problem (2.13) has a solution if and only if the desired expected wealth  $z$  takes a value in this interval.

PROPOSITION 3.3. *If  $a < \frac{x_0}{z} < b$ , then there must be a feasible solution to (2.13).*

*Conversely, if (2.13) has a feasible solution, then it must be that  $a \leq \frac{x_0}{z} \leq b$ .*

*Proof.* Assume  $a < \frac{x_0}{z} < b$ . Again we only need to show the feasibility of the problem (2.15). By the definition of  $a$  and  $b$ , for any  $x_0 > 0$  and  $z > 0$  with  $a < \frac{x_0}{z} < b$  there exist  $Y_1, Y_2 \in \{Y \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}) : Y \geq 0, EY > 0\}$  such that

$$\frac{E[\rho(T)Y_1]}{EY_1} < \frac{x_0}{z} < \frac{E[\rho(T)Y_2]}{EY_2}.$$

Define a function

$$f(\lambda) := \frac{E[\rho(T)(\lambda Y_1 + (1 - \lambda)Y_2)]}{E[\lambda Y_1 + (1 - \lambda)Y_2]} = \frac{\lambda E[\rho(T)Y_1] + (1 - \lambda)E[\rho(T)Y_2]}{\lambda EY_1 + (1 - \lambda)EY_2}, \quad \lambda \in [0, 1].$$

Then  $f$  is continuous on  $[0, 1]$  with  $f(1) < \frac{x_0}{z} < f(0)$ , so there exists a  $\lambda_0 \in (0, 1)$  such that  $\frac{x_0}{z} = f(\lambda_0) = \frac{E[\rho(T)(\lambda_0 Y_1 + (1-\lambda_0)Y_2)]}{E[\lambda_0 Y_1 + (1-\lambda_0)Y_2]}$ . Set  $Y_0 := \lambda_0 Y_1 + (1 - \lambda_0)Y_2$  and  $Y^* := zY_0/E[Y_0]$ . Then clearly  $Y^* \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R})$ ,  $Y^* \geq 0$ ,  $E(Y^*) = z$ , and

$$E[\rho(T)Y^*] = zf(\lambda_0) = x_0.$$

This shows that  $Y^*$  is a feasible solution of (2.15).

Conversely, if there is a feasible solution of (2.13), then (2.15) also has a feasible solution, say  $Y^*$ . Hence  $Y^* \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R})$ ,  $Y^* \geq 0$ , and  $E[Y^*] = z$ . Thus,

$$\frac{x_0}{z} = \frac{E[\rho(T)Y^*]}{EY^*} \geq a.$$

Similarly,  $\frac{x_0}{z} \leq b$ .  $\square$

One naturally wonders what can be said about the feasibility of (2.13) when  $\frac{x_0}{z} = a$  or  $b$ . The answer is that at these “boundary” points, (2.13) may *or* may not be feasible, as can be seen from the following example.

**EXAMPLE 3.1.** First consider the process  $\theta(\cdot)$  as given in Remark 3.1, namely  $\theta(t) = \mathbf{1}_{t \leq \tau}$ , where  $\tau$  is defined by (3.3) for a one-dimensional standard Brownian motion  $W(t)$  and a given real number  $K > 0$ . Let  $r(t) := -\frac{|\theta(t)|^2}{2}$ . Then it follows from (2.12) that  $\rho(T) = e^{-\int_0^T \theta(t)dW(t)} = e^{-W(\tau)}$ . Now

$$a = \inf\{\eta \in \mathbb{R} : P(\rho(T) < \eta) > 0\} = e^{-K},$$

whereas

$$P(\rho(T) = a) = P(W(\tau) = K) = 1 - P\left(\sup_{0 \leq t \leq T} |W(t)| < K\right) > 0.$$

Take  $Y := \mathbf{1}_{\rho(T)=a}$ . Then  $Y \geq 0$ ,  $EY > 0$  and  $E[\rho(T)Y] = aP(\rho(T) = a)$ . Hence with  $x_0 := aP(\rho(T) = a) > 0$  and  $z := EY = P(\rho(T) = a) > 0$ , we have  $\frac{x_0}{z} = a$  while  $Y$  is a feasible solution to (2.15).

Next, let  $\theta(\cdot)$  be the same as above, and  $r(t) = -\frac{|\theta(t)|^2}{2} + \arctan(W(t))$ . (Recall that the range of  $\arctan(\cdot)$  is  $(-\frac{\pi}{2}, \frac{\pi}{2})$ .) Then  $\rho(T) = e^{-W(\tau) + \arctan(W(T))}$ ,  $a = \inf\{\eta \in \mathbb{R} : P(\rho(T) < \eta) > 0\} = e^{-\frac{\pi}{2} - K}$ , and

$$(3.4) \quad P(\rho(T) > a) \geq P(W(\tau) \leq K) = 1.$$

If there is a feasible solution  $Y$  to (2.15) for certain  $x_0 > 0$  and  $z > 0$  with  $\frac{x_0}{z} = a$ , or  $\frac{E[\rho(T)Y]}{EY} = a$ , then

$$E[(\rho(T) - a)Y] = 0,$$

implying  $Y = 0$  a.s. in view of (3.4). Thus,  $EY = 0$  leading to a contradiction. So (2.15) has no feasible solution when  $\frac{x_0}{z} = a$ .  $\square$

We summarize most of the results in this section as follows:

**THEOREM 3.1.** *If  $a < \frac{x_0}{z} < b$ , then the minimizing variance problem (2.13) is feasible and must admit a unique optimal solution. In particular, if the process  $\theta(\cdot)$  is deterministic with  $\int_0^T |\theta(t)|^2 dt > 0$ , then (2.13) must have a unique optimal solution for any  $x_0 > 0, z > 0$ .*

## 4 SOLUTION TO (2.15): THE OPTIMAL ATTAINABLE WEALTH

In this section we present the complete solution to the auxiliary problem (2.15). First a preliminary result involving Lagrange multipliers follows.



PROPOSITION 4.1. Let  $D \subset L^2_{\mathcal{F}_T}(\Omega; \mathbb{R})$  be a convex set,  $a_i \in \mathbb{R}$ , and  $\xi_i \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R})$ ,  $i = 1, 2, \dots, l$ , be given, and let  $f$  be a scalar-valued convex function on  $\mathbb{R}$ . If the problem

$$(4.1) \quad \begin{array}{ll} \text{minimize} & Ef(Y), \\ \text{subject to} & \left\{ \begin{array}{l} E[\xi_i Y] = a_i, \quad i = 1, 2, \dots, l, \\ Y \in D \end{array} \right. \end{array}$$

has a solution  $Y^*$ , then there exists an  $l$ -dimensional deterministic vector  $(\lambda_1, \dots, \lambda_l)$  such that  $Y^*$  also solves the following

$$(4.2) \quad \begin{array}{ll} \text{minimize} & E[f(Y) - Y \sum_{i=1}^l \lambda_i \xi_i], \\ \text{subject to} & Y \in D. \end{array}$$

Conversely, if  $Y^*$  solves (4.2) for some  $(\lambda_1, \dots, \lambda_l)$ , then it must also solve (4.1) with  $a_i = E[\xi_i Y^*]$ .

*Proof.* Let  $Y^*$  solve (4.1). Define a set  $\Delta := \{(E[\xi_1 Y], \dots, E[\xi_l Y]) : Y \in D\} \subseteq \mathbb{R}^l$ , which is clearly a convex set, and a function

$$g(x) \equiv g(x_1, \dots, x_l) := \inf_{E[\xi_i Y] = x_i, i=1, \dots, l, Y \in D} E[f(Y)], \quad x \in \Delta.$$

In view of the assumptions,  $g$  is a convex function on  $\Delta$ . By the convex separation theorem, for the given  $a = (a_1, \dots, a_l)'$ , there exists an  $l$ -dimensional vector  $\lambda = (\lambda_1, \dots, \lambda_l)'$  such that  $g(x) \geq g(a) + \lambda'(x - a)$ ,  $\forall x \in \Delta$ . Equivalently,  $g(x) - \lambda'x \geq g(a) - \lambda'a$ . Now, for any

$Y \in D$ ,

$$\begin{aligned}
E[f(Y) - Y \sum_{i=1}^l \lambda_i \xi_i] &\geq g(E[\xi_1 Y], \dots, E[\xi_l Y]) - \sum_{i=1}^l \lambda_i E[\xi_i Y] \\
&\geq g(a) - \lambda' a \\
&= E[f(Y^*) - Y^* \sum_{i=1}^l \lambda_i \xi_i],
\end{aligned}$$

implying that  $Y^*$  solves (4.2).

Conversely, if  $Y^*$  solve (4.2), then for any  $Y \in D$  satisfying  $E[\xi_i Y] = E[\xi_i Y^*]$ , we have

$$E[f(Y^*) - Y^* \sum_{i=1}^l \lambda_i \xi_i] \leq E[f(Y) - Y \sum_{i=1}^l \lambda_i \xi_i] = E[f(Y) - Y^* \sum_{i=1}^l \lambda_i \xi_i].$$

Hence  $E[f(Y^*)] \leq E[f(Y)]$ , thereby proving the desired result.  $\square$

We now solve problem (2.15) by using Proposition 4.1 to transform it to an equivalent problem that has two Lagrange multipliers and only one constraint:  $X \geq 0$ .

**THEOREM 4.1.** *If problem (2.15) admits a solution  $X^*$ , then  $X^* = (\lambda - \mu\rho(T))^+$ , where the pair of scalars  $(\lambda, \mu)$  solves the system of equations*

$$(4.3) \quad \begin{cases} E[(\lambda - \mu\rho(T))^+] = z, \\ E[\rho(T)(\lambda - \mu\rho(T))^+] = x_0. \end{cases}$$

*Conversely, if  $(\lambda, \mu)$  satisfies (4.3), then  $X^* := (\lambda - \mu\rho(T))^+$  must be an optimal solution of (2.15).*

*Proof.* If  $X^*$  solves problem (2.15), then by Proposition 4.1 there exists a pair of constants

$(2\lambda, -2\mu)$  such that  $X^*$  also solves

$$(4.4) \quad \begin{aligned} & \text{minimize} && E[X^2 - 2\lambda X + 2\mu\rho(T)X] - z^2, \\ & \text{subject to} && X \geq 0, \text{ a.s.} \end{aligned}$$

However, the objective function of (4.4) equals

$$E[X - (\lambda - \mu\rho(T))]^2 - z^2 - E[\lambda - \mu\rho(T)]^2.$$

Hence problem (4.4) has an obvious unique solution  $(\lambda - \mu\rho(T))^+$  which must then coincide with  $X^*$ . In this case, the two equations in (4.3) are nothing else than the two equality constraints in problem (2.15).

The converse result of the theorem can be proved similarly in view of Proposition 4.1.  $\square$

Observe that if the non-negativity constraint  $X \geq 0$  is removed from problem (2.15), then the optimal solution to such a relaxed problem is simply  $X^* = \lambda - \mu\rho(T)$ , with the constants  $\lambda$  and  $\mu$  satisfying

$$(4.5) \quad \begin{cases} E[\lambda - \mu\rho(T)] = z, \\ E[\rho(T)(\lambda - \mu\rho(T))] = x_0. \end{cases}$$

Since these equations are linear, the solution is immediate:

$$\lambda = \frac{zE[\rho(T)^2] - x_0E[\rho(T)]}{\text{Var } \rho(T)}, \quad \mu = \frac{zE[\rho(T)] - x_0}{\text{Var } \rho(T)}.$$

But for problem (2.15) the existence and uniqueness of Lagrange multipliers  $\lambda$  and  $\mu$  satisfying (4.3) is a more delicate issue, which we discuss in the following section.

## 5 EXISTENCE AND UNIQUENESS OF LAGRANGE MULTIPLIERS

By virtue of Theorem 4.1 an optimal solution to (2.15) is obtained explicitly if the system of equations (4.3) for Lagrange multipliers admits solutions. In this section we study the unique solvability of (4.3). For notational simplicity we rewrite (4.3) as

$$(5.1) \quad \begin{cases} E[(\lambda - \mu Z)^+] = z, \\ E[(\lambda - \mu Z)^+ Z] = x_0, \end{cases}$$

where  $Z := \rho(T)$ . First we have three preliminary lemmas.

LEMMA 5.1. *For any random variable  $X$  and real number  $c$ ,*

$$E[X(c - X)] - E[X]E[c - X] \leq 0, \quad E[X(X - c)] - E[X]E[X - c] \geq 0.$$

*Proof.* We have

$$E[X(c - X)] - E[X]E[c - X] = -E[X^2] + (EX)^2 \leq 0,$$

$$E[X(X - c)] - E[X]E[X - c] = E[X^2] - (EX)^2 \geq 0.$$

□

LEMMA 5.2. *The function  $R_1(\eta) := \frac{E[(\eta - Z)^+ Z]}{E[(\eta - Z)^+]}$  is continuous and strictly increasing for  $\eta \in (a, +\infty)$ , and the function  $R_2(\eta) := \frac{E[(Z - \eta)^+ Z]}{E[(Z - \eta)^+]}$  is continuous and strictly decreasing for  $\eta \in (-\infty, b)$ , where  $a$  and  $b$  are given in (3.2).*

*Proof.* Let us first observe that in view of characterization (3.2) we have that  $P(Z < \eta) > 0$  for any  $\eta > a$ , and that  $P(Z > \eta) > 0$  for any  $\eta < b$ . Consequently,  $P((\eta - Z)^+ > 0) > 0$  for any  $\eta > a$  and  $P((Z - \eta)^+ > 0) > 0$  for any  $\eta < b$ . Thus the following inequalities are satisfied:  $E[(\eta - Z)^+] > 0$  for  $\eta > a$ , and  $E[(Z - \eta)^+] > 0$  for  $\eta < b$ . This verifies continuity of both functions.

To prove the strict monotonicity of  $R_1(\cdot)$ , take any  $\eta_1 > \eta_2 > a$ . Then we have

$$\begin{aligned}
(5.2) \quad \frac{E[(\eta_2 - Z)^+ Z]}{E[(\eta_2 - Z)^+]} &= \frac{E[(\eta_2 - Z)Z | Z < \eta_2]}{E[(\eta_2 - Z) | Z < \eta_2]} \\
&\leq E(Z | Z < \eta_2) \quad (\text{by Lemma 5.1}) \\
&= \frac{E[Z \mathbf{1}_{Z < \eta_2}]}{E[\mathbf{1}_{Z < \eta_2}]} \\
&< \eta_2 \\
&\leq \frac{E[(\eta_1 - Z)Z \mathbf{1}_{\eta_2 \leq Z \leq \eta_1}]}{E[(\eta_1 - Z) \mathbf{1}_{\eta_2 \leq Z \leq \eta_1}]}.
\end{aligned}$$

Note that in particular the above inequalities imply that

$$(5.3) \quad \frac{E[(\eta_1 - \eta_2)Z \mathbf{1}_{Z < \eta_2}]}{E[(\eta_1 - \eta_2) \mathbf{1}_{Z < \eta_2}]} = \frac{E[Z \mathbf{1}_{Z < \eta_2}]}{E[\mathbf{1}_{Z < \eta_2}]} < \frac{E[(\eta_1 - Z)Z \mathbf{1}_{\eta_2 \leq Z \leq \eta_1}]}{E[(\eta_1 - Z) \mathbf{1}_{\eta_2 \leq Z \leq \eta_1}]}.$$

On the other hand,

$$\begin{aligned}
(5.4) \quad \frac{E\{[(\eta_1 - Z)^+ - (\eta_2 - Z)^+] Z\}}{E\{[(\eta_1 - Z)^+ - (\eta_2 - Z)^+]\}} &= \frac{E[(\eta_1 - \eta_2)Z \mathbf{1}_{Z < \eta_2}] + E[(\eta_1 - Z)Z \mathbf{1}_{\eta_2 \leq Z \leq \eta_1}]}{E[(\eta_1 - \eta_2) \mathbf{1}_{Z < \eta_2}] + E[(\eta_1 - Z) \mathbf{1}_{\eta_2 \leq Z \leq \eta_1}]} \\
&> \frac{E[(\eta_1 - \eta_2)Z \mathbf{1}_{Z < \eta_2}]}{E[(\eta_1 - \eta_2) \mathbf{1}_{Z < \eta_2}]} \\
&\geq \frac{E[(\eta_2 - Z)^+ Z]}{E[(\eta_2 - Z)^+]},
\end{aligned}$$

where the first inequality is due to (5.3) and the familiar inequality

$$(5.5) \quad \frac{x_1 + x_2}{y_1 + y_2} > \frac{x_1}{y_1} \quad \text{if} \quad \frac{x_2}{y_2} > \frac{x_1}{y_1} \quad \text{and} \quad y_1, y_2 > 0,$$

and the last inequality follows from (5.2). Finally,

$$(5.6) \quad \begin{aligned} \frac{E[(\eta_1 - Z)^+ Z]}{E[(\eta_1 - Z)^+]} &= \frac{E[(\eta_2 - Z)^+ Z] + E\{[(\eta_1 - Z)^+ - (\eta_2 - Z)^+] Z\}}{E[(\eta_2 - Z)^+] + E[(\eta_1 - Z)^+ - (\eta_2 - Z)^+]} \\ &> \frac{E[(\eta_2 - Z)^+ Z]}{E[(\eta_2 - Z)^+]}, \end{aligned}$$

owing to (5.4) and inequality (5.5). This shows that  $R_1(\cdot)$  is strictly increasing. Similarly, we can prove that  $R_2(\cdot)$  is strictly decreasing.  $\square$

LEMMA 5.3. *We have the following interval representations of the respective sets:*

$$(5.7) \quad \{R_1(\eta) : \eta > a\} = (a, E[Z]),$$

$$(5.8) \quad \{R_2(\eta) : \eta < 0\} = (E[Z], \frac{E[Z^2]}{E[Z]}),$$

$$(5.9) \quad \{R_2(\eta) : 0 \leq \eta < b\} = [\frac{E[Z^2]}{E[Z]}, b).$$

*Proof.* By the definition of  $a$  we have  $P(Z < a) = 0$ . In other words  $Z \geq a$ , a.s. Hence

$$(5.10) \quad R_1(\eta) = \frac{E[(\eta - Z)^+ Z]}{E[(\eta - Z)^+]} \geq a, \quad \forall \eta > a.$$

Meanwhile,

$$E[(\eta - Z)^+ Z] \leq E[(\eta - Z)^+ \eta] = \eta E[(\eta - Z)^+],$$

leading to

$$(5.11) \quad R_1(\eta) \leq \eta, \quad \forall \eta > a.$$

Combining (5.10) and (5.11) we conclude

$$(5.12) \quad \lim_{\eta \rightarrow a^+} R_1(\eta) = a.$$

On the other hand,

$$(5.13) \quad \begin{aligned} \lim_{\eta \rightarrow +\infty} R_1(\eta) &= \lim_{\eta \rightarrow +\infty} \frac{E[(\eta - Z)^+ Z]}{E[(\eta - Z)^+]} \\ &= \lim_{\eta \rightarrow +\infty} \frac{E[(1 - Z/\eta)^+ Z]}{E[(1 - Z/\eta)^+]} \\ &= E[Z]. \end{aligned}$$

Hence, (5.7) follows from the fact that  $R_1(\eta)$  is continuous and strictly increasing.

Next, observe that since  $Z$  is almost surely positive, then for every  $\eta \leq 0$  we have that  $E[(Z - \eta)^+ Z] = E[(Z - \eta)Z]$  and  $E[(Z - \eta)^+] = E(Z - \eta)$ . Consequently, we obtain that

$$\lim_{\eta \rightarrow -\infty} R_2(\eta) = \lim_{\eta \rightarrow -\infty} \frac{E[(Z - \eta)^+ Z]}{E[(Z - \eta)^+]} = \lim_{\eta \rightarrow -\infty} \frac{E[Z^2] - \eta E[Z]}{E[Z] - \eta} = E[Z],$$

and

$$R_2(0) = \frac{E[Z^2]}{E[Z]}.$$

The above as well as the strict monotonicity of  $R_2(\cdot)$  imply (5.8). Finally, an argument analogous to the one that lead to (5.12) yields

$$\lim_{\eta \rightarrow b^-} R_2(\eta) = b,$$

and this implies (5.9).  $\square$

Now we are in a position to present our main results on the unique solvability of equations (5.1). In particular, we characterize the signs of the two Lagrange multipliers.

THEOREM 5.1. Equations (5.1) have a unique solution  $(\lambda, \mu)$  for any  $x_0 > 0, z > 0$

satisfying  $a < \frac{x_0}{z} < b$ . Moreover,

- (1)  $\lambda = z, \mu = 0$  if  $\frac{x_0}{z} = E[Z]$ ;
- (2)  $\lambda > 0, \mu > 0$  if  $a < \frac{x_0}{z} < E[Z]$ ;
- (3)  $\lambda \leq 0, \mu < 0$  if  $\frac{E[Z^2]}{E[Z]} \leq \frac{x_0}{z} < b$ ;
- (4)  $\lambda > 0, \mu < 0$  if  $E[Z] < \frac{x_0}{z} < \frac{E[Z^2]}{E[Z]}$ .

*Proof.* First off, if  $EZ^2 = (EZ)^2$ , then the variance of  $Z$  is zero or  $Z$  is a deterministic constant almost surely. Hence  $a = b$  by (3.1), which violates the assumption of the theorem. Consequently,  $E[Z^2] > (E[Z])^2$ . On the other hand, again by (3.1) we have immediately (by letting  $Y = 1$  and  $Y = Z$  in  $\frac{E[Z^2Y]}{E[Y]}$ , respectively)

$$a \leq E[Z] < \frac{E[Z^2]}{E[Z]} \leq b,$$

where it is important to note the strict inequality above.

We now examine the four cases. Case (1) is easy, for when  $\frac{x_0}{z} = E[Z]$ , one directly verifies that  $\lambda = z, \mu = 0$  solve (5.1).

For the other three cases we must have  $\mu^* \neq 0$  for any solution  $(\lambda^*, \mu^*)$  of (5.1), for otherwise in view of (5.1) we have  $\lambda^* = z$  and  $\lambda^*E[Z] = x_0$  leading to  $\frac{x_0}{z} = E[Z]$  which is Case (1).

Next, observe that if  $\mu^* > 0$ , then  $(\eta, \mu) := (\frac{\lambda^*}{\mu^*}, \mu^*)$  is a solution of the following equations

$$(5.14) \quad \begin{cases} \frac{E[(\eta - Z)^+ Z]}{E[(\eta - Z)^+]} = \frac{x_0}{z}, \\ E[(\eta - Z)^+] = \frac{z}{\mu}. \end{cases}$$



Likewise, if  $\mu^* < 0$ , then  $(\eta, \mu) := (\frac{\lambda^*}{\mu^*}, \mu^*)$  is a solution of the following equations

$$(5.15) \quad \begin{cases} \frac{E[(Z-\eta)^+Z]}{E[(Z-\eta)^+]} = \frac{x_0}{z} \\ E[(Z-\eta)^+] = -\frac{z}{\mu}. \end{cases}$$

Now for case (2) where  $a < \frac{x_0}{z} < E[Z]$  it follows from Lemma 5.3 that the first equation of (5.14) admits a unique solution  $\eta^* > a \geq 0$  and (5.15) admits no solution. Set

$$\mu^* := \frac{z}{E[(\eta^* - Z)^+]} > 0, \quad \lambda^* := \eta^* \mu^* > 0.$$

Then  $(\lambda^*, \mu^*)$  is the unique solution for (5.1).

If  $\frac{E[Z^2]}{E[Z]} \leq \frac{x_0}{z} < b$ , which is case (3), then by Lemma 5.3 the first equation of (5.15) admits a unique solution  $\eta^* \geq 0$  and (5.14) admits no solution. Set

$$\mu^* := -\frac{z}{E[(Z - \eta^*)^+]} < 0, \quad \lambda^* := \eta^* \mu^* \leq 0.$$

Then  $(\lambda^*, \mu^*)$  is the unique solution for (5.1).

Finally, in case (4) where  $E[Z] < \frac{x_0}{z} < \frac{E[Z^2]}{E[Z]}$ , Lemma 5.3 yields that the first equation of (5.15) admits a unique solution  $\eta^* < 0$  and (5.14) admits no solution. Letting

$$\mu^* := -\frac{z}{E[(Z - \eta^*)^+]} < 0, \quad \lambda^* := \eta^* \mu^* > 0,$$

we get that  $(\lambda^*, \mu^*)$  uniquely solves (5.1).  $\square$

Observe that the Lagrange multipliers have a homogeneous property, for if one denotes by  $(\lambda(x_0, z), \mu(x_0, z))$  the solution to (5.1) when taking  $x_0 > 0$  and  $z > 0$  as parameters, then clearly

$$\lambda(x_0, z) = x_0 \lambda(1, \frac{z}{x_0}), \quad \mu(x_0, z) = x_0 \mu(1, \frac{z}{x_0}).$$

In other words, the solution really depends only on the ratio  $z/x_0$ , which is essentially the expected return desired by the investor.

## 6 EFFICIENT PORTFOLIOS AND EFFICIENT FRONTIER

In this section we derive the efficient portfolios and efficient frontier of our mean–variance portfolio selection problem based on the variance minimizing portfolios and variance minimizing frontier. We fix the initial capital level  $x_0 > 0$  for the rest of this section.

First we give the following definition, following p.6 in Markowitz (1987).

DEFINITION 6.1. *The mean–variance portfolio selection problem with bankruptcy prohibition is formulated as the following multi-objective optimization problem*

$$(6.1) \quad \begin{array}{l} \text{Minimize} \\ \text{subject to} \end{array} \left\{ \begin{array}{l} (J_1(\pi(\cdot)), J_2(\pi(\cdot))) := (\text{Var } x(T), -Ex(T)), \\ x(T) \geq 0, \text{ a.s.}, \\ \pi(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m), \\ (x(\cdot), \pi(\cdot)) \text{ satisfies equation (2.7)}. \end{array} \right.$$

An admissible portfolio  $\pi^*(\cdot)$  is called an *efficient portfolio* if there exists no admissible portfolio  $\pi(\cdot)$  satisfying (6.1) such that

$$(6.2) \quad J_1(\pi(\cdot)) \leq J_1(\pi^*(\cdot)), \quad J_2(\pi(\cdot)) \leq J_2(\pi^*(\cdot)),$$

with at least one of the inequalities holds strictly. In this case, we call  $(J_1(\pi^*(\cdot)), -J_2(\pi^*(\cdot))) \in$

$\mathbb{R}^2$  an *efficient point*. The set of all efficient points is called the *efficient frontier*.

In words, an efficient portfolio is one for which there does not exist another portfolio that has higher mean and no higher variance, and/or has less variance and no less mean at the terminal time  $T$ . In other words, an efficient portfolio is one that is *Pareto optimal*. The problem then is to identify *all* the efficient portfolios along with the efficient frontier.

By their very definitions the efficient frontier is a subset of the variance minimizing frontier, and efficient portfolios must be variance minimizing portfolios. In fact, an alternative definition of an efficient portfolio is the following. A variance minimizing portfolio  $\tilde{\pi}_z$  corresponding to the terminal expected wealth  $z$  is called efficient if it is also *mean maximizing* in the following sense:  $Ex^\pi(T) \leq Ex^{\tilde{\pi}_z}(T)$  for all portfolios  $\pi$  that satisfy the conditions

$$(6.3) \quad \left\{ \begin{array}{l} \pi(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m), \\ (x^\pi(\cdot), \pi(\cdot)) \text{ satisfies equation (2.7)}, \\ x^\pi(T) \geq 0, \text{ a.s.}, \\ \text{Var } x^\pi(T) = \text{Var } x^{\tilde{\pi}_z}(T) , \end{array} \right.$$

where  $x^\pi(\cdot)$  denotes the wealth process under a portfolio  $\pi(\cdot)$  and with the initial wealth  $x_0$ .

The preceding discussion shows that our first task is to obtain variance minimizing portfolios, namely, the optimal trading strategies for problem (2.13).

**THEOREM 6.1.** *The unique variance minimizing portfolio for (2.13) corresponding to*

$z > 0$ , where  $a < \frac{x_0}{z} < b$ , is given by

$$(6.4) \quad \pi^*(t) = (\sigma(t)')^{-1} z^*(t),$$

where  $(x^*(\cdot), z^*(\cdot))$  is the unique solution to the BSDE

$$(6.5) \quad \begin{cases} dx(t) = [r(t)x(t) + \theta(t)z(t)]dt + z(t)'dW(t) \\ x(T) = (\lambda - \mu\rho(T))^+, \end{cases}$$

with  $(\lambda, \mu)$  being the solution to (4.3).

*Proof.* Since  $\rho(\cdot)$  is the solution to (2.11),  $\rho(T) \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R})$ . Meanwhile by Theorem 5.1 equation (4.3) admits a unique solution  $(\lambda, \mu)$ . By standard linear BSDE theory, (6.5) has a unique solution  $(x^*(\cdot), z^*(\cdot)) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ . Thus, the portfolio defined by (6.4) must be admissible. Now, the pair  $(x^*(\cdot), \pi^*(\cdot))$  satisfies (2.16) with  $X^* = (\lambda - \mu\rho(T))^+$ , the latter being the optimal solution of (2.15) by virtue of Theorem 4.1. Thus, Theorem 2.1 stipulates that  $\pi^*(\cdot)$  must be optimal for (2.13).  $\square$

Theorem 6.1 asserts that a variance minimizing portfolio is the one that replicates the time- $T$  payoff of the contingent claim  $(\lambda - \mu\rho(T))^+$ . Note that computing solutions of BSDE's like (6.5) is reasonably standard; see, for example, Ma, Protter, and Yong (1994) or Ma and Yong (1999). In particular, if the market coefficients are deterministic, then it is possible to solve (6.5) explicitly via some partial differential equations; see Section 7 for details.

Our next result pinpoints the value of  $z$  corresponding to the riskless investment in our economy.

**THEOREM 6.2.** *The variance minimizing portfolio corresponding to  $z = \frac{x_0}{E[\rho(T)]}$  is a risk-free portfolio.*

*Proof.* By Theorem 5.1,  $\lambda = z$  and  $\mu = 0$  when  $\frac{x_0}{z} = E[\rho(T)]$ . The terminal wealth under the corresponding variance minimizing portfolio, say  $\pi_0(\cdot)$ , is therefore  $x_0(T) = (\lambda - \mu\rho(T))^+ = \lambda = z$ . Hence this portfolio is risk-free.  $\square$

In view of Theorem 6.2, the risk-free portfolio  $\pi_0(\cdot)$  exists even when all the market parameters are random. Under  $\pi_0(\cdot)$  a terminal payoff  $\frac{x_0}{E[\rho(T)]}$  is guaranteed. Hence  $E[\rho(T)]$  can be regarded as the risk-adjusted discount factor between 0 and  $T$ . We may explain this from another angle. Note in this case  $x_0 = s_0 E_Q[S_0(T)^{-1}z]$ , namely, the initial wealth  $x_0$  is equal to the present value of a (sure) cash flow of  $z$  units at time  $t = T$ . Since our market is complete, there must be a portfolio having initial value  $x_0$  and replicating this cash deterministic flow. Our portfolio  $\pi_0(\cdot)$  is such a replicating portfolio. Note, however, that  $\pi_0(\cdot)$  might involve exposure to the stocks. When the spot interest rates  $r(t)$  are random, it is necessary to hedge the interest rate risk by taking a suitable position in the stocks; since the market is complete, this risk can be eliminated.

Due to the availability of the risk-free portfolio, it is sensible to restrict attention to values of the expected payoff satisfying  $z \geq \frac{x_0}{E[\rho(T)]}$  when considering problem (2.13). On the other hand, by Proposition 3.3,  $z$  will be too large for the mean–variance problem to be feasible if  $z > \frac{x_0}{a}$  ( $\frac{x_0}{a}$  is defined to be  $\infty$  if  $a = 0$ ). Hence it is sensible to focus on values of the parameter  $z$  (the targeted mean terminal payoff) satisfying  $\frac{x_0}{E[\rho(T)]} \leq z < \frac{x_0}{a}$  (In particular, in the special case where the interest rate process  $r(\cdot)$  and the other parameters in the model are deterministic, then the relevant interval for the mean terminal payoff  $z$  is simply  $[x_0 e^{\int_0^T r(t)dt}, \infty)$ ). For such values of  $z$  we then have the following economic interpretation of the optimal terminal wealth.

PROPOSITION 6.1. *The unique variance minimizing portfolio for (2.13) corresponding to  $z$  with  $\frac{x_0}{E[\rho(T)]} \leq z < \frac{x_0}{a}$  is a replicating portfolio for a European put option written on the fictitious asset  $\mu\rho(\cdot)$  with a strike price  $\lambda > 0$  and maturity  $T$ .*

*Proof.* By Theorem 5.1,  $\lambda > 0$  and  $\mu \geq 0$  for  $\frac{x_0}{E[\rho(T)]} \leq z < \frac{x_0}{a}$ . Thus the result follows immediately from Theorem 6.1.  $\square$

The following lemma implies that the portion of the variance minimizing frontier corresponding to  $\frac{x_0}{E[\rho(T)]} \leq z < \frac{x_0}{a}$  is *exactly* the efficient frontier that we are ultimately interested in.

LEMMA 6.1 *Denote by  $J_1^*(z)$  the optimal value of (2.13) corresponding to  $z > 0$ , where  $a < \frac{x_0}{z} < b$ . Then  $J_1^*(z)$  is strictly increasing for  $z \in [\frac{x_0}{E[\rho(T)]}, \frac{x_0}{a})$ , and strictly decreasing for  $z \in (\frac{x_0}{b}, \frac{x_0}{E[\rho(T)]}]$ .*

*Proof.* For any  $z_1$  and  $z_2$  with  $\frac{x_0}{a} > z_2 > z_1 \geq z_0 := \frac{x_0}{E[\rho(T)]}$ , denote by  $x_i^*(\cdot)$  the optimal wealth process of (2.13) corresponding to  $z_i$ ,  $i = 0, 1, 2$ . Notice that  $z_1$  can be represented as

$$z_1 = kz_2 + (1 - k)z_0$$

where  $k := \frac{z_1 - z_0}{z_2 - z_0} \in [0, 1)$ . Define

$$x(t) := kx_2^*(t) + (1 - k)x_0^*(t), \quad \forall t \in [0, T].$$

Then  $x(\cdot)$  is a feasible wealth process corresponding to  $z_1$  due to the linearity of the system (2.7). Thus, noting that  $0 \leq k < 1$ ,

$$J_1^*(z_1) \leq \text{Var } x(T) = k^2 \text{Var } x_2^*(T) < J_1^*(z_2).$$

This shows that  $J_1^*(z)$  is strictly increasing for  $z \in [\frac{x_0}{E[\rho(T)]}, \frac{x_0}{a})$ . Similarly we can prove the second assertion of the lemma.  $\square$

We are now ready to state the final result of this section.

**THEOREM 6.3.** *Let  $x_0$  be fixed. The efficient frontier for (6.1) is determined by the following parameterized equations:*

$$(6.6) \quad \begin{cases} E[x^*(T)] = z, \\ \text{Var } x^*(T) = \lambda(z)z - \mu(z)x_0 - z^2, \quad \frac{x_0}{E[\rho(T)]} \leq z < \frac{x_0}{a}, \end{cases}$$

where  $(\lambda(z), \mu(z))$  is the unique solution to (4.3) (parameterized by  $z$ ). Moreover, all the efficient portfolios are those variance minimizing portfolios corresponding to  $z \in [\frac{x_0}{E[\rho(T)]}, \frac{x_0}{a})$ .

*Proof.* First let us determine the variance minimizing frontier. Let  $x^*(\cdot)$  be the wealth process under the variance minimizing portfolio corresponding to  $z = E[x^*(T)]$ . Then

$$\begin{aligned} \text{Var } x^*(T) &= E[x^*(T)^2] - z^2 \\ &= E[(\lambda(z) - \mu(z)\rho(T))x^*(T)] - z^2 \\ &= \lambda(z)E[x^*(T)] - \mu(z)E[\rho(T)x^*(T)] - z^2 \\ &= \lambda(z)z - \mu(z)x_0 - z^2, \end{aligned}$$

where the second equality followed from the general fact that  $x^2 = \alpha x$  if  $x = \alpha^+$ . Now, Lemma 6.1 yields that the efficient frontier is the portion of the variance minimizing frontier corresponding to  $\frac{x_0}{E[\rho(T)]} \leq z < \frac{x_0}{a}$ . This completes the proof.  $\square$

We remark that for  $z$  as in (6.6) the equality  $E[x(T)] = z$  in (2.13) can be replaced by the inequality  $E[x(T)] \geq z$ , and one will get the same solution.

To conclude this section, we remark that the approaches and results of this paper on the no-bankruptcy problem (2.13) can easily be adapted to the problem with a *benchmark floor*:

$$(6.7) \quad \begin{array}{l} \text{Minimize} \\ \text{subject to} \end{array} \quad \left\{ \begin{array}{l} \text{Var } x(T) \equiv Ex(T)^2 - z^2, \\ Ex(T) = z, \\ x(t) \geq \underline{x}(t), \text{ a.s.}, \\ \pi(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m), \\ (x(\cdot), \pi(\cdot)) \text{ satisfies equation (2.7),} \end{array} \right.$$

where  $\underline{x}(\cdot)$  is the wealth process of a *benchmark portfolio* (which is an admissible portfolio but not necessarily starting with the same initial wealth  $x_0$ ).

For the model (6.7) the condition  $x(t) \geq \underline{x}(t)$  implies that  $\underline{x}(0) \leq x_0$  and  $E\underline{x}(T) \leq z$ . A similar argument as in Proposition 2.1 yields that this condition is equivalent to  $x(T) \geq \underline{x}(T)$ .

The counterpart of problem (2.15) corresponding to problem (2.7) is

$$(6.8) \quad \begin{array}{l} \text{Minimize} \\ \text{subject to} \end{array} \quad \left\{ \begin{array}{l} EX^2 - z^2, \\ EX = z, \\ E[\rho(T)X] = x_0, \\ X \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}), \quad X \geq \underline{x}(T), \text{ a.s.} \end{array} \right.$$



The above problem is equivalent to

$$(6.9) \quad \begin{array}{l} \text{Minimize} \\ \text{subject to} \end{array} \quad \left\{ \begin{array}{l} E[Y + \underline{x}(T)]^2 - z^2, \\ EY = \underline{z}, \\ E[\rho(T)Y] = \underline{y}_0, \\ Y \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}), \quad Y \geq 0, \text{ a.s.}, \end{array} \right.$$

where  $\underline{z} = z - E\underline{x}(T)$  and  $\underline{y}_0 = x_0 - \underline{x}(0)$ . Compared with problem (2.15), the cost function of (6.9) involves a first-order term of  $Y$ . However, (6.9) can be readily solved using exactly the same approach as in the proof of Theorem 4.1. Details are left to the interested readers.

An interesting special case of this model is when  $\underline{x}(T) = \underline{x}_T$ , where  $\underline{x}_T$  is a positive deterministic constant. In this case  $\underline{x}(\cdot)$  is the wealth process under a risk-free portfolio (similar to the one in Theorem 6.2) with the terminal wealth  $\underline{x}_T$  (alternatively, one may regard  $\underline{x}(t) = \underline{x}_T B(t, T)$  where  $B(t, T)$  is the time- $t$  price of a unit discount Treasury bond maturing at time  $T$ ). Thus, the process  $\underline{x}(\cdot)$  provides a natural floor for the wealth process of an investor who wishes that his/her terminal wealth is at least  $\underline{x}_T$  with probability one.

Obviously, the benchmark portfolio cannot be chosen arbitrarily. It must be selected so that the above problem is feasible. A feasibility study similar to the one in Section 3 will lead to proper conditions. Again it is left to the readers.

# 7 SPECIAL CASE OF DETERMINISTIC MARKET COEFFICIENTS

For the general case of a market with random coefficients, we have (see Proposition 6.1) derived the efficient portfolios as ones that replicate certain European put options with exercise price  $\lambda$  and expiration date  $T$  and written on a fictitious security having time- $T$  price  $\mu\rho(T)$ . Moreover, to find this replicating portfolio it suffices to find a trading strategy  $\pi^*(\cdot)$  along with a wealth process  $x^*(\cdot)$  satisfying the BSDE

$$(7.1) \quad \begin{cases} dx^*(t) = [r(t)x^*(t) + B(t)\pi^*(t)]dt + \pi^*(t)'\sigma(t)dW(t), \\ x^*(T) = (\lambda - \mu\rho(T))^+. \end{cases}$$

By the BSDE theory we know there exist a unique admissible portfolio  $\pi^*(\cdot)$  along with a wealth process  $x^*(\cdot)$  satisfying this BSDE, but actually solving this BSDE is sometimes easier said than done. This is because, in general, one is not able to express  $(x^*(\cdot), \pi^*(\cdot))$  in a closed form. However, if all the market coefficients are deterministic (albeit time-varying), then, as will be shown in this section, an explicit form for  $(x^*(\cdot), \pi^*(\cdot))$  is obtainable. In particular, we shall obtain analytical representations of the efficient portfolios via the Black–Scholes equation.

Throughout this section, in addition to all the basic assumptions specified earlier, we assume that  $r(\cdot)$  and  $\theta(\cdot)$  are deterministic functions (although  $b(\cdot)$  and  $\sigma(\cdot)$  themselves do not need to be deterministic). Notice that, according to Theorem 6.3 in the present case, the efficient portfolios are the variance minimizing portfolios corresponding to  $z \geq x_0 e^{\int_0^T r(s)ds}$ .

THEOREM 7.1. Assume that  $\int_0^T |\theta(t)|^2 dt > 0$ . Then there is a unique efficient portfolio for (2.13) corresponding to any given  $z \geq x_0 e^{\int_0^T r(s) ds}$ . Moreover, the efficient portfolio and the associated wealth process are given respectively as

$$\begin{aligned}
(7.2) \quad \pi^*(t) &= N(-d_+(t, y(t))) (\sigma(t) \sigma(t)')^{-1} B(t)' y(t) \\
&= -(\sigma(t) \sigma(t)')^{-1} B(t)' [x^*(t) - \lambda N(-d_-(t, y(t))) e^{-\int_t^T r(s) ds}]
\end{aligned}$$

and

$$(7.3) \quad x^*(t) = \lambda N(-d_-(t, y(t))) e^{-\int_t^T r(s) ds} - N(-d_+(t, y(t))) y(t),$$

where  $N(\cdot)$ , with  $N(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{v^2}{2}} dv$ , is the cumulative distribution function of the standard normal distribution,

$$\begin{aligned}
(7.4) \quad y(t) &:= \mu \exp\left\{-\int_0^T [2r(s) - |\theta(s)|^2] ds\right\} \exp\left\{\int_0^t [r(s) - \frac{3}{2}|\theta(s)|^2] ds - \int_0^t \theta(s) dW(s)\right\}, \\
d_+(t, y) &:= \frac{\ln(y/\lambda) + \int_t^T [r(s) + \frac{1}{2}|\theta(s)|^2] ds}{\sqrt{\int_t^T |\theta(s)|^2 ds}},
\end{aligned}$$

$$d_-(t, y) := d_+(t, y) - \sqrt{\int_t^T |\theta(s)|^2 ds},$$

and  $(\lambda, \mu)$ , with  $\lambda > 0, \mu \geq 0$ , is the unique solution to (4.3).

*Proof.* First of all, in view of Remark 3.1,  $a = 0$  and  $b = +\infty$  under the given assumptions. Moreover, taking expectation on equation (2.11) and solving the resulting ordinary differential equation we get immediately that  $E[\rho(T)] = e^{-\int_0^T r(s) ds}$ . Thus a specialization of Theorem 6.3 establishes that the unique efficient portfolio exists for (2.13) corresponding to any  $z \geq x_0 e^{\int_0^T r(s) ds}$ .

Now consider the fictitious security process  $y(\cdot)$  explicitly given in (7.4). Ito's formula

shows that  $y(\cdot)$  satisfies

$$(7.5) \quad \begin{cases} dy(t) = y(t)[(r(t) - |\theta(t)|^2)dt - \theta(t)dW(t)], \\ y(0) = \mu \exp\{-\int_0^T [2r(s) - |\theta(s)|^2]ds\}, \quad y(T) = \mu\rho(T). \end{cases}$$

By virtue of Proposition 6.1, the efficient portfolio  $\pi^*(\cdot)$  corresponding to a  $z \geq x_0 e^{\int_0^T r(s)ds}$  is a replicating portfolio for a European put option written on  $y(\cdot)$  with the strike  $\lambda$  and expiration date  $T$ . Now, we need to find  $(x^*(\cdot), \pi^*(\cdot))$  that satisfies (7.1). Write  $x^*(t) = f(t, y(t))$  for some function  $f(\cdot, \cdot)$  (to be determined). Applying Ito's formula to  $f$  and (7.5) and then comparing with (7.1) in terms of both the drift and diffusion terms, we obtain

$$(7.6) \quad \pi^*(t) = -(\sigma(t)\sigma(t)')^{-1}B(t)' \frac{\partial f}{\partial y}(t, y(t))y(t),$$

whereas  $f$  satisfies the following partial differential equation

$$(7.7) \quad \begin{cases} \frac{\partial f}{\partial t}(t, y) + r(t)y \frac{\partial f}{\partial y}(t, y) + \frac{1}{2}|\theta(t)|^2 y^2 \frac{\partial^2 f}{\partial y^2}(t, y) = r(t)f(t, y), \\ f(T, y) = (\lambda - y)^+. \end{cases}$$

This is exactly the Black–Scholes equation (generalized to deterministic but not necessarily constant coefficients) for a European put option; hence one can write down its solution explicitly as<sup>4</sup>

$$(7.8) \quad f(t, y) = \lambda N(-d_-(t, y))e^{-\int_t^T r(s)ds} - N(-d_+(t, y))y.$$

---

<sup>4</sup>There are at least two ways to obtain the solution (7.8). One is to use the more familiar European call option formula and then use the put-call parity. The other is simply to check that the solution given by (7.8) indeed satisfies the Black–Scholes equation (7.7).

Finally, simple (yet non-trivial) calculations lead to

$$\frac{\partial f}{\partial y}(t, y) = -N(-d_+(t, y)).$$

Thus the desired results (7.2) and (7.3) follow from (7.6) as well as the fact that  $x^*(t) = f(t, y(t))$ .  $\square$

REMARK 7.1. The second expression of the efficient portfolio in (7.2) is in a feedback form, namely, it is a function of the wealth. In the case where bankruptcy is allowed (see Zhou and Li (2000)), the efficient portfolio is

$$(7.9) \quad \pi^*(t) = -(\sigma(t)\sigma(t)')^{-1}B(t)'[x^*(t) - \gamma e^{-\int_t^T r(s)ds}],$$

where

$$\gamma := \frac{z - x_0 e^{\int_0^T [r(t) - |\theta(t)|^2] dt}}{1 - e^{-\int_0^T |\theta(t)|^2 dt}}$$

Note the striking resemblance in form between (7.2) and (7.9).  $\square$

REMARK 7.2. The discounted price process of any financial security must be a martingale under the risk neutral probability measure  $Q$ . Since it can be easily verified that the process  $y(\cdot)$  given in (7.4) satisfies  $y(T) = \mu\rho(T)$  and  $y(t) = S_0(t)E_Q[S_0(T)^{-1}y(T)|\mathcal{F}_t]$  for  $t \in [0, T]$ , it follows that the process  $y(\cdot)$  can be interpreted as the price process of a *fictitious* security that takes the value  $\mu\rho(T)$  at the maturity date  $T$ . We say fictitious security, as the price process  $y(\cdot)$  does not belong to our underlying market, which is comprised of the securities with price processes  $S_i(\cdot)$ ,  $i = 0, 1, 2, \dots, m$ .  $\square$

REMARK 7.3. It appears that expression (7.2) for the optimal trading strategy  $\pi^*(\cdot)$  is not convenient for practical implementation because it is in terms of the fictitious security process  $y(\cdot)$  which, in fact, is not directly observable. There are at least two ways to deal

with this issue. First, simple manipulation shows that equation (7.5) is nothing else but the wealth equation (2.7) under the portfolio

$$(7.10) \quad \hat{\pi}(t) := -(\sigma(t)\sigma(t)')^{-1}B(t)'y(t).$$

Notice that, with the initial endowment  $y(0) = \mu \exp\{-\int_0^T [2r(s) - |\theta(s)|^2] ds\}$ , the above  $\hat{\pi}(\cdot)$  is a legitimate, implementable continuous-time portfolio because it is a feedback of the wealth process  $y(\cdot)$ . The portfolio  $\hat{\pi}(\cdot)$  is also called a (continuous-time) *mutual fund* or a *basket of stocks*. Thus, one may compose, actually or virtually (via a simulation, say), a portfolio using the initial wealth  $y(0)$  and the strategy  $\hat{\pi}(\cdot)$ , and the corresponding wealth process as determined via (2.7) is exactly the fictitious security process  $y(\cdot)$  which is observable. The efficient portfolio is then the replicating portfolio for a European put option (with strike  $\lambda$  and maturity  $T$ ) written on this basket of stocks. Another way is based on the observation that, since the market is complete, the “auxiliary” process  $y(\cdot)$  can be inferred from the returns of the risky securities. To see this, define  $DS(t) := (\frac{dS_1(t)}{S_1(t)}, \dots, \frac{dS_m(t)}{S_m(t)})'$  and  $b(t) := (b_1(t), \dots, b_m(t))'$ . Then one can solve for  $dW(t)$  from equation (2.2), obtaining

$$dW(t) = \sigma(t)^{-1}[DS(t) - b(t)dt].$$

Consequently, one can compute the value of  $y(t)$  for every  $t \geq 0$  by combining the above with (7.4). In practice, this can provide an approximation of  $y(\cdot)$  in terms of discrete-time asset returns.  $\square$

REMARK 7.4. Continuing with the second approach discussed in the preceding remark, we can express the fictitious process  $y(\cdot)$  explicitly in terms of the stock prices if all the

coefficients are time-invariant. In fact, in this case Ito's formula yields

$$\begin{aligned}\ln S_i(t) - \ln S_i(0) &= (b_i - \frac{1}{2} \sum_{j=1}^m |\sigma_{ij}|^2)t + \sum_{j=1}^m \sigma_{ij} W^j(t) \\ &= (r - \frac{1}{2} \sum_{j=1}^m |\sigma_{ij}|^2)t + (b_i - r)t + \sum_{j=1}^m \sigma_{ij} W^j(t).\end{aligned}$$

Solving for  $W(t)$  we get

$$W(t) = \sigma^{-1}V(t) - \theta't$$

where  $V(t) := (v_1(t), \dots, v_m(t))'$  with  $v_i(t) := \ln S_i(t) - \ln S_i(0) - (r - \frac{1}{2} \sum_{j=1}^m |\sigma_{ij}|^2)t$ . Substituting the above to (7.4) we obtain

$$\begin{aligned}y(t) &= y(0) \exp\{(r - \frac{3}{2}|\theta|^2)t - \theta W(t)\} \\ &= y(0) \exp\{(r - \frac{1}{2}|\theta|^2)t - \theta \sigma^{-1}V(t)\}.\end{aligned}$$

In particular, in the simple Black–Scholes case where the interest rate is constant and there is a single risky asset whose price process  $S_1(\cdot)$  is taken as geometric Brownian motion:  $S_1(t) = S_1(0) \exp\{(b - \sigma^2/2)t + \sigma W(t)\}$ , by the preceding formula the fictitious security process is of the form  $y(t) = \alpha e^{\beta t} [S_1(t)]^{-\theta/\sigma}$ , where  $\alpha > 0$  and  $\beta$  are two computable scalars. But since  $\theta > 0$  and  $\sigma > 0$  it is apparent that this contingent claim has a positive payoff (i.e., is “in the money”) if and only if the terminal price  $S_1(T)$  is greater than some positive constant (the “strike price”). In this respect the contingent claim resembles a conventional call, and it is in accordance with economic intuition: the bigger the terminal price  $S_1(T)$  of the risky asset, the better for the investor.  $\square$

REMARK 7.5. The terminal wealth under an efficient portfolio is of the form  $(\lambda - \mu\rho(T))^+$ , which may take zero value with *positive* probability. Nevertheless, by risk neutral valuation for each  $t < T$  the portfolio value is strictly positive with probability one, and

so a trading strategy that replicates this contingent claim is well-defined for  $t < T$  as a proportional strategy. However, for the reasons discussed in Section 2, it is not clear whether such a proportional strategy will satisfy a reasonable condition of admissibility, such as the ones found in Cvitanic and Karatzas (1992) and Karatzas and Shreve (1998).  $\square$

In Theorem 7.1,  $(\lambda, \mu)$  is the unique solution to (4.3), a solution that is ensured by Theorem 5.1. It turns out that, in the case of deterministic coefficients, (4.3) has a more explicit form.

**PROPOSITION 7.1.** *Under the assumptions of Theorem 7.1, if  $z > x_0 e^{\int_0^T r(s) ds}$ , then*

*$(\lambda, \mu)$  is the unique solution to the following system of equations:*

$$(7.11) \quad \begin{cases} \lambda N \left( \frac{\ln(\lambda/\mu) + \int_0^T [r(s) - \frac{1}{2}|\theta(s)|^2] ds}{\sqrt{\int_0^T |\theta(s)|^2 ds}} \right) - \mu e^{-\int_0^T [r(s) - |\theta(s)|^2] ds} N \left( \frac{\ln(\lambda/\mu) + \int_0^T [r(s) - \frac{3}{2}|\theta(s)|^2] ds}{\sqrt{\int_0^T |\theta(s)|^2 ds}} \right) = x_0 e^{\int_0^T r(s) ds}, \\ \lambda N \left( \frac{\ln(\lambda/\mu) + \int_0^T [r(s) + \frac{1}{2}|\theta(s)|^2] ds}{\sqrt{\int_0^T |\theta(s)|^2 ds}} \right) - \mu e^{-\int_0^T r(s) ds} N \left( \frac{\ln(\lambda/\mu) + \int_0^T [r(s) - \frac{1}{2}|\theta(s)|^2] ds}{\sqrt{\int_0^T |\theta(s)|^2 ds}} \right) = z. \end{cases}$$

*Proof.* First note that when  $z > x_0 e^{\int_0^T r(s) ds}$ , it follows from Theorem 5.1 that  $\lambda > 0$  and  $\mu > 0$ . We start with the second equation in (4.3):

$$(7.12) \quad E[\rho(T)(\lambda - \mu\rho(T))^+] = x_0.$$

By the proof of Theorem 7.1,  $x_0 = x^*(0) = f(0, y(0))$ . Using the expressions for  $f(\cdot, \cdot)$  and  $y(0)$  as given in (7.8) and (7.5) respectively, we conclude that  $f(0, y(0))e^{\int_0^T r(s) ds}$  equals the left hand side of the first equation in (7.11). Hence the first equation in (7.11) follows.

Next, the first equation in (4.3) can be rewritten as

$$(7.13) \quad E[\rho(T)\left(\frac{\lambda}{\rho(T)} - \mu\right)^+] = z.$$



Drawing an analog between (7.13) and (7.12), we see that equation (7.13) is nothing else than a statement that  $z$  is the initial price of a European *call* option on  $\frac{\lambda}{\rho(T)}$  with strike  $\mu$  and maturity  $T$ . Define

$$(7.14) \quad \bar{y}(t) := \lambda \exp\left\{\int_0^t \left[r(s) + \frac{1}{2}|\theta(s)|^2\right]ds + \int_0^t \theta(s)dW(s)\right\},$$

which satisfies

$$(7.15) \quad \begin{cases} d\bar{y}(t) = \bar{y}(t)[(r(t) + |\theta(t)|^2)dt + \theta(t)dW(t)], \\ \bar{y}(0) = \lambda, \quad \bar{y}(T) = \frac{\lambda}{\rho(T)}. \end{cases}$$

The well-known Black–Scholes call option formula (or going through a similar derivation to that in the proof of Theorem 7.1) implies that  $z = g(0, \bar{y}(0))$  where

$$(7.16) \quad g(t, y) = N(\bar{d}_+(t, y))y - \mu N(\bar{d}_-(t, y))e^{-\int_t^T r(s)ds},$$

with

$$(7.17) \quad \begin{aligned} \bar{d}_+(t, y) &:= \frac{\ln(y/\mu) + \int_t^T [r(s) + \frac{1}{2}|\theta(s)|^2]ds}{\sqrt{\int_t^T |\theta(s)|^2 ds}}, \\ \bar{d}_-(t, y) &:= \bar{d}_+(t, y) - \sqrt{\int_t^T |\theta(s)|^2 ds}. \end{aligned}$$

This leads to the second equation in (7.11).  $\square$

We now turn to the representation of the efficient frontier. For the general case this is provided by Theorem 6.3, where we represented the minimal variance  $\text{Var } x^*(T)$  as a function of the expected terminal wealth  $E[x^*(T)] (= z)$ . But there is the drawback to representation (6.6) in Theorem 6.3, namely, the minimal variance  $\text{Var } x^*(T)$  is an implicit function of  $z$ , because the Lagrange multipliers  $\lambda(z)$  and  $\mu(z)$  are, in general, implicit functions of  $z$ . It

turns out that in the deterministic coefficient case we can write the efficient frontier in an explicit parametric form, as a function of a positive scalar variable that we denote by  $\eta$ .

**THEOREM 7.2.** *Under the assumptions of Theorem 7.1, the efficient frontier is the following*

$$(7.18) \quad \begin{cases} E[x^*(T)] = \frac{\eta e^{\int_0^T r(t)dt} N_1(\eta) - N_2(\eta)}{\eta N_2(\eta) - e^{-\int_0^T [r(s) - |\theta(s)|^2]ds} N_3(\eta)} x_0, \\ \text{Var } x^*(T) = \left[ \frac{\eta}{\eta N_1(\eta) - e^{-\int_0^T r(t)dt} N_2(\eta)} - 1 \right] [E x^*(T)]^2 - \frac{x_0}{\eta N_1(\eta) - e^{-\int_0^T r(t)dt} N_2(\eta)} E x^*(T), \quad \eta \in (0, \infty], \end{cases}$$

where

$$(7.19) \quad \begin{aligned} N_1(\eta) &:= N \left( \frac{\ln \eta + \int_0^T [r(s) + \frac{1}{2} |\theta(s)|^2] ds}{\sqrt{\int_0^T |\theta(s)|^2 ds}} \right), \\ N_2(\eta) &:= N \left( \frac{\ln \eta + \int_0^T [r(s) - \frac{1}{2} |\theta(s)|^2] ds}{\sqrt{\int_0^T |\theta(s)|^2 ds}} \right), \\ N_3(\eta) &:= N \left( \frac{\ln \eta + \int_0^T [r(s) - \frac{3}{2} |\theta(s)|^2] ds}{\sqrt{\int_0^T |\theta(s)|^2 ds}} \right). \end{aligned}$$

*Proof.* Set  $\eta := \frac{\lambda}{\mu}$ . From the proof of Theorem 5.1, it follows that as  $z$  runs from  $x_0 e^{\int_0^T r(t)dt}$  (inclusive) to  $\infty$  (exclusive),  $\eta$  changes decreasingly from  $\infty$  (inclusive) to 0 (exclusive). Therefore  $\eta \in (0, \infty]$ . Dividing the second equation by the first one in (7.11) we get the first equation of (7.18). Now, replacing  $\lambda$  by  $\eta\mu$  in the second equation of (7.11) and solving for  $\mu$ , we obtain

$$(7.20) \quad \mu = \frac{z}{\eta N_1(\eta) - e^{-\int_0^T r(t)dt} N_2(\eta)}.$$

Thus, appealing to (6.6), we have

$$\text{Var } x^*(T) = \lambda z - \mu x_0 - z^2 = \eta \mu z - z^2 - \mu x_0.$$

Using (7.20) and noting  $z \equiv E[x^*(T)]$ , we get the second equation of (7.18).  $\square$

REMARK 7.6. Although the efficient frontier does not have a closed analytical form, equation (7.18) is “explicit” enough in the sense that it has only one parameter  $\eta \in (0, \infty]$ . It is easy to numerically draw the curve based on (7.18).  $\square$

Analogous to the single-period case, the efficient frontier in continuous time will induce the so-called *capital market line* (CML). Specifically, define  $r^*(t) := \frac{x^*(t) - x_0}{x_0}$ , the return rate of an efficient strategy at time  $t$ . Then in the case where bankruptcy is allowed, the capital market line is the following straight line in the terminal mean–standard deviation plane (see Zhou (2003)):

$$(7.21) \quad Er^*(T) = r_f(T) + \sqrt{e^{\int_0^T |\theta(t)|^2 dt} - 1} \sigma_{r^*(T)},$$

where  $r_f(T) := e^{\int_0^T r(t) dt} - 1$  is the risk-free return rate over  $[0, T]$ , and  $\sigma_{r^*(T)}$  denotes the standard deviation of  $r^*(T)$ . In the present case of bankruptcy prohibition, we can easily obtain the corresponding CML via the efficient frontier (7.18). Clearly the CML is no longer a straight line, as seen from (7.18).

EXAMPLE 7.1. Take the same example as in Zhou and Li (2000) where a market has a bank account with  $r(t) = 0.06$  and only one stock with  $b(t) = 0.12$  and  $\sigma(t) = 0.15$ . An agent starts with an endowment  $x_0 = \$1$  million and expects a terminal mean payoff  $z = \$1.2$  million at  $T = 1$  (year). Bankruptcy is not allowed (as opposed to Zhou and Li (2000)). In this case  $\theta(t) = 0.4$ . Thus the system of equations (7.11) reduces to

$$(7.22) \quad \begin{cases} \lambda N\left(\frac{\ln(\lambda/\mu) - 0.02}{0.4}\right) - \mu e^{0.1} N\left(\frac{\ln(\lambda/\mu) - 0.18}{0.4}\right) = e^{0.06}, \\ \lambda N\left(\frac{\ln(\lambda/\mu) + 0.14}{0.4}\right) - \mu e^{-0.06} N\left(\frac{\ln(\lambda/\mu) - 0.02}{0.4}\right) = 1.2. \end{cases}$$

Solving this equation we get

$$\lambda = 2.0220, \quad \mu = 0.8752.$$

Therefore the corresponding efficient portfolio is the replicating portfolio of a European put option on the following fictitious stock

$$(7.23) \quad \begin{cases} dy(t) = y(t)[-0.1dt - 0.4dW(t)], \\ y(0) = \$0.9109 \end{cases}$$

with a strike price \$2.0220 maturing at the end of the year.

The CML when bankruptcy is allowed has been obtained in Zhou and Li (2000) as

$$(7.24) \quad Er^*(1) = 0.0618 + 0.4165\sigma_{r^*(1)}.$$

In the current case of no bankruptcy, the CML is the following based on (7.18):

$$Er^*(1) = \frac{e^{0.06} \eta N\left(\frac{\ln \eta + 0.14}{0.4}\right) - N\left(\frac{\ln \eta - 0.02}{0.4}\right)}{\eta N\left(\frac{\ln \eta - 0.02}{0.4}\right) - e^{0.1} N\left(\frac{\ln \eta - 0.18}{0.4}\right)} - 1,$$

$$(7.25) \quad \sigma_{r^*(1)}^2 = \left[ \frac{\eta}{\eta N\left(\frac{\ln \eta + 0.14}{0.4}\right) - e^{-0.06} N\left(\frac{\ln \eta - 0.02}{0.4}\right)} - 1 \right] [Er^*(1) + 1]^2 - \frac{1}{\eta N\left(\frac{\ln \eta + 0.14}{0.4}\right) - e^{-0.06} N\left(\frac{\ln \eta - 0.02}{0.4}\right)} [Er^*(1) + 1].$$

Both (7.24) and (7.25) are plotted on the same plane; see Figure 1. We see that (7.25) falls below (7.24), which is certainly expected. In particular, if an agent is expecting an annual return rate of 20%, then the corresponding standard deviation with bankruptcy allowed is 33.1813%, whereas the one without bankruptcy is 33.3540%.

## 8 CONCLUDING REMARKS

This paper investigates a continuous-time mean–variance portfolio selection problem with

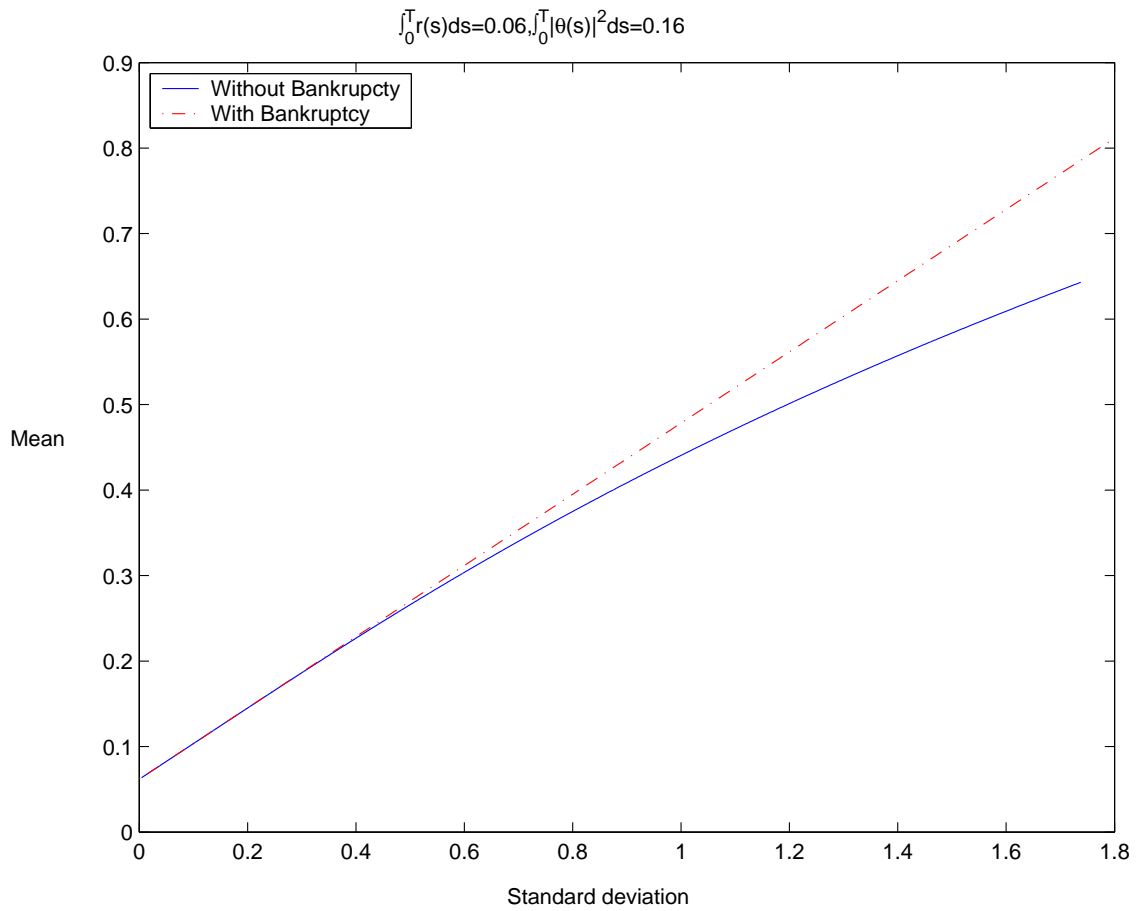


Figure 1: Mean-standard deviation of the terminal return rate (CML)

stochastic parameters under a no bankruptcy constraint. The problem has been completely solved in the following sense. First, the range of the ratio between the expected terminal payoff and the initial wealth is specified which ensures the feasibility of the problem. Second, the efficient portfolios and efficient frontier are obtained based on a BSDE and a system of algebraic equations; the unique solvability of the latter is, for the first time, proved. Third, in the deterministic parameter case, complete, explicit results are obtained, with the efficient portfolio presented in a closed feedback form and the efficient frontier expressed as a system of parameterized equations.

The main idea of the paper is the decomposition of the continuous-time portfolio selection problem. We first identify the optimal terminal wealth attainable by those constrained portfolios, and then replicate this optimal wealth. This idea in fact applies to a more general class of constrained continuous-time portfolio selection problem: first translate all the constraints to the ones imposed on the terminal wealth, solve this constrained optimization problem on random variables, and then replicate the contingent claim represented by the optimal terminal wealth.

As we emphasize in Section 2 and elsewhere, by defining trading strategies in terms of the amount of money invested in individual assets, rather than in terms of the proportion of wealth invested in individual assets, we can allow for strategies where the portfolio's value becomes zero before the terminal date with positive probability. Hence our approach, which includes an explicit constraint on nonnegative portfolio value, leads to a strictly bigger set of admissible trading strategies than with the proportional strategy approach. It is an open question whether this larger class of admissible strategies gives a strictly better value of the optimal objective value than with the smaller class, although we conjecture that the two values are the same. And if the two optimal objective values are indeed the same, it is another open question whether this common value is attained by some proportional trading strategy. This is an open question because if you try to convert our optimal strategy to a proportional strategy, then it might be well defined for  $t < T$ , but even so it might not be admissible because the ratio of the money in a risky asset to the total wealth might not be well-behaved. Since the optimal attainable wealth takes the value zero with positive probability, it is clear it cannot be replicated by a proportional trading strategy satisfying the admissibility condition given immediately before (2.8). However, since the attainable

wealth process is strictly positive with probability one for all  $t < T$ , it is an open question whether some other reasonable definition of admissibility might lead to a proportional trading strategy that does replicate the optimal attainable wealth.

**Acknowledgment.** We are grateful to Jun Sekine of Osaka University for useful discussions.

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