Interplay between dividend rate and business constraints for a financial corporation

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Abstract

We study a model of a corporation which has possibility to choose various production/business policies with different expected profits and risks. In the model there are restrictions on the dividend distribution rates, as well as restrictions on the risk the company can undertake. The objective is to maximize the expected present value of the total dividend distributions. We outline the corresponding Hamilton-Jacobi-Bellman equation and compute explicitly the optimal return function and determine the optimal policy. As a consequence of these results the way the dividend rate and business constraints affects the optimal policy is revealed. In particular we show that under certain relationship between the constraints and the exogenous parameters of the random processes governing the returns, some business activities might be redundant, i.e., under the optimal policy they will be never used in any scenario.

Short Title. Dividend rate and business constraints

1   Introduction

In recent years we saw a lot of new results in application of diffusion optimization models to financial mathematics. Together with portfolio optimization models, the dividend distribution and risk control models have undergone a major development.

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In typical model of this type (see Jeanblanc Piqué and Shiryaev [11], Asmussen and Taksar [2], Radner and Shepp [14], Boyle et al. [3], Højgaard and Taksar [8], [9], [10], Paulsen and Gjessing [13], and Taksar and Zhou [16]) the liquid assets of the company are governed by a Brownian motion with constant drift and diffusion coefficients. The drift term corresponds to the expected (potential) profit per unit time, while the diffusion term is interpreted as risk. The decrease of the risk from the business activities, corresponds to a decrease in potential profits. Different business activities in these models correspond to changing *simultaneously* the drift and the diffusion coefficients of the underlying process. This sets a scene for an optimal stochastic control model where the controls affect not only the drift, but also the diffusion part of the dynamic of the system.

In this paper we study a model with an explicit restriction on risk control and on the rate at which the dividends are paid out. In addition there may exist liability which the company has to pay out at a constant rate no matter what the business plan is.

The controls are described by two functionals a_t and c_t. The first represents the degree of the business activity which the company assumes. The process a_t takes on values in the interval $[\alpha, \beta]$, $0 < \alpha < \beta \leq +\infty$. The risk, which in our model is associated with the diffusion coefficient and the potential profit associated with the drift coefficient of the corresponding process, are both proportional to a_t. The constraints for the values of a_t reflect institutional or statutory restrictions (e.g., for a public company) that the risk it can assume cannot exceed a certain level or that its business activities cannot be reduced to zero unless the company goes bankrupt.

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The value c_t of second control functional shows the rate at which the dividends are paid out at time t. The dividends are paid out from the liquid reserve and are distributed to shareholders. That corresponds to c_t entering the drift coefficient of the reserve process with negative sign. The dividend rate is bounded by a constant M given a priori.

In our model we also assume existence of a constant rate liability payment, such as a mortgage payment on a property or amortization of bonds. The results of this model can be viewed as an extension of the results of Choulli, Taksar and Zhou [4]. The presence of dividend rate constraints, however, adds a whole new dimension to the analysis as well as to the qualitative structure of the results obtained.

What is the most interesting is the interplay between the constraints and the exogenous parameters governing the process of returns. Depending on the relationship between those parameters, we get several distinct cases of qualitative behavior of the company under the optimal policy. The paper is structured as following. In the next section we present a rigorous mathematical formulation of the problem and state general properties of the *optimal return* or the *value* function. We also write down the Hamilton-Jacobi-Bellman (HJB) equation this function must satisfy. In Section 3 we find a bounded smooth solution to the HJB equation. In Section 4 we construct the optimal policy and present our main findings in a table form. Finally, in Section 5 we describe some economic interpretation of the results and conclude the paper.

2 Mathematical Model

We start with a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and a one-dimensional standard Brownian motion W_t (with $W_0 = 0$) on it, adapted to the filtration $\mathcal{F}_t$. We denote by R_t^π the reserve of the company at time t under a control policy $\pi = (a_t^\pi, c_t^\pi; t \geq 0)$ (to be specified below). The dynamic of the reserve process R_t^π is described by

$$dR_t^\pi = (a_t^\pi \mu - \delta)dt + a_t^\pi \sigma dW_t - c_t^\pi dt, \qquad R_0^\pi = x, \tag{2.1}$$

where μ is the expected profit per unit time (profit rate) and σ is the volatility rate of the reserve process (in the absence of any risk control), δ represents the amount of money the company has to pay per unit time (the debt rate) irrespective of what business activities it chooses, and x is the initial reserve.

The control in this model is described by a pair of $\mathcal{F}_t$-adapted processes $\pi = (a_t^\pi, c_t^\pi; t \geq 0)$. A control $\pi = (a_t^\pi, c_t^\pi; t \geq 0)$ is admissible if $\alpha \leq a_t^\pi \leq \beta$, and $0 \leq c_t^\pi \leq M$, $\forall t \geq 0$, where $0 < \alpha < \beta < +\infty$ and $0 < M < +\infty$ are given scalars. We denote the set of all admissible controls by $\mathcal{A}$. The control component a_t^π represents one of the possible business activities available for the company at time t, and the component c_t^π corresponds to the dividend pay-out rate at time t.

Given a control policy π, the time of bankruptcy is defined as

$$\tau^\pi = \inf\{t \geq 0 : R_t^\pi = 0\}. \tag{2.2}$$

The *performance functional* associated with each control π is

$$J_x(\pi) = E\left(\int_0^{\tau^\pi} e^{-\gamma t} c_t^\pi dt\right), \tag{2.3}$$

where $\gamma > 0$ is an a priori given discount factor (used in calculating the present value of the future dividends), and the subscript x denotes the initial state x. The objective is to find

$$v(x) = \sup_{\pi \in \mathcal{A}} J_x(\pi) \tag{2.4}$$

and the optimal policy π^* such that

$$J_x(\pi^*) = v(x). \tag{2.5}$$

The exogenous parameters of the problem are $\mu, \sigma, \delta, \alpha, \beta$ and γ. The aim of this paper is to obtain the optimal return function v and the optimal policy *explicitly* in terms of these parameters.

The main tools for solving the problem are the dynamic programming and Hamilton-Jacobi-Bellman (HJB) equation (see Fleming and Rishel [6], Fleming and Soner [7], and Yong and Zhou [17], as well as relevant discussions in [2], [9] and [16]). We start with stating the following properties of the optimal return function v.

Proposition 2.1 *The optimal return function v is a concave, non-decreasing function subject to $v(0) = 0$ and*

$$0 \leq v(x) \leq \frac{M}{\gamma}, \quad \forall x > 0. \tag{2.6}$$

Proof. The proof of the concavity and the monotonicity as well as the boundary condition, $v(0) = 0$, is similar to the one in [4]. To show (2.6), consider

$$0 \leq E \left(\int_0^{\tau^\pi} e^{-\gamma t} c_t^\pi dt \right) \leq M \int_0^\infty e^{-\gamma t} dt = \frac{M}{\gamma}.$$

$\square$

If the optimal return function v is twice continuously differentiable, then it must be a solution to the following HJB equation

$$
\begin{aligned}
0 = \ & \max_{\alpha \leq a \leq \beta, 0 \leq c \leq M} \left(\tfrac{1}{2}\sigma^2 a^2 V''(x) + (a\mu - \delta - c)V'(x) - \gamma V(x) + c \right) \\
\equiv \ & \max_{\alpha \leq a \leq \beta} \left(\tfrac{1}{2}\sigma^2 a^2 V''(x) + (a\mu - \delta)V'(x) - \gamma V(x) + M \left(1 - V'(x)\right)^+ \right), \quad\quad (2.7) \\
0 = \ & V(0),
\end{aligned}
$$

where $x^+ = \max(x, \, 0)$. This equation is rather standard and its derivation can be found in [7], [6], [17]; see also [9] and [10].

Note that we do not know a priori whether the HJB equation has any solution other than the optimal return function. However, the following verification theorem, which says that *any* concave solution V to the HJB equation (2.7) whose derivative is finite at 0 majorizes the performance functional for any policy π, is sufficient for us to identify optimal policies.

Theorem 2.2 *Let V be a concave, twice continuously differentiable solution of (2.7), such that $V'(0) < +\infty$. Then for any policy $\pi = (a_t^\pi, c_t^\pi; t \geq 0)$,*

$$
V(x) \geq J_x(\pi). \quad\quad (2.8)
$$

Proof. Let R_t^π be the reserve process given by (2.1). Denote the operator

$$
L^a = \frac{1}{2}\sigma^2 a^2 \frac{d^2}{dx^2} + (a\mu - \delta)\frac{d}{dx} - \gamma.
$$

Then applying Ito's formula (see Dellacherie and Meyer [5, Theorem VIII.27]) to the process $e^{-\gamma t}V(R_t^\pi)$, we get

$$
e^{-\gamma t \wedge \tau}V(R_{t\wedge\tau}^\pi) = V(x) + \int_0^{t\wedge\tau} e^{-\gamma s}\sigma a_s^\pi V'(R_s^\pi)dW_s \qu\quad (2.9)
$$

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$$+ \int_0^{t\wedge\tau} e^{-\gamma s} L^{a_s^\pi} V(R_s^\pi) ds - \int_0^{t\wedge\tau} e^{-\gamma s} V'(R_s^\pi) c_s^\pi ds.$$

Since V is non-decreasing, concave with finite derivative at the origin, $V'(x)$ is bounded and the stochastic integral in (2.9) is a square integrable martingale whose expectation vanishes. In view of the HJB equation (2.7) and the inequality $c_s^\pi \leq M$, we have

$$L^{a_s^\pi} V(R_s^\pi) \leq -c_s^\pi \left(1 - V'(R_s^\pi)\right)^+ . \tag{2.10}$$

Taking expectations of both sides of (2.9), in view of (2.10) we get

$$E(e^{-\gamma(t\wedge\tau)} V(R_{t\wedge\tau}^\pi)) \leq V(x) - E \int_0^{t\wedge\tau} e^{-\gamma s} c_s^\pi \left[V'(R_s^\pi) + (1 - V'(R_s^\pi))^+\right] ds. \tag{2.11}$$

Combining (2.11) with the fact that $y + (1 - y)^+ \geq 1$, we get

$$E(e^{-\gamma(t\wedge\tau)} V(R_{t\wedge\tau}^\pi)) + E \int_0^{t\wedge\tau} e^{-\gamma s} c_s^\pi ds \leq V(x). \tag{2.12}$$

Note that in view of boundedness of V',

$$e^{-\gamma(t\wedge\tau)} V(R_{t\wedge\tau}^\pi) \leq e^{-\gamma t} K(1 + R_{t\wedge\tau}^\pi) \leq e^{-\gamma t} K(1 + |R_t^\pi|)$$

for some constant K. Since R_t^π is a diffusion process with uniformly bounded drift and diffusion coefficient, standard arguments yield $E|R_t^\pi| \leq x + K_1 t$ for some constant K_1. Therefore

$$E e^{-\gamma(t\wedge\tau)} V(R_{t\wedge\tau}^\pi) \to 0 \tag{2.13}$$

as $t \to \infty$. Thus taking limit in (2.12) as $t \to \infty$ we arrive at

$$V(x) \geq E \int_0^\tau e^{-\gamma s} c_s^\pi ds = J_\pi(x).$$

$\square$

The idea of solving the original optimization problem is to first find a concave, smooth function to the HJB equation (2.7), and then construct a control policy (via solving an SDE; for details see Section 4) whose performance functional can be shown to coincide with the solution to (2.7). Then, the above verification theorem establishes the optimality of the constructed control policy. As a by-product, there is no other concave solution to (2.7) than the optimal return function.

3   A Smooth Solution to the HJB Equation

In this section, we are looking for a concave, smooth solution to (2.7). Assume that such a solution, V, has been found. Let

$$x_1 = \inf\{x \geq 0 \; : \; V'(x) \leq 1\}. \tag{3.1}$$

Then for $0 \leq x < x_1$, (2.7) becomes

$$0 = \max_{\alpha \leq a \leq \beta} \left(\frac{1}{2}\sigma^2 a^2 V''(x) + (a\mu - \delta)V'(x) - \gamma V(x) \right), \tag{3.2}$$

while for $x \geq x_1$, (2.7) can be rewritten as

$$0 = \max_{\alpha \leq a \leq \beta} \left(\frac{1}{2}\sigma^2 a^2 V''(x) + (a\mu - \delta - M)V'(x) - \gamma V(x) + M \right). \tag{3.3}$$

We start with seeking a smooth solution to (3.3). Obviously if $V'(0) \leq 1$, then $x_1 = 0$ and (2.7) is equivalent to (3.3) for all $x \geq 0$.

Proposition 3.1 *If $\beta\mu \leq \delta$, then $V'(0) < 1$.*

Proof. It follows from (2.7) that there exits $\tilde{a} \in [\alpha, \; \beta]$ such that

$$0 = \frac{1}{2}\sigma^2\tilde{a}^2 V''(0) + (\tilde{a}\mu - \delta)V'(0) + M\left(1 - V'(0)\right)^+. \tag{3.4}$$

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If $\beta\mu < \delta$, then each of the first two terms on the right hand side of (3.4) is non-positive with the second being strictly negative. Therefore $M\left(1 - V'(0)\right)^+ > 0$, which implies $V'(0) < 1$. The same argument goes if $\beta\mu = \delta$ and $V''(0) < 0$. In this case either the first or the second term on the right hand side of (3.4) is strictly negative. If $\beta\mu = \delta$ and $V''(0) = 0$, then the maximizer of the right hand side of (2.7) is equal to β for all x in a right neighborhood of 0 (recall that $V'(0) > 0$). Substituting $a = \beta$ either into (3.2) or into (3.3) and solving the resulting linear ODE with constant coefficients we get a function V whose second derivative at 0 does not vanish, which is a contradiction. $\qquad\square$

Remark 3.2 When the dividend rates are unrestricted, the condition $\beta\mu \leq \delta$ makes the problem trivial (see [4], Theorem 4.1). This is not the case when the dividend rates are bounded. Even if $\beta\mu \leq \delta$ the second derivative of V at 0 is strictly negative which makes the problem nontrivial in contrast to a similar situation in the case of unrestricted dividends.

Now we analyze the solution to (3.3) under the condition $\beta\mu > \delta$. As we will see later the qualitative nature of this solution depends on whether $a(x_1) < \alpha$ or $\alpha \leq a(x_1) < \beta$ or $a(x_1) \geq \beta$, where $a(x)$ is defined by

$$a(x) \equiv -\frac{\mu V'(x)}{\sigma^2 V''(x)} > 0, \quad x < x_1. \tag{3.5}$$

To this end, we need the following proposition

Proposition 3.3 *(i) If $a(x_1) \geq \alpha$, then for each $x \geq x_1$,*

$$a(x) \geq \alpha. \tag{3.6}$$

(ii) If $a(x_1) \geq \beta$, then for each $x \geq x_1$,

$$a(x) \geq \beta. \qquad (3.7)$$

Proof. (i) Suppose there exists $x_0 > x_1$ such that $a(x_0) < \alpha$. Then there exists $\varepsilon > 0$ such that $a(x) < \alpha$ for each x with $|x - x_0| < \varepsilon$. Let $x' = \sup\{x_1 \leq x < x_0 : \quad a(x) = \alpha\}$. Then $x_1 \leq x' < x_0 < x_0 + \varepsilon$ and $a(x') = \alpha$. Since $a(x) \leq \alpha$ for all $x \in [x', \ x_0 + \varepsilon)$, the function V satisfies (3.3) with the maximum there attained at $a = \alpha$. Therefore

$$V(x) = \frac{M}{\gamma} + K_1 e^{\tilde{r}_+(\alpha)(x-x')} + K_2 e^{\tilde{r}_-(\alpha)(x-x')}, \quad \forall x \in [x', \ x_0 + \varepsilon). \qquad (3.8)$$

Here

$$\tilde{r}_+(z) \equiv \frac{-(z\mu - \delta - M) + \sqrt{(z\mu - \delta - M)^2 + 2\gamma\sigma^2 z^2}}{z^2 \sigma^2}, \qquad (3.9)$$

$$\tilde{r}_-(z) \equiv \frac{-(z\mu - \delta - M) - \sqrt{(z\mu - \delta - M)^2 + 2\gamma\sigma^2 z^2}}{z^2 \sigma^2}, \quad z > 0. \qquad (3.10)$$

From (3.8) and (3.5), the equation $a(x') = \alpha$ can be rewritten as

$$K_1 \tilde{r}_+(\alpha) = -K_2 \tilde{r}_-(\alpha) \frac{\mu + \alpha\sigma^2 \tilde{r}_-(\alpha)}{\mu + \alpha\sigma^2 \tilde{r}_+(\alpha)},$$

which establishes a relation between the constants K_1 and K_2. Using this relation, we calculate

$$a(x) = \frac{-\mu V'(x)}{\sigma^2 V''(x)} = \frac{-\mu \left(e^{(\tilde{r}_+(\alpha) - \tilde{r}_-(\alpha))(x-x')} - \frac{\mu + \alpha\sigma^2 \tilde{r}_+(\alpha)}{\mu + \alpha\sigma^2 \tilde{r}_-(\alpha)} \right)}{\sigma^2 \left(\tilde{r}_+(\alpha) e^{(\tilde{r}_+(\alpha) - \tilde{r}_-(\alpha))(x-x')} - \tilde{r}_-(\alpha) \frac{\mu + \alpha\sigma^2 \tilde{r}_+(\alpha)}{\mu + \alpha\sigma^2 \tilde{r}_-(\alpha)} \right)}, \quad \forall x \in [x', \ x_0 + \varepsilon) \quad (3.11)$$

However, we have $a(x) < \alpha$ for $x > x'$, which after a simple algebraic transformation of (3.11) is equivalent to $e^{(\tilde{r}_+(\alpha) - \tilde{r}_-(\alpha))(x-x')} < 1$. This leads to a contradiction. Therefore (3.6) holds.

(ii) By virtue of the assertion (i), $a(x) \geq \alpha$ for all $x \geq x_1$. Suppose there exists $x' > x_1$ such that $a(x') < \beta$. Then there exists $\varepsilon > 0$ such that $a(x) < \beta$ for all $x < x' + \varepsilon$. Let

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$\bar{x} = \sup\{x_1 \le x < x' : a(x) = \beta\}$. Then $x_1 \le \bar{x} < x'$ and $a(\bar{x}) = \beta$. In addition $\alpha \le a(x) < \beta$ for all $\bar{x} < x \le x'$. Substituting $a \equiv a(x)$ and $V''(x) = -\frac{\mu V'(x)}{\sigma^2 a(x)}$ into (3.3), we get

$$0 = \frac{\mu a(x)}{2} V'(x) - (\delta + M)V'(x) - \gamma V(x) + M. \tag{3.12}$$

Differentiating (3.12) and again substituting $V''(x) = -\frac{\mu V'(x)}{\sigma^2 a(x)}$ into the resulting equation, we obtain

$$a(x)a'(x) = (a(x) - \tilde{c})\frac{\mu^2 + 2\sigma^2\gamma}{\mu\sigma^2}. \tag{3.13}$$

Then integrating (3.13) we get

$$a(x) - a(\bar{x}) + \tilde{c}\,\log\left(\frac{a(x) - \tilde{c}}{a(\bar{x}) - \tilde{c}}\right) = (x - \bar{x})\frac{\mu^2 + 2\sigma^2\gamma}{\mu\sigma^2} > 0, \quad \forall \bar{x} < x < x', \tag{3.14}$$

which is a contradiction. Hence (3.7) holds and this ends the proof of the proposition. $\qquad\square$

First suppose that $a(x_1) \ge \beta$. In view of Proposition 3.3-(i), we deduce that $a(x) \ge \beta$ for each $x \ge x_1$. Substituting $a = \beta$ into (3.3) and solving the resulting equation, we get

$$V(x) = \frac{M}{\gamma} + K_\beta e^{\tilde{r}_-(\beta)(x - x_1)}, \quad \forall x \ge x_1.$$

Here K_β is a constant which takes on either the value $-\frac{M}{\gamma}$ or $\frac{1}{\tilde{r}_-(\beta)}$ depending respectively on the fact whether x_1 in (3.1) is zero or not. Then straightforward calculations show that $a(x) = \frac{-\mu}{\sigma^2\tilde{r}_-(\beta)}$. Thus, the condition $a(x_1) \ge \beta$ is equivalent to

$$\tilde{c} \equiv \frac{2\mu(\delta + M)}{\mu^2 + 2\gamma\sigma^2} \ge \beta. \tag{3.15}$$

Next suppose $\alpha \le a(x_1) < \beta$. By virtue of Proposition 3.3-(i), $a(x) \ge \alpha$ for all $x \ge x_1$. As a result, $\alpha \le a(x) < \beta$ in a right neighborhood of x_1. Substituting $a \equiv a(x)$ and $V''(x) = -\frac{\mu V'(x)}{\sigma^2 a(x)}$

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into (3.3), we deduce that $a(x)$ satisfies (3.12). Then, following the same analysis there, we derive the equation (3.13) for $a(x)$.

Suppose there exists $x' \geq x_1$ such that $a(x') < \tilde{c}$ (respectively $a(x') > \tilde{c}$), then from (3.13) we deduce that $a(x) < \tilde{c}$ (respectively $a(x) > \tilde{c}$), for each $x \geq x'$. Thus, by integrating (3.13) we derive (3.14) for all $x \geq x'$, with $\bar{x}$ replaced by x'. From (3.12) and (2.6), we see that $a(x) \leq \frac{2(\delta + M)}{\mu}$, $\forall x \geq x_1$. Therefore, the left hand side of (3.14) is bounded. This is a contradiction and we conclude that $a(x) = \tilde{c}$, for each $x \geq x_1$. In view of the above, the condition $\alpha \leq a(x_1) < \beta$ can be rewritten as $\alpha \leq \tilde{c} < \beta$. Now, substituting $a = \tilde{c}$ into (3.3) and solving the resulting equation (noting that $\tilde{r}_-(\tilde{c}) = -\frac{\sigma^2 \tilde{c}}{\mu}$), we get

$$V(x) = \frac{M}{\gamma} + \tilde{K} e^{-\frac{\sigma^2 \tilde{c}}{\mu}(x - x_1)}, \quad \forall x \geq x_1,$$

where $\tilde{K}$ is a constant which takes on either the value of $-\frac{M}{\gamma}$ or $-\frac{\mu}{\sigma^2 \tilde{c}}$ depending respectively on whether $x_1 = 0$ or $x_1 > 0$.

Finally, suppose that $a(x_1) < \alpha$. Then it follows from the above that $\tilde{c} < \alpha$. Therefore $a(x) < \alpha$ for all x in a right neighborhood of x_1. Substituting $a = \alpha$ into (3.3) and solving the resulting linear differential equation, we get

$$V(x) = \frac{M}{\gamma} + K_1(\alpha) e^{\tilde{r}_+(\alpha)(x - x_1)} + K_2(\alpha) e^{\tilde{r}_-(\alpha)(x - x_1)}, \tag{3.16}$$

where $K_1(\alpha)$ and $K_2(\alpha)$ are free constants. If $K_1(\alpha) > 0$, then the right hand side of (3.16) is unbounded on $[x_1, \infty)$, which contradicts (2.6). If $K_1(\alpha) < 0$, then the right hand side of (3.16) becomes negative for x large enough, which again is a contradiction. Hence $K_1(\alpha) = 0$.

On the other hand, we have $K_2(\alpha) < 0$ in view of $V''(0) < 0$. Therefore

$$V(x) = \frac{M}{\gamma} + K_\alpha e^{\tilde{r}_-(\alpha)(x-x_1)},$$

where K_α is a constant that takes on either the value $-\frac{M}{\gamma}$ or $\frac{1}{\tilde{r}_-(\alpha)}$ depending on whether x_1 is

zero or not. Combining the above results, we can formulate the following theorem.

Theorem 3.4 *Let $\tilde{r}_-(\alpha)$, $\tilde{r}_-(\beta)$ and $\tilde{c}$ be the constants given by (3.10) and (3.15) respectively.*

Let x_1 be defined by (3.1). Then for $x_1 = 0$ (respectively for $x_1 > 0$) the following assertions

hold.

(i) If $\tilde{c} \geq \beta$, then

$$V(x) = \frac{M}{\gamma} + K_\beta e^{\tilde{r}_-(\beta)(x-x_1)}, \quad x \geq x_1 \tag{3.17}$$

is a concave, twice differentiable solution of the HJB equation (3.3) on $[x_1, \infty)$, where K_β is

equal to $-\frac{M}{\gamma}$ (respectively to $\frac{1}{\tilde{r}_-(\beta)}$).

(ii) If $\alpha \leq \tilde{c} < \beta$, then

$$V(x) = \frac{M}{\gamma} + \tilde{K} e^{\frac{-\mu}{\sigma^2 \tilde{c}}(x-x_1)}, \quad x \geq x_1 \tag{3.18}$$

is a concave, twice differentiable solution of the HJB equation (3.3) on $[x_1, \infty)$, where $\tilde{K}$ is

equal to $-\frac{M}{\gamma}$ (respectively to $-\frac{\mu}{\sigma^2 \tilde{c}}$).

(iii) If $\tilde{c} < \alpha$, then

$$V(x) = \frac{M}{\gamma} + K_\alpha e^{\tilde{r}_-(\alpha)(x-x_1)}, \quad x \geq x_1 \tag{3.19}$$

is a concave, twice differentiable solution of the HJB equation (3.3) on $[x_1, \infty)$, where K_α a

constant equal to $-\frac{M}{\gamma}$ (respectively to $\frac{1}{\tilde{r}_-(\alpha)}$).

Corollary 3.5 *If $x_1 = 0$, then the solution to (2.7) subject to (2.6) is given by*

$$V(x) = \begin{cases} \frac{M}{\gamma}\left(1 - e^{\tilde{r}_-(\beta)x}\right), & \text{if } \tilde{c} \geq \beta, \\ \frac{M}{\gamma}\left(1 - e^{\frac{-\mu}{\sigma^2 \tilde{c}}x}\right), & \text{if } \alpha \leq \tilde{c} < \beta, \quad \forall x \geq 0. \\ \frac{M}{\gamma}\left(1 - e^{\tilde{r}_-(\alpha)x}\right), & \text{if } \tilde{c} < \alpha, \end{cases}$$

Corollary 3.5 shows that the qualitative nature of the solution depends on the relation between $\frac{2\delta}{\mu}$, α and β. Accordingly, we will consider three cases. However, in contrast to the situation with the unbounded dividend rates, each case here will consist of several subcases, each subcase being associated with a different range for the value of M.

Remark 3.6 *If neither $-\frac{M}{\gamma}\tilde{r}_-(\beta) \leq 1$ when $\tilde{c} \leq \beta$, nor $\frac{M}{\gamma}\frac{\mu}{\sigma^2 \tilde{c}} \leq 1$ when $\alpha \leq \tilde{c} < \beta$ nor $-\frac{M}{\gamma}\tilde{r}_-(\alpha) \leq 1$ when $\tilde{c} < \alpha$ is satisfied, then the solution to (2.7) satisfies*

$$V'(0) > 1$$

The main purpose of the remaining part is to derive the solution to (3.2), and then to combine the latter with Theorem 3.4. The solution to (3.2) is based mainly on the value of $a(0)$. Thus, first of all, we will present an analysis of $a(0)$.

Proposition 3.7 *Suppose the assumptions of Remark 3.6 hold. Then*

(i) $\frac{2\delta}{\mu} < \alpha$ if and only if $a(0) < \alpha$. In this case $a(0) = \frac{\mu\alpha^2}{2(\mu\alpha-\delta)}$.

(ii) $\alpha \leq \frac{2\delta}{\mu} < \beta$ if and only if $\alpha \leq a(0) < \beta$. In this case $a(0) = 2\frac{\delta}{\mu}$.

(iii) $\beta \leq \frac{2\delta}{\mu}$ if and only if $a(0) \geq \beta$. In this case $a(0) = \frac{\mu\beta^2}{2(\mu\beta-\delta)}$.

Proof. The assumption of Remark 3.6, assume that x_1 is positive. Let $\tilde{a} \in [\alpha, \beta]$ be such that

$$0 = \max_{\alpha \leq a \leq \beta}\left(\frac{1}{2}\sigma^2 a^2 V''(0) + (a\mu - \delta)V'(0)\right) = \frac{1}{2}\sigma^2\tilde{a}^2 V''(0) + (\tilde{a}\mu - \delta)V'(0). \qquad (3.20)$$

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Comparing (3.20) with (3.5) we obtain

$$\tilde{a}^2 - 2a(0)\tilde{a} + \frac{2\delta}{\mu}a(0) = 0. \tag{3.21}$$

From (3.21), it follows $a(0) \geq \frac{2\delta}{\mu}$. Moreover, by definition, $a(0) \in [\alpha, \ \beta]$ is equivalent to $\tilde{a} = a(0)$, which is further equivalent to $a(0) = \frac{2\delta}{\mu} \in [\alpha, \ \beta]$. Thus we conclude:

(i) If $a(0) < \alpha$, then $\frac{2\delta}{\mu} \leq a(0) < \alpha$. Conversely, suppose $\frac{2\delta}{\mu} < \alpha$. If $a(0) \in [\alpha, \ \beta]$, then by the above $a(0) = \frac{2\delta}{\mu} < \alpha$ which is a contradiction. Thus either $a(0) < \alpha$ or $a(0) > \beta$. Suppose $a(0) > \beta$, then $\tilde{a} = \beta$ and by (3.21), $a(0) = \frac{\mu\beta^2}{2(\mu\beta - \delta)} < \beta$ (due to $\frac{2\delta}{\mu} < \alpha < \beta$). This is again a contradiction. Hence we have $a(0) < \alpha$. Then $\tilde{a} = \alpha$ and in view of (3.21), we get $a(0) = \frac{\mu\alpha^2}{2(\alpha\mu - \delta)}$.

(ii) Suppose $\alpha \leq \frac{2\delta}{\mu} < \beta$. Then due to (i) we have $a(0) \geq \alpha$. Now we proceed to prove that $a(0) \leq \frac{2\delta}{\mu} < \beta$. Suppose $a(0) > \frac{2\delta}{\mu}$. Then $a(0) > \beta \equiv \tilde{a}$. On the other hand, in view of (3.21) we have $a(0) = \frac{\mu\beta^2}{2(\beta\mu - \delta)}$, thus $\frac{\mu\beta^2}{2(\beta\mu - \delta)} \geq \beta$, which is equivalent to $2\frac{\delta}{\mu} \geq \beta$ This however is a contradiction and therefore $a(0) = \frac{2\delta}{\mu} \in [\alpha, \ \beta)$. Conversely, if $a(0) \in [\alpha, \ \beta)$, then $a(0) = \frac{2\delta}{\mu} \in [\alpha, \ \beta)$.

(iii) Suppose $\beta \leq \frac{2\delta}{\mu}$. Then $a(0) \geq \frac{2\delta}{\mu} \geq \beta$, leading to $\tilde{a} = \beta$ and $a(0) = \frac{\mu\beta^2}{2(\beta\mu - \delta)} \geq \beta$. Conversely, if $a(0) \geq \beta$, then $\tilde{a} = \beta$ and $a(0) = \frac{\mu\beta^2}{2(\mu\beta - \delta)} \geq \beta$, which is equivalent to $\frac{2\delta}{\mu} \geq \beta$. $\qquad\square$

3.1 Case of $\frac{2\delta}{\mu} < \alpha$

To resolve the equation (3.2), we begin our analysis with an observation that in this case, in view of Proposition 3.7-(i), $a(x) < \alpha$ for all x in the right neighborhood of 0. We also

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suppose that $a(x_1) > \beta$. This assumption is not a restriction, but gives us the solution of (3.2) that corresponds to the maximal interval $[0, \ x_1)$. Substituting $a \equiv \alpha$ in (3.2) and solving the resulting second-order linear ODE, we obtain

$$V(x) = k_1(\alpha, \ \beta)(e^{r_+(\alpha)x} - e^{r_-(\alpha)x}), \tag{3.22}$$

where $k_1(\alpha, \ \beta)$ is a free constant to be determined, and

$$
\begin{aligned}
r_+(z) &= \frac{-(z\mu - \delta) + [(z\mu - \delta)^2 + 2\sigma^2 z^2 \gamma]^{1/2}}{\sigma^2 z^2}, \\
r_-(z) &= \frac{-(z\mu - \delta) - [(z\mu - \delta)^2 + 2\sigma^2 z^2 \gamma]^{1/2}}{\sigma^2 z^2}, \ z > 0.
\end{aligned}
\tag{3.23}
$$

Due to (3.5) and (3.22),

$$a'(x) = \frac{-\mu}{\sigma^2} \frac{(V''(x))^2 - V'(x)V^{(3)}(x)}{(V''(x))^2} = \frac{-\mu r_+(\alpha) r_-(\alpha) e^{(r_+(\alpha) + r_-(\alpha))x} \left(r_+(\alpha) - r_-(\alpha)\right)^2}{\sigma^2 (V''(x))^2} > 0$$

for each x in the right neighborhood of 0. Therefore $a(x)$ increases and reaches α at the point x_α given by

$$x_\alpha = \frac{1}{r_+(\alpha) - r_-(\alpha)} \log \left(\frac{r_-(\alpha) \left(\mu + \alpha\sigma^2 r_-(\alpha)\right)}{r_+(\alpha) \left(\mu + \alpha\sigma^2 r_+(\alpha)\right)} \right) > 0. \tag{3.24}$$

By virtue of Proposition 3.3 (i), $\alpha \leq a(x) < \beta$ in the right neighborhood of x_α. In this case we substitute $a \equiv a(x)$ and

$$V''(x) = \frac{-\mu V'(x)}{\sigma^2 a(x)} \tag{3.25}$$

into (3.2), differentiating the resulting equation and substituting $V''(x) = \frac{-\mu V'(x)}{\sigma^2 a(x)}$ once more, we arrive at $\frac{\mu a'(x)}{2} + \frac{\mu \delta}{\sigma^2 a(x)} = \frac{\mu^2 + 2\gamma\sigma^2}{2\sigma^2}$. As a result

$$a'(x) = \frac{\mu^2 + 2\gamma\sigma^2}{\mu\sigma^2}(1 - \frac{c}{a(x)}) \tag{3.26}$$

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with

$$c \equiv 2\delta\mu/(\mu^2 + 2\gamma\sigma^2). \tag{3.27}$$

Integrating equation (3.26) we get $G(a(x)) \equiv \frac{\mu^2 + 2\gamma\sigma^2}{\mu\sigma^2}(x - x_\alpha) + G(\alpha)$, where

$$G(u) = u + c\log(u - c). \tag{3.28}$$

Therefore

$$a(x) = G^{-1}\left(\frac{\mu^2 + 2\gamma\sigma^2}{\mu\sigma^2}(x - x_\alpha) + G(\alpha)\right). \tag{3.29}$$

Thus $a(x)$ is increasing and $a(x_\beta) = \beta$ for

$$x_\beta \equiv \frac{\mu\sigma^2}{\mu^2 + 2\gamma\sigma^2}[G(\beta) - G(\alpha)] + x_\alpha = \frac{\mu\sigma^2}{\mu^2 + 2\gamma\sigma^2}(\beta - \alpha) + \frac{\mu\sigma^2 c}{\mu^2 + 2\gamma\sigma^2}\log(\frac{\beta - c}{\alpha - c}). \tag{3.30}$$

Solving equation (3.25) we obtain

$$V(x) = V(x_\alpha) + V'(x_\alpha)\int_{x_\alpha}^x \exp\left(-\frac{\mu}{\sigma^2}\int_{x_\alpha}^y \frac{du}{a(u)}\right) dy, \quad x_\alpha \le x < x_\beta, \tag{3.31}$$

where $V(x_\alpha)$ and $V'(x_\alpha)$ are free constants. Choosing $V(x_\alpha)$ and $V'(x_\alpha)$ as the value and the derivative respectively of the right hand side of (3.22) at x_α, we can ensure that the function V given by (3.22) and (3.31) is continuous with its first and second derivatives at the point x_α no matter what the choice of $k(\alpha, \beta)$ is. (Note that due to the HJB equation, continuity of V and its first derivative at x_α automatically implies continuity of the second derivative as well.) Next we simplify (3.31). First, changing variables $a(u) = \theta$ we get

$$\int_{x_\alpha}^x \exp\left(-\mu/\sigma^2\int_{x_\alpha}^y \frac{du}{a(u)}dy\right) = \frac{\mu\sigma^2}{\mu^2 + 2\gamma\sigma^2}\int_\alpha^{a(x)}(1 + \frac{c}{\theta - c})\left(\frac{\theta - c}{\alpha - c}\right)^{-\Gamma} d\theta, \quad x_\alpha \le x < x_\beta.$$

On the other hand, relations (3.24) and (3.22) imply

$$V(x_\alpha) = \frac{\alpha\mu - 2\delta}{2\gamma}V'(x_\alpha).$$

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Simple algebraic transformations yield

$$\left(\frac{\mu\sigma^2}{\mu^2 + 2\gamma\sigma^2}\right)\left(\frac{c}{\Gamma} - \frac{z-c}{1-\Gamma}\right) = \frac{z\mu - 2\delta}{2\gamma}, \ \forall z > 0, \tag{3.32}$$

where c is given by (3.27) and

$$\Gamma = \frac{\mu^2}{\mu^2 + 2\gamma\sigma^2}. \tag{3.33}$$

Therefore

$$V(x) = V'(x_\alpha)\frac{\mu a(x) - 2\delta}{2\gamma}\left(\frac{a(x) - c}{\alpha - c}\right)^{-\Gamma}, \ \ x_\alpha \le x < x_\beta. \tag{3.34}$$

The same arguments as in Proposition 3.3 (ii) show that $a(x) \ge \beta$ for each $x \ge x_\beta$. Thus, substituting $a \equiv \beta$ into (3.2) and solving the resulting ODE, we get

$$V(x) = k_1(\beta)e^{r_+(\beta)(x-x_1)} + k_2(\beta)e^{r_-(\beta)(x-x_1)}, \ \ \ x_\beta \le x < x_1, \tag{3.35}$$

where $k_1(\beta)$ and $k_2(\beta)$ are two free constants to be determined. The continuity of (3.31) at x_β, together with simple but tedious algebraic transformation (similar to the ones used above to simplify (3.31) to (3.34)) lead to

$$V'(x_\alpha) = V'(x_\beta)\left(\frac{\beta - c}{\alpha - c}\right)^\Gamma. \tag{3.36}$$

Let $\tilde{c}$ be given by (3.15) and

$$M_z = \left(z - \frac{2\delta\mu}{\mu^2 + 2\sigma^2\gamma}\right)\frac{\mu^2 + 2\sigma^2\gamma}{2\mu}, \ \ z > 0. \tag{3.37}$$

Then, $\tilde{c} \ge \beta$ (respectively $\tilde{c} = \beta$) is equivalent to $M \ge M_\beta$ (respectively $M = M_\beta$). This is the first subcase we will consider.

3.1.1 Case of $M > M_\beta$

Our assumptions imply that in this case, $a(x_1) > \beta$, which is equivalent to $x_1 > x_\beta$.

Combining (3.22), (3.34), (3.35) and Theorem 3.4-(i) we can write a general form of the solution

to (2.7) and (2.6):

$$
V(x) = \begin{cases}
K_1(\alpha, \ \beta)\left(e^{r_+(\alpha)x} - e^{r_-(\alpha)x}\right), & 0 \leq x < x_\alpha, \\
V'(x_\alpha)\frac{\mu a(x) - 2\delta}{2\gamma}\left(\frac{a(x)-c}{\alpha-c}\right)^{-\Gamma}, & x_\alpha \leq x < x_\beta, \\
K_1(\beta)e^{r_+(\beta)(x-x_1)} + K_2(\beta)e^{r_-(\beta)(x-x_1)}, & x_\beta \leq x < x_1, \\
\frac{M}{\gamma} + \frac{1}{\tilde{r}_-(\beta)}e^{\tilde{r}_-(\beta)(x-x_1)}, & x \geq x_1,
\end{cases}
\tag{3.38}
$$

where $r_+(\alpha)$, $r_-(\alpha)$, $r_+(\beta)$ and $r_-(\beta)$, x_α and x_β are given by (3.23), (3.24) and (3.30) respec-

tively, and $K_1(\beta)$, $K_2(\beta)$, $K_1(\alpha, \beta)$ and x_1 are unknown constants to be determined. Continuity

of the first and the second derivatives at x_1 results in

$$
V'(x_1) = 1, \quad V''(x_1) = \tilde{r}_-(\beta).
$$

This gives us two equations

$$
1 = K_1(\beta)r_+(\beta) + K_2(\beta)r_-(\beta), \quad \tilde{r}_-(\beta) = K_1(\beta)r_+^2(\beta) + K_2(\beta)r_-^2(\beta),
$$

whose solutions are

$$
K_1(\beta) = \frac{\tilde{r}_-(\beta) - r_-(\beta)}{r_+(\beta)\left(r_+(\beta) - r_-(\beta)\right)}, \quad K_2(\beta) = \frac{r_+(\beta) - \tilde{r}_-(\beta)}{r_-(\beta)\left(r_+(\beta) - r_-(\beta)\right)}.
\tag{3.39}
$$

Put $\Delta = x_\beta - x_1$. As before, using the principle of smooth fit at x_β, we get

$$
x_\beta - x_1 = \Delta = \frac{1}{\left(r_+(\beta) - r_-(\beta)\right)}\log\left(-\frac{\left(r_+(\beta) - \tilde{r}_-(\beta)\right)\left(\mu + \beta\sigma^2 r_-(\beta)\right)}{\left(\tilde{r}_-(\beta) - r_-(\beta)\right)\left(\mu + \beta\sigma^2 r_+(\beta)\right)}\right).
\tag{3.40}
$$

The expression on the right hand side of (3.40) is negative due to $\tilde{c} > \beta$. In view of (3.40) and (3.39) we can derive a simplified expression for $V'(x_\beta)$:

$$V'(x_\beta) = \beta\sigma^2 e^{r_-(\beta)\Delta} \frac{r_+(\beta) - \tilde{r}_-(\beta)}{\mu + \beta\sigma^2 r_+(\beta)}.$$

The continuity of V' at x_α yields $V'(x_\alpha) = K_1(\alpha,\ \beta)\left(r_+(\alpha)e^{r_+(\alpha)x_\alpha} - r_-(\alpha)e^{r_-(\alpha)x_\alpha}\right)$. Combining this equality with (3.36), we get

$$K_1(\alpha,\ \beta) = \frac{V'(x_\beta)\left(\frac{\beta-c}{\alpha-c}\right)^\Gamma}{r_+(\alpha)e^{r_+(\alpha)x_\alpha} - r_-(\alpha)e^{r_-(\alpha)x_\alpha}}. \tag{3.41}$$

Theorem 3.8 *Let $a(x)$ be a function given by (3.29) and $r_+(\alpha)$, $r_-(\alpha)$, $r_+(\beta)$, $r_-(\beta)$, x_α, x_β, c, Γ, $\tilde{r}_-(\beta)$, $K_1(\beta)$, $K_2(\beta)$, $K_1(\alpha,\ \beta)$, and x_1 be given by (3.23), (3.24), (3.30), (3.27), (3.33), (3.10), (3.39), (3.41), and (3.40) respectively. If $\frac{2\delta}{\mu} < \alpha$ and $M > M_\beta$, then V given by (3.38) is a concave, twice differentiable solution of the HJB equation (2.7), subject to (2.6).*

Proof. From the way we constructed V, it is a twice continuously differentiable solution to the HJB equation (3.2). What remains to show is the concavity. From (3.38), we deduce that

$$V'''(x) = k_1(\alpha,\ \beta)\left(r_+^3(\alpha)e^{r_+(\alpha)x} - r_-^3(\alpha)e^{r_-(\alpha)x}\right) > 0, \quad \forall 0 \le x < x_\alpha,$$

due to $r_-(\alpha) < 0 < k_1(\alpha,\ \beta)$. Hence on this interval V'' is increasing and

$$V''(x) < V''(x_\alpha) = k_1(\alpha,\ \beta)\left(r_+^2(\alpha)e^{r_+(\alpha)x_\alpha} - r_-^2(\alpha)e^{r_-(\alpha)x_\alpha}\right) < 0,$$

due to $\frac{r_-(\alpha)}{r_+(\alpha)} = e^{(r_+(\alpha)-r_-(\alpha))x_\alpha}$ and $|r_-(\alpha)| > r_+(\alpha)$.

For $x_\alpha \le x < x_\beta$, $V''(x) = \frac{-\mu V'(x)}{\sigma^2 a(x)} < 0$. For $x_\beta \le x < x_1$,

$$V'''(x) = k_1(\beta)r_+^3(\beta)e^{r_+(\beta)(x-x_1)} + k_2(\beta)r_-^3(\beta)e^{r_-(\beta)(x-x_1)} > 0,$$

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since $k_2(\beta)$ and $r_-(\beta)$ are of the same sign. Thus $V''(x) < V''(x_1) < 0$, $\forall x_\beta \le x < x_1$. Finally, $V''(x) < 0$, $\forall x \ge x_1$. This establishes the concavity of V. Since $V'(x) > 1$ for $x < x_1$ and $V'(x) \le 1$ for $x \ge x_1$, it is clear that V satisfies (2.7). $\qquad\square$

3.1.2 Case of $M_\alpha < M \le M_\beta$

Expression (3.37) shows that $\beta \ge \tilde{c} > \alpha$ if and only if $M_\beta \ge M > M_\alpha$. From (3.29) we see that the condition $\beta \ge \tilde{c} > \alpha$ is equivalent to $\beta \ge a(x_1) > \alpha$. In view of Theorem 3.4, $a(x) \le a(x_1) \le \beta$, for all $x \ge 0$. This also implies $V'(x_1) = 1$. As a result $\tilde{K} = -\frac{\sigma^2 \tilde{c}}{\mu}$. Taking into account, (3.22), (3.34) and (3.18) we can write the expression for V as follows

$$
V(x) = \begin{cases}
K_1(\alpha,\ \beta)\left(e^{r_+(\alpha)x} - e^{r_-(\alpha)x}\right), & 0 \le x < x_\alpha, \\
V'(x_\alpha)\frac{\mu a(x)-2\delta}{2\gamma}\left(\frac{a(x)-c}{\frac{2\delta}{\mu}-c}\right)^{-\Gamma}, & x_\alpha \le x < x_1, \\
\frac{M}{\gamma} - \frac{\sigma^2 \tilde{c}}{\mu}e^{-\frac{\mu}{\sigma^2 \tilde{c}}(x-x_1)}, & x \ge x_1.
\end{cases}
$$

For $a(x)$ given by (3.29), the root of the equation $a(x_1) = \tilde{c}$ can be written as

$$
x_1 = \frac{\mu\sigma^2}{\mu^2+2\sigma^2\gamma}\int_{\frac{2\delta}{\mu}}^{\tilde{c}}\frac{u\,du}{u-c} = \frac{\mu\sigma^2(\tilde{c}-\frac{2\delta}{\mu})}{\mu^2+2\sigma^2\gamma} + \frac{\mu\sigma^2}{\mu^2+2\sigma^2\gamma}\log\left(\frac{\tilde{c}-c}{\frac{2\delta}{\mu}-c}\right). \tag{3.42}
$$

The continuity of V' at x_1 leads to $V'(x_\alpha) = \left(\frac{\tilde{c}-c}{\frac{2\delta}{\mu}-c}\right)^{\Gamma}$. Consequently

$$
K_1(\alpha,\ \beta) = \frac{\left(\frac{\tilde{c}-c}{\frac{2\delta}{\mu}-c}\right)^{\Gamma}}{r_+(\alpha)e^{r_+(\alpha)x_\alpha} - r_-(\alpha)e^{r_-(\alpha)x_\alpha}}. \tag{3.43}
$$

Theorem 3.9 *Let $a(x)$ be a function given by (3.29) and $r_+(\alpha)$, $r_-(\alpha)$, $K_1(\alpha,\ \beta)$, x_α, x_1, c, Γ, and $\tilde{c}$ are given by (3.23), (3.43), (3.24), (3.42), (3.27), (3.33), and (3.15) respectively. If*

$\frac{2\delta}{\mu} < \alpha$ and $M_\alpha < M \leq M_\beta$, then

$$V(x) = \begin{cases} K_1(\alpha, \ \beta) \left(e^{r_+(\alpha)x} - e^{r_-(\alpha)x} \right), & 0 \leq x < x_\alpha, \\ \frac{\mu a(x) - 2\delta}{2\gamma} \left(\frac{a(x) - c}{\tilde{c} - c} \right)^{-\Gamma}, & x_\alpha \leq x < x_1, \\ \frac{M}{\gamma} - \frac{\sigma^2 \tilde{c}}{\mu} e^{-\frac{\mu}{\sigma^2 \tilde{c}}(x - x_1)}, & x \geq x_1 \end{cases} \tag{3.44}$$

is a concave, twice differentiable solution of the HJB equation (2.7) subject to (2.6).

Proof. The proof of this theorem follows the same lines of those of Theorem 3.8. $\qquad \square$

Now suppose that $M \leq M_\alpha$. Then $a(x) \leq a(x_1) \leq \alpha$ for each $x \geq 0$ (since $a(x)$ is increasing on $[0, \ x_1)$ and is constant for $x \geq x_1$; see Theorem 3.4). If $V'(0) > 1$, then $x_1 > 0$ and $V'(x_1) = 1$. As a result $\tilde{K}_\alpha = \frac{1}{\tilde{r}_-(\alpha)}$. In view of (3.22) and (3.19), the function V is given by

$$V(x) = \begin{cases} k_1(\alpha, \ \beta) \left(e^{r_+(\alpha)x} - e^{r_-(\alpha)x} \right), & 0 \leq x < x_1, \\ \frac{M}{\gamma} + \frac{1}{\tilde{r}_-(\alpha)} e^{\tilde{r}_-(\alpha)(x - x_1)}, & x \geq x_1. \end{cases} \tag{3.45}$$

The smoothness of V requires

$$V'(x_1-) = 1, \quad V''(x_1-) = \tilde{r}_-(\alpha),$$

which translates into

$$k_1(\alpha, \ \beta) \left(r_+(\alpha) e^{r_+(\alpha)x_1} - r_-(\alpha) e^{r_-(\alpha)x_1} \right) = 1, \tag{3.46}$$

$$k_1(\alpha, \ \beta) \left(r_+^2(\alpha) e^{r_+(\alpha)x_1} - r_-^2(\alpha) e^{r_-(\alpha)x_1} \right) = \tilde{r}_-(\alpha).$$

Excluding $k_1(\alpha, \beta)$, we get an equation for x_1

$$e^{(r_+(\alpha) - r_-(\alpha))x_1} = \frac{r_-(\alpha) \left(r_-(\alpha) - \tilde{r}_-(\alpha) \right)}{r_+(\alpha) \left(r_+(\alpha) - \tilde{r}_-(\alpha) \right)}. \tag{3.47}$$

This equation has a positive solution if and only if

$$M > M_0(\alpha) \equiv \frac{\alpha^2\sigma^2\gamma}{2\,(\alpha\mu - \delta)}. \tag{3.48}$$

This proves the following

Proposition 3.10 *If* $\frac{2\delta}{\mu} < \alpha$, *then*

$$V'(0) > 1 \quad \text{iff} \quad M > \frac{\alpha^2\sigma^2\gamma}{2(\alpha\mu - \delta)}.$$

Let M_z be given by (3.37) and

$$M_0(z) \equiv \frac{z^2\sigma^2\gamma}{2\,(z\mu - \delta)}, \quad z > \frac{\delta}{\mu}.$$

A simple analysis shows that $f(z) \equiv M_0(z) - M_z$ is a decreasing function of z and $f(\frac{2\delta}{\mu}) = 0$.

Similarly, we claim that $M_0(z)$ is decreasing for $z \le \frac{2\delta}{\mu}$ and increasing for $z \ge \frac{2\delta}{\mu}$. Thus, we

derive the following inequalities

$$M_0(\frac{2\delta}{\mu}) < M_0(\alpha) < M_\alpha < M_\beta, \quad \text{if} \quad \frac{2\delta}{\mu} < \alpha. \tag{3.49}$$

$$M_\alpha \le M_0(\alpha) \le M_0(\frac{2\delta}{\mu}) < M_\beta, \quad \text{if} \quad \alpha \le \frac{2\delta}{\mu} < \beta. \tag{3.50}$$

$$M_\alpha < M_\beta \le M_0(\frac{2\delta}{\mu}) \le M_0(\beta) < M_0(\alpha), \quad \text{if} \quad \beta \le \frac{2\delta}{\mu}. \tag{3.51}$$

Since the qualitative behavior of the solution to (3.2) (respectively to (3.3)) depends on the

value of $a(0)$ (respectively of $a(x_1)$), in accordance with (3.49) we will distinguish and study

the remaining subcases in the following subsections.

3.1.3 Case of $M_0(\alpha) < M \leq M_\alpha$

This is the case when (3.47) has a positive solution x_1 given by

$$x_1 = \frac{1}{r_+(\alpha) - r_-(\alpha)} \log \left(\frac{r_-(\alpha)\,(r_-(\alpha) - \tilde{r}_-(\alpha))}{r_+(\alpha)\,(r_+(\alpha) - \tilde{r}_-(\alpha))} \right). \tag{3.52}$$

Theorem 3.11 *Let $r_+(\alpha)$, $r_-(\alpha)$, $\tilde{r}_-(\alpha)$ and x_1 are given by (3.23), (3.10) and (3.52) respectively and $k_1(\alpha,\ \beta)$ be determined by (3.46). If $\frac{2\delta}{\mu} < \alpha$ and $M_0(\alpha) < M \leq M_\alpha$, then V given by (3.45) is a concave, twice continuously differentiable solution of (2.7) subject to (2.6).*

Proof. The proof of this theorem follows from that of Theorem 3.9. $\qquad\square$

3.1.4 Case of $M \leq M_0(\alpha)$

By virtue of Proposition 3.10, this assumption results in $V'(0) \leq 1$. As a consequence, $x_1 = 0$. As shown in Theorem 3.4, this leads to $a(x) = a(0)$ for each $x \geq 0$. Since $M_\alpha > M_0(\alpha)$, we can apply Corollary 3.5 to deduce $a(0) < \alpha$.

Theorem 3.12 *Let $\tilde{r}_-(\alpha)$ be a constant given by (3.10). If $\frac{2\delta}{\mu} < \alpha$ and $M \leq M_0(\alpha)$, then*

$$V(x) = \frac{M}{\gamma} \left(1 - e^{\tilde{r}_-(\alpha)x} \right), \qquad x \geq 0, \tag{3.53}$$

is a concave, twice continuously differentiable solution of (2.7) subject to (2.6).

Proof. See Corollary 3.5. $\qquad\square$

3.2 Case of $\alpha \le \frac{2\delta}{\mu} < \beta$

In this subsection, we will investigate the second main case of $\alpha \le a(0) < \beta$. As in the precedent

section, if we assume $a(x_1) > \beta$, then (3.2) admits the following solution

$$V(x) = \begin{cases} V'(0)\frac{\mu a(x)-2\delta}{2\gamma}\left(\frac{a(x)-c}{\frac{2\delta}{\mu}-c}\right)^{-\Gamma}, & 0 \le x < x_\beta, \\ K_1(\beta)e^{r+(\beta)(x-x_1)} + K_2(\beta)e^{r-(\beta)(x-x_1)}, & x_\beta \le x < x_\beta \end{cases} \qquad (3.54)$$

Here

$$x_\beta = \frac{\mu\sigma^2}{\mu^2 + 2\gamma\sigma^2}[G(\beta) - G(2\delta/\mu)] = \frac{\mu\sigma^2}{\mu^2 + 2\gamma\sigma^2}(\beta - 2\delta/\mu) + \frac{2\delta\mu c}{\mu^2 + 2\gamma\sigma^2}\log(\frac{\beta-c}{2\delta/\mu-c}), \quad (3.55)$$

and the function $a(x)$ is defined by

$$a(x) = G^{-1}\left(\frac{\mu^2 + 2\gamma\sigma^2}{\mu\sigma^2}x + G(2\delta/\mu)\right) \in [2\delta/\mu, \ \infty), \qquad (3.56)$$

where G is given by (3.28).

As before, the solution to (2.7) is derived by combining (3.54) and (3.3), by distinguishing

subcases as follows.

3.2.1 Case of $M > M_\beta$

As in Subsection 3.1.1, consider the case of $x_1 > x_\beta$. This case is characterized by $M > M_\beta$,

which is also equivalent to $\tilde{c} > \beta$. As a result, we get $V'(x_1) = 1$. This leads to $\tilde{K}_\beta = \frac{1}{\tilde{r}_-(\beta)}$

(see Theorem 3.4-(i)). Then using (3.54) and Theorem 3.4-(i), V can be represented in the

following form

$$V(x) = \begin{cases} V'(0)\frac{\mu a(x)-2\delta}{2\gamma}\left(\frac{a(x)-c}{\frac{2\delta}{\mu}-c}\right)^{-\Gamma}, & 0 \le x < x_\beta, \\ K_1(\beta)e^{r+(\beta)(x-x_1)} + K_2(\beta)e^{r-(\beta)(x-x_1)}, & x_\beta \le x < x_1, \\ \frac{M}{\gamma} + \frac{1}{\tilde{r}_-(\beta)}e^{\tilde{r}_-(\beta)(x-x_1)}, & x \ge x_1, \end{cases} \qquad (3.57)$$

where $a(x)$, $K_1(\beta)$, $K_2(\beta)$ and x_1 are given by (3.56), (3.39) and (3.40) respectively. Let $\Delta = x_\beta - x_1$ be given by (3.30). Continuity of V at x_β yields

$$V'(0) = \frac{\frac{2\gamma}{\mu\beta - 2\delta}\left(\frac{\beta - c}{\frac{2\delta}{\mu} - c}\right)^\Gamma}{K_1(\beta)e^{r_+(\beta)\Delta} + K_2(\beta)e^{r_-(\beta)\Delta}}. \tag{3.58}$$

Theorem 3.13 *Let $V'(0)$, $a(x)$, c, Γ, $K_1(\beta)$, $K_2(\beta)$, x_1, and $\tilde{c}$ are given by (3.58), (3.56), (3.27), (3.33), (3.39), (3.40) and (3.15) respectively. If $\alpha \leq \frac{2\delta}{\mu} < \beta$ and $M \geq M_\beta$, then $V(x)$ given by (3.57) is a concave, twice continuously differentiable solution of (2.7) subject to (2.6).*

Proof. The proof results from combining (3.54) and Theorem 3.4-(ii). $\qquad\qquad\square$

To classify the remaining cases, suppose $M \leq M_\beta$. Let $V'(0) > 1$. In this case x_1 defined by (3.1) is positive. Therefore, using (3.54) and Theorem 3.4 (ii), we can represent V as

$$V(x) = \begin{cases} V'(0)\frac{\mu a(x) - 2\delta}{2\gamma}\left(\frac{a(x) - c}{\frac{2\delta}{\mu} - c}\right)^{-\Gamma}, & 0 \leq x < x_1, \\ V(x_1) + \frac{\sigma^2\tilde{c}}{\mu} - \frac{\sigma^2\tilde{c}}{\mu}e^{-\frac{\mu}{\sigma^2\tilde{c}}(x - x_1)}, & x \geq x_1, \end{cases} \tag{3.59}$$

where $a(x)$ and $\tilde{c}$ are given by (3.56) and (3.15) respectively. As a consequence, we get

$$a(x_1) = \tilde{c}. \tag{3.60}$$

Continuity of $V'(x)$ at $x = x_1$ (see (3.26) for the expression of the derivatives of $a(x)$) along with (3.60) results in

$$V'(0) = \left(\frac{\tilde{c} - c}{\frac{2\delta}{\mu} - c}\right)^\Gamma. \tag{3.61}$$

Substituting (3.61), (3.60) and (3.37) into (3.59), we obtain

$$V(x_1) = \frac{M}{\gamma} - \frac{\sigma^2\tilde{c}}{\mu}. \tag{3.62}$$

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The unknown constant x_1 is the root of the equation (3.60). Recalling (3.56), we see that (3.60) admits a positive solution if and only if

$$M > M_0(\frac{2\delta}{\mu}) = \frac{2\delta^2\sigma^2\gamma}{\mu^2}. \tag{3.63}$$

Proposition 3.14 *Suppose $\alpha \leq \frac{2\delta}{\mu} < \beta$. Then*

$$V'(0) > 1 \quad iff \quad M > M_0(\frac{2\delta}{\mu}).$$

In view of this proposition, we distinguish the remaining subcases as follows.

3.2.2 Case of $M_0(\frac{2\delta}{\mu}) < M \leq M_\beta$

Substituting (3.56) into (3.60), we obtain

$$x_1 = \frac{\mu\sigma^2}{\mu^2 + 2\sigma^2\gamma} \int_{\frac{2\delta}{\mu}}^{\tilde{c}} \frac{u\,du}{u - c} = \frac{\mu\sigma^2(\tilde{c} - \frac{2\delta}{\mu})}{\mu^2 + 2\sigma^2\gamma} + \frac{\mu\sigma^2}{\mu^2 + 2\sigma^2\gamma} \log\left(\frac{\tilde{c} - c}{\frac{2\delta}{\mu} - c}\right). \tag{3.64}$$

Theorem 3.15 *Let $a(x)$, c, Γ, $\tilde{c}$, and x_1 be given by (3.56), (3.27), (3.33), (3.15), and (3.64) respectively. If $\alpha \leq \frac{2\delta}{\mu} < \beta$ and $M_0(\frac{2\delta}{\mu}) < M \leq M_\beta$, then*

$$V(x) = \begin{cases} \frac{\mu a(x) - 2\delta}{2\gamma} \left(\frac{a(x) - c}{\tilde{c} - c}\right)^{-\Gamma}, & 0 \leq x < x_1, \\ \frac{M}{\gamma} - \frac{\sigma^2 \tilde{c}}{\mu} e^{-\frac{\mu}{\sigma^2 \tilde{c}}(x - x_1)}, & x \geq x_1 \end{cases} \tag{3.65}$$

is a concave, twice continuously differentiable solution of (2.7) subject to (2.6).

Proof. The proof of this theorem follows from that of Theorem (3.54). □

3.2.3 Case of $M \leq M_0(\frac{2\delta}{\mu})$

Note that in this case, $V'(0) \leq 1$ due to Proposition 3.14. From (3.50), it follows that $a(0) = \tilde{c} < \beta$.

Theorem 3.16 *Suppose $\alpha \le \frac{2\delta}{\mu} < b$ and $M \le M_0(\frac{2\delta}{\mu})$. Let $\tilde{c}$ and $\tilde{r}_-(\alpha)$ be given by (3.15) and (3.10) respectively.*

(i) If $\alpha < \frac{2\delta}{\mu}$ and $\alpha \le \tilde{c}$, then

$$V(x) = \frac{M}{\gamma}\left(1 - e^{-\frac{\mu}{\sigma^2 \tilde{c}}x}\right), \quad x \ge 0 \tag{3.66}$$

is a concave, twice continuously differentiable solution of (2.7) subject to (2.6).

(ii) If $\alpha = \frac{2\delta}{\mu}$ or $\tilde{c} < \alpha$, then

$$V(x) = \frac{M}{\gamma}\left(1 - e^{\tilde{r}_-(\alpha)x}\right), x \ge 0 \tag{3.67}$$

is a concave, twice continuously differentiable solution of (2.7) subject to (2.6).

Proof. See Corollary 3.5. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

3.3 Case of $\frac{\delta}{\mu} < \beta \le \frac{2\delta}{\mu}$

We now investigate the final main case, $\frac{\delta}{\mu} < \beta \le \frac{2\delta}{\mu}$. Suppose $V'(0) > 1$. Then x_1 defined by (3.1) is positive. In view of Proposition 3.7 (iii) and Proposition 3.3 (i), $a(x) \ge \beta$ for all $x \ge 0$. Therefore, using (3.62) and Theorem 3.4-(i), V can be represented as follows

$$V(x) = \begin{cases} K\left(e^{r_+(\beta)x} - e^{r_-(\beta)x}\right), & 0 \le x < x_1, \\ \frac{M}{\gamma} + \frac{1}{\tilde{r}_-(\beta)}e^{\tilde{r}_-(\beta)(x-x_1)}, & x \ge x_1. \end{cases} \tag{3.68}$$

The principle of smooth fit for V at x_1 yields

$$V'(x_1-) = K\left(r_+(\beta)e^{r_+(\beta)x_1} - r_-(\beta)e^{r_-(\beta)x_1}\right) = 1, \tag{3.69}$$

$$V''(x_1-) = \tilde{r}_-(\beta).$$

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Thus

$$e^{(r_+(\beta)-r_-(\beta))x_1} = \frac{r_-(\beta)\left(r_-(\beta) - \tilde{r}_-(\beta)\right)}{r_+(\beta)\left(r_+(\beta) - \tilde{r}_-(\beta)\right)}, \tag{3.70}$$

which admits a positive solution x_1 iff

$$M > M_0(\beta) = \frac{\sigma^2 \beta^2 \gamma}{2(\beta\mu - \delta)}. \tag{3.71}$$

Proposition 3.17 *If $\frac{\delta}{\mu} < \beta \le \frac{2\delta}{\mu}$, then*

$$V'(0) > 1 \quad iff \quad M > M_0(\beta).$$

Proof. The proof of this proposition follows from the calculations in this and in the previous

subsections. □

In view of the above proposition we need only to treat two subcases, namely, $M > M_0(\beta)$ and

$M \le M_0(\beta)$, to complete our analysis.

3.3.1 Case of $M > M_0(\beta)$

In this case, (3.70) has a positive solution x_1 given by

$$x_1 = \frac{1}{r_+(\beta) - r_-(\beta)} \log\left(\frac{r_-(\beta)\left(r_-(\beta) - \tilde{r}_-(\beta)\right)}{r_+(\beta)\left(r_+(\beta) - \tilde{r}_-(\beta)\right)}\right). \tag{3.72}$$

Theorem 3.18 *Let x_1, $r_+(\beta)$, and $r_-(\beta)$ and $\tilde{r}_-(\beta)$ be given by (3.72), (3.23), and (3.10)*

respectively and let K be a constant determined from (3.69). If $\frac{\delta}{\mu} < \beta \le \frac{2\delta}{\mu}$ and $M > M_0(\beta)$,

then V given by (3.68) is a concave, twice continuously differentiable solution of (2.7) subject

to (2.6).

Proof. By differentiating the expression (3.68), we obtain

$$V^{(3)}(x) = K\left(r_+^3(\beta)e^{r+(\beta)x} - r_-^3(\beta)e^{r-(\beta)x}\right) > 0, \quad 0 \le x < x_1.$$

As a result, $V''(x) < V''(x_1) = 0$ and $V'(x) > V'(x_1) = 1$. This proves that V is concave. $\quad\square$

3.3.2 Case of $M \le M_0(\beta)$

In this case, in view of Proposition 3.17, $V'(0) \le 1$. Therefore, x_1 defined by (3.1) equals zero.

Theorem 3.19 *Suppose that either $\beta \le \frac{\delta}{\mu}$, or $\frac{\delta}{\mu} < \beta \le \frac{2\delta}{\mu}$ and $M \le M_0(\beta)$. Let $\tilde{r}_-(\beta)$, $\tilde{r}_-(\alpha)$ and $\tilde{c}$ be given by (3.10) and (3.15) respectively.*

(i) If $M \ge M_\beta$, then

$$V(x) = \frac{M}{\gamma}\left(1 - e^{\tilde{r}_-(\beta)x}\right), \qquad x \ge 0 \tag{3.73}$$

is a concave, twice continuously differentiable solution of (2.7)subject to (2.6).

(ii) If $M_\alpha \le M < M_\beta$, then

$$V(x) = \frac{M}{\gamma}\left(1 - e^{-\frac{\mu}{\sigma^2\tilde{c}}x}\right), \qquad x \ge 0$$

is a concave, twice continuously differentiable solution of (2.7) subject to (2.6).

(iii) If $M < M_\alpha$ then

$$V(x) = \frac{M}{\gamma}\left(1 - e^{\tilde{r}_-(\alpha)x}\right), \qquad x \ge 0$$

is a concave, twice continuously differentiable solution of (2.7) subject to (2.6).

Proof. In the case of $\beta \le \frac{\delta}{\mu}$, the inequality $V'(0) \le 1$ holds due to Proposition 3.17. Then by applying Corollary 3.5, the desired result follows. On the other hand, if $\frac{\delta}{\mu} < \beta \le \frac{2\delta}{\mu}$ and $M \le M_0(\beta)$, then $V'(0) \le 1$ (see Proposition 3.1). Thus, in view of (3.51), Corollary 3.5 can be applied again to obtain the results. $\quad\square$

4 Optimal Policies

In this section we construct the optimal control policies based on the solutions to the HJB equations obtained in the previous section. The derivation of the results of this section is simpler than the corresponding one in [4], in view of the fact that no Skorohod problem has to be involved in this case.

Suppose V is a concave solution to the HJB equation (2.7). Define

$$a^*(x) \equiv \arg \max_{\alpha \le a \le \beta} \left(\frac{1}{2}\sigma^2 a^2 V''(x) + (a\mu - \delta)V'(x) - \gamma V(x) + M\left(1 - V'(x)\right)^+ \right), \qquad (4.1)$$

and

$$M^*(x) \equiv M 1_{\{x \ge x_1\}},$$

where x_1 is defined by (3.1). The function $a^*(x)$ is the optimal feedback risk control function while the function $M^*(x)$ represents the optimal dividend rate payments, when the level of the reserve is x.

Theorem 4.1 *Let $R_t^*; t \ge 0$, be a solution to the following stochastic differential equation*

$$\begin{aligned} dR_t^* &= \left[a^*(R_t^*)\mu - \delta - M^*(R_t^*) \right] dt + a^*(R_t^*)\sigma dW_t, \\ R_0^* &= x. \end{aligned} \qquad (4.2)$$

Then for $\pi^* \equiv (a_t^*, c_t^*; t \ge 0) = (a^*(R_t^*), M^*(R_t^*); t \ge 0)$, we have

$$J_x(\pi^*) = V(x), \quad \forall x \ge 0. \qquad (4.3)$$

Proof. For simplicity assume that the initial position $x \le x_1$. In this case the process R_t^* as a solution to (4.2) is continuous. In view of (4.1) and (2.7)

$$L^{a^*(R_s^*)}V(R_s^*) - M^*(R_s^*)V'(R_s^*) + M^*(R_t^*) = 0, \qquad (4.4)$$

(since $M(1 - V'(x))^+ = M^*(x)(1 - V'(x))$) where the operator L^a is defined in the proof of Theorem 2.2. Repeating the arguments of the proof of Theorem 2.2 and applying (4.4), we see that (we write τ instead of τ^π below, since there would be no confusion)

$$E(e^{-\gamma(t\wedge\tau)}V(R^*_{t\wedge\tau})) = V(x) - E\int_0^{t\wedge\tau} e^{-\gamma s}c^*_s ds. \tag{4.5}$$

Taking limit as $t \to \infty$, and applying (2.13), we obtain the desired result. $\qquad\square$

Combining Theorems 2.2 and 4.1, we get the following result immediately.

Corollary 4.2 *The function V presented in Section 3 is the optimal return function and π^* is the optimal policy.*

Next we summarize all the results we obtained in the following tables for easy reference.

Range for M	x_α	x_β	$a^*(x)$	Risk α ever attained	Risk β ever attained	x_1	x_1 is the first point the possible maximal risk is attained at
$M > M_\beta$	positive and finite; see (3.24)	positive and finite; see (3.30)	(i) α, for $x \in [0, x_\alpha]$; (ii) increases from α to β on $[x_\alpha,\ x_\beta]$; see (3.29); (iii) β, for $x \geq x_\beta$	yes	yes	positive; see (3.40)	no
$\dfrac{M_\alpha < M < M_\beta}{M = M_\beta}$	positive and finite; see (3.24)	∞	(i) α, for $x \in [0, x_\alpha]$; (ii) increases from α to $\frac{2\mu(\delta+M)}{\mu^2+2\gamma\sigma^2}$ on $[x_\alpha, x_1]$; see (3.29); (iii) $\frac{2\mu(\delta+M)}{\mu^2+2\gamma\sigma^2}$, for $x \geq x_1$	yes	$\dfrac{no}{yes}$	positive; see (3.42)	yes
$M_0(\alpha) < M \leq M_\alpha$	∞	∞	α	yes	no	positive; see (3.52)	no
$M \leq M_0(\alpha)$	∞	∞	α	yes	no	0	yes

$$\textbf{Table 1: The case of } \frac{2\delta}{\mu} < \alpha$$

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Range for M	x_α	x_β	$a^*(x)$	Risk α ever attained	Risk β ever attained	x_1	x_1 is the first point the possible maximal risk is attained at
$M > M_\beta$	0	positive and finite; see (3.55)	(i) increases from $\frac{2\delta}{\mu}$ to β on $[0,\ x_\beta]$; see (3.56); (ii) β, for $x \geq x_\beta$	yes, if $\alpha = \frac{2\delta}{\mu}$; no, if $\alpha > \frac{2\delta}{\mu}$	yes	positive; see (3.40)	no
$\dfrac{M_0(\frac{2\delta}{\mu}) < M < M_\beta}{M = M_\beta}$	0	∞	(i) increases from $\frac{2\delta}{\mu}$ to $\frac{2\mu(\delta+M)}{\mu^2+2\gamma\sigma^2}$ on $[0,\ x_1]$; see (3.56); (ii) $\frac{2\mu(\delta+M)}{\mu^2+2\gamma\sigma^2}$, for $x \geq x_1$	yes, if $\alpha = \frac{2\delta}{\mu}$; no, if $\alpha > \frac{2\delta}{\mu}$	$\dfrac{\text{no}}{\text{yes}}$	positive; see (3.64)	yes
$M_\alpha < M \leq M_0(\frac{2\delta}{\mu})$	0	∞	$\frac{2\mu(\delta+M)}{\mu^2+2\gamma\sigma^2}$	no	no	0	yes
$M \leq M_\alpha$	0	∞	α	yes	no	0	yes

$$\text{Table 2: The case of } \alpha \leq \frac{2\delta}{\mu} < \beta.$$

Range for M	x_α	x_β	$a^*(x)$	Risk α ever attained	Risk β ever attained	x_1	x_1 is the first point the possible maximal risk is attained at
$M > M_0(\beta)$	0	0	β	no	yes	positive; see (3.72)	no
$M_\beta \leq M \leq M_0(\beta)$	0	0	β	no	yes	0	yes
$\dfrac{M_\alpha < M < M_\beta}{M = M_\alpha}$	0	∞	$\frac{2\mu(\delta+M)}{\mu^2+2\gamma\sigma^2}$	$\dfrac{\text{no}}{\text{yes}}$	no	0	yes
$M < M_\alpha$	∞	∞	α	no	no	0	yes

$$\text{Table 3: The case of } \frac{\delta}{\mu} < \beta \leq \frac{2\delta}{\mu}.$$

5 Economic Interpretation and Conclusions

The optimal policies obtained in the previous sections have clear economic meaning and are very easy to implement. Let us now elaborate.

The risk control policy is characterized by two critical reserve levels: x_α and x_β. The values of these two levels are further determined by the four parameters: the minimum risk allowed (α), the maximum risk allowed (β), the ratio between the debt rate and profit rate ($\frac{\delta}{\mu}$), and the maximum dividend rate allowed (M). Specifically, there are three different cases to consider. The first case is when the company has very little debt compared to the potential profit (so that $\frac{2\delta}{\mu} < \alpha$). In this case, if the maximum dividend rate M is large enough ($M > M_\beta$), then both the critical reserve levels, x_α and x_β, are positive and finite. In other words, the company will minimize the business activity (i.e., take the minimum risk α) when the reserve is below the level x_α, then gradually increase the business activity when the reserve is between x_α and x_β, and then maximize the business activity (i.e., take the maximum risk β) when the reserve ever reaches or goes beyond the level x_β. This policy is the same as that obtained in [4] for the case of unbounded dividend rate. Next, if the maximum dividend rate M is at a medium level ($M_\alpha < M < M_\beta$), then x_α remains positive and finite while x_β becomes infinity. This implies that the company will become less aggressive, in particular it will never take the maximum risk, due to a more restrictive dividend pay-out upper bound. Finally, if M is so small that $M \leq M_\alpha$, then both x_α and x_β turn out to be infinity, meaning that the business activities will be carried out at the minimum level or, those business activities are redundant.

The second case is when the company has a higher debt-profit ratio (so that $\alpha \leq \frac{2\delta}{\mu} < \beta$). In

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this case $x_\alpha = 0$. This means that no matter how small the reserve is the company will never take the minimum risk; rather it will start with a bit higher risk level and gradually increase it. On the other hand, whether it will ever increase to the maximum possible risk (ie., whether x_β is finite or infinite) depends on the value of the maximum possible dividend rate M, in the same way as in the first case discussed above. Therefore, in the second case the company overall has to be a bit more aggressive than in the first case. This can be explained by the fact that when the debt rate is high one needs to gamble on the higher potential profits in order to get out of the "bankruptcy zone" as fast as possible, even at the expense of assuming higher risk.

The company becomes even more aggressive in the third case when the debt-profit ratio is even higher (precisely when $\frac{\delta}{\mu} < \beta \leq \frac{2\delta}{\mu}$). In this case, when the maximum dividend rate M is large enough ($M \geq M_\beta$), the maximum allowable risk β is taken throughout while the two critical levels x_α and x_β are both zero. On the other hand, when M is small enough so that $M < M_\alpha$, the business activities are carried out at the minimum level α throughout.

On the other hand, the optimal dividend policy is always of a threshold type with the threshold being equal to x_1 (which is positive or zero). Namely, the dividend distribution takes place only when the reserve exceeds the critical level x_1, in which case the dividend pay-out rate is M.

It is interesting to note that in the case of unbounded dividend rate, the maximum business activity is always taken *before* dividend distributions ever take place; see [4]. However, in the present case of bounded dividend rate, the company may need to pay dividends *before* the maximum risk level β is ever taken; refer to Tables 1–4 for details. This represents a striking

difference between the cases of unbounded and bounded dividend rate. The economic reason for such a behavior is the following. When there is a significant constraint on the dividend rate there might be not necessary to pursue the business aggressively because the accumulated liquid assets could not be paid out as dividends fast enough anyway.

In conclusion, we would like to point out at an intricate interplay between the restriction on the dividend distribution rate and that on the risk control of a financial company. The sheer number of qualitatively different optimal policies, which appears due to different possible relationship between exogenous parameters, shows the multiplicity of different economic environments which a financial company faces depending on the size of the debt, on the constraint on the dividend rate, and on the size of available business activity.

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