Stochastic LQ Control with Conic Control Constraints on an Infinite Time Horizon

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Abstract

This paper is concerned with a stochastic linear–quadratic (LQ) control problem in infinite time horizon where the control is constrained in a given, arbitrary closed cone, the cost weighting matrices are allowed to be indefinite, and the state is scalar-valued. First, the (mean–square, conic) stabilizability of the system is defined, which is then characterized by a set of simple conditions involving linear matrix inequalities (LMI's). Next, the issue of well-posedness of the underlying optimal LQ control, which is necessitated by the indefiniteness of the problem, is addressed in great details, and necessary and sufficient conditions of the well-posedness are presented. On the other hand, to address the LQ optimality two new algebraic equations *à la* Riccati, called extended algebraic Riccati equations (EARE's), along with the notion of their stabilizing solutions, are introduced for the first time. Optimal feedback control as well as the optimal value are explicitly derived in terms of the stabilizing solutions to the EARE's. Moreover, several cases when the stabilizing solutions do exist are discussed and algorithms of computing the solutions are presented. Finally, numerical examples are provided to illustrate the theoretical results established.

Keywords - stochastic linear–quadratic (LQ) control, infinite time horizon, constrained control, cone, (conic) stabilizability, well-posedness, extended algebraic Riccati equations (EARE's)

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1 Introduction

Linear–quadratic (LQ) control, pioneered by Kalman [17] for deterministic systems and extended to stochastic systems by Wonham [31, 32] and Bismut [5], constitutes an extremely important, in both theory and applications, class of control problems. In recent years, there have been considerable renewed interest in stochastic LQ control. In particular, the notion of mean–square stabilizability and detectability was introduced in [12]. On the other hand, initiated by Chen, Li and Zhou [10], extensive research has been carried out in the so-called *indefinite* stochastic LQ control, where, quite contrary to the conventional belief, the cost weighting matrices are allowed to be indefinite; see [11, 2, 1, 33]. Moreover, this new theory has found applications in financial portfolio selection;

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see [35, 19, 18]. For systematic accounts of the deterministic and stochastic LQ theory, refer to [4] and [34] respectively.

A key assumption in the LQ theory at large, deterministic and stochastic alike, is that the control variable is unconstrained. This assumption renders the feedback control constructed via the Riccati equation *automatically* admissible, and in turn (along with the underlying linear–quadratic structure) makes possible the elegant explicit solution to the optimal LQ control problem. Because of this, the whole conventional LQ approach would collapse in the presence of any control constraint.

On the other hand, from a practical point of view, LQ control with control constraints is a well-defined and sensible problem. For example, in many real applications the control variable is required to take only non-negative values. The mean–variance portfolio selection problem with short-selling prohibition exemplifies such problems. Other applications include models in medical, chemical, and economics where system inputs are inherently constrained.

There have been some attempts in dealing with *deterministic* LQ problems with positive controls, or more general, with controls contained in a given cone. For example, controllability for linear system $\dot{x} = Ax + Bu$ with positive/conic controls was studied in [26, 7, 25, 14, 8, 22]. These papers investigated the necessary and sufficient conditions of different types of controllability (null-controllability, global controllability, differential controllability, etc.). Later, conic stabilization was addressed in [23]. In a recent work on positive feedback stabilization [30], a stabilizing positive feedback controller was derived based on the pole placement technique.

Deterministic continuous-time LQ optimal control problems with positive controllers were studied in [21, 9, 13]. Discrete-time versions can be found in [27, 28]. In these works, however, only some necessary and sufficient conditions for optimality were derived based on Pontryagin's maximum principle and/or Bellman's dynamic programming, and some numerical schemes suggested. The special LQ structure was not fully taken advantage of and no explicit result comparable to those of unconstrained control was obtained.

As for the constrained *stochastic* LQ control, to our best knowledge it has never been studied by other authors in the literature. A related, albeit specific, problem is the mean–portfolio selection model with no-shorting constraint solved in Li, Zhou and Lim [18], which was formulated as a stochastic LQ control problem with positive controls in a finite time horizon. The approach developed in [18] is nevertheless rather *ad hoc* (via Hamilton–Jacobi–Bellman equation and viscosity solution theory), and by no means suggests a remedy for a more general problem. More recently, a stochastic LQ control problem with conic control constraint, random coefficients as well as possibly singular cost weighting matrices in a finite time horizon was solved by Hu and Zhou [15], with explicit solutions based on Tanaka's formula and the backward stochastic differential equation theory.

In this paper, we study stochastic LQ control in the infinite time horizon, where the control variable is constrained in a cone (which includes the problem with positive controls as a special case). Moreover, the problem is allowed to be indefinite in the sense that the cost weighting matrices are possibly indefinite. A main assumption of the paper is that the state variable is scalar-valued. Note that this assumption is valid in many meaningful practical applications, in particular in financial area where the one-dimensional wealth process is typically taken as the state. The investigation in this paper centers around several key issues associated with the problem, viz conic stabilizability, well-posedness and optimality. The conic stabilizability refers to the question whether the system can be stabilized by a control satisfying the given conic constraint. It arises from the infinity of the time horizon under consideration, and is quite different from the normal stabilizability for unconstrained control. In this paper we will derive simple necessary and sufficient conditions for the conic stabilizability. The second issue, well-posedness of the LQ problem, becomes an *issue* because the problem is indefinite. To ensure the well-posedness the problem data must coordinate well, which will be characterized in this paper by the nonemptiness of certain sets in the real space. Finally, for the optimality, we aim to obtain *explicit* solutions comparable to those classical unconstrained-control counterparts. To this end, we will introduce, for the first time in this paper, two algebraic equations termed the extended algebraic Riccati equations (EARE's) along with the notion of their stabilizing solutions. Then it will be shown that the existence of the stabilizing solutions is *sufficient* for the existence of optimal feedback control of the constrained LQ problem, and explicit forms of the optimal feedback control as well as the optimal cost value will be derived in terms of the stabilizing solutions. Furthermore, several important cases, including that of the definite LQ control, will be discussed where stabilizing solutions to the EARE's do exist, and algorithms for computing these solutions will be presented. To demonstrate the theoretical results obtained, numerical examples will be given at last.

The rest of the paper is organized as follows. In Section 2, the constrained stochastic LQ control problem is formulated and conic stabilizability of the system defined. As the prelude to the main analysis, two technical lemmas are presented in Section 3. The subsequent three sections, Sections 4, 5, and 6, are devoted to the three major issues, namely stabilizability, well-posedness, and optimality, respectively. Numerical examples are reported in Section 7. Finally, Section 8 concludes the paper

2 Problem Formulation

Notation: We make use of the following basic notation in this paper:

$\mathbf{R}^n$	:	the set of n-dimensional column vectors.
$\mathbf{R}$	:	$= \mathbf{R}^1$.
$\mathbf{R}^n_+$	:	the subset of $\mathbf{R}^n$ consisting of elements with nonnegative components.
$\mathbf{R}^{m\times n}$	:	the set of all $m \times n$ matrices.
$\mathbf{S}^{n\times n}$	:	the set of all $n \times n$ symmetric matrices.
$\lvert v\rvert$	:	$= \sqrt{v'v}, \quad v \in \mathbf{R}^n$.
x^+	:	$= \max\{x, 0\}, \quad x \in \mathbf{R}$.
x^-	:	$= \max\{-x, 0\}, \quad x \in \mathbf{R}$.
M'	:	the transpose of a matrix M.
$M > 0$	:	the square matrix M is positive definite.
$M \geq 0$	:	the square matrix M is positive semidefinite.
$\mathbf{E}[x]$	:	the expectation of a random variable x.

Let $(\Omega, \mathcal{F}, \mathbf{P}; \mathcal{F}_t)$ be a given filtered probability space with a standard $\mathcal{F}_t$-adapted, k-dimensional Brownian motion $w(t) \equiv (w_1(t), w_2(t), \cdots, w_k(t))'$ on $[0, +\infty)$. Let $\Gamma \subseteq \mathbf{R}^m$ be a given closed cone; i.e., Γ is closed, and if $u \in \Gamma$ then $\alpha u \in \Gamma \ \forall \alpha \geq 0$. Typical examples of such a cone are $\Gamma = \mathbf{R}^m_+$, $\Gamma = \{u \in \mathbf{R}^m | Mu \leq 0\}$ and $\Gamma = \{u \in \mathbf{R}^m | Mu = 0\}$ where $M \in \mathbf{R}^{n\times m}$, or the so-called second-order cone (cf., e.g., [20, p.221]) $\Gamma = \{(t, u) \in \mathbf{R} \times \mathbf{R}^{m-1} | t \geq |u|\}$. Next, for $M \in \mathbf{S}^{m\times m}$ if there is $\delta > 0$ so that $K'MK \geq \delta |K|^2 \ \forall 0 \neq K \in \Gamma$, then we denote $M\Big|_\Gamma > 0$. Similarly we write $M\Big|_\Gamma \geq 0$ if $K'MK \geq 0 \ \forall K \in \Gamma$. Clearly $M > 0$ (respectively $M \geq 0$) implies $M\Big|_\Gamma > 0$ (respectively $M\Big|_\Gamma \geq 0$). Finally, define the following Hilbert space:

$$L^2_{\mathcal{F}}(\Gamma) = \{\phi(\cdot,\cdot) : \quad [0, +\infty) \times \Omega \to \Gamma \Big| \phi(\cdot,\cdot) \text{ is } \mathcal{F}_t-\text{adapted, measurable,} \\ \text{and } \mathbf{E}\int_0^{+\infty} |\phi(t, \omega)|^2 dt < +\infty\}$$

with the norm $\|\phi(\cdot,\cdot)\| := (\mathbf{E}\int_0^{+\infty} |\phi(t,\omega)|^2 dt)^{\frac{1}{2}}$.

Consider the following Itô stochastic differential equation (SDE)

$$\begin{cases} dx(t) = [Ax(t) + Bu(t)]dt + \sum_{j=1}^{k}[C_j x(t) + D_j u(t)]dw_j(t), \quad t \in [0,\infty), \\ x(0) = x_0 \in \mathbf{R}, \end{cases} \tag{1}$$

where $A, C_j \in \mathbf{R}$ and $B, D_j \in \mathbf{R}^{1\times m}$. A process $u(\cdot)$ is called a control (with conic constraint) if $u(\cdot) \in L^2_{\mathcal{F}}(\Gamma)$.

Definition 2.1 A control $u(\cdot) \in L^2_{\mathcal{F}}(\Gamma)$ is called (mean–square, conic) stabilizing with respect to x_0 if the corresponding state $x(\cdot)$ of (1) with the initial state x_0 satisfies $\lim_{t\to+\infty} \mathbf{E}|x(t)|^2 = 0$.

Definition 2.2 System (1) is said to be (mean–square, conic) stabilizable if there is a feedback control of the form $u(t) = K_+ x^+(t) + K_- x^-(t)$, where K_+ and K_- are constant vectors with $K_+ \in \Gamma$ and $K_- \in \Gamma$, which is (mean–square, conic) stabilizing with respect to *every* initial state x_0.

Now, for any $x_0 \in \mathbf{R}$, we define the set of admissible controls

$$\mathcal{U}_{x_0} := \{u(\cdot) \in L^2_{\mathcal{F}}(\Gamma) | u(\cdot) \text{ is stabilizing with respect to } x_0\}. \tag{2}$$

If $u(\cdot) \in \mathcal{U}_{x_0}$ and $x(\cdot)$ is the corresponding solution of (1), then $(x(\cdot), u(\cdot))$ is called an admissible pair (with respect to x_0). For each $(x_0, u(\cdot)) \in \mathbf{R} \times \mathcal{U}_{x_0}$ the associated cost to system (1) is

$$J(x_0; u(\cdot)) := \mathbf{E}\int_0^{+\infty} [Qx(t)^2 + u(t)' R u(t)]dt, \tag{3}$$

where $Q \in \mathbf{R}$ and $R \in \mathbf{S}^{m\times m}$. Note here Q and R are *not* assumed to be nonnegative/positive semidefinite. As a result, $J(x_0; u(\cdot))$ is not necessarily bounded below.

The indefinite LQ control problem with conic constraint is to minimize the cost functional (3), for a given x_0, subject to (1) and $u(\cdot) \in \mathcal{U}_{x_0}$. Such a problem is denoted as Problem (LQ). An admissible control $u(\cdot) \in \mathcal{U}_{x_0}$ is called optimal (with respect to x_0) if $u(\cdot)$ achieves the infimum of (3), and in this case Problem (LQ) is also referred to as attainable (with respect to x_0).

The value function V is defined as

$$V(x_0) := \inf_{u(\cdot)\in\mathcal{U}_{x_0}} J(x_0; u(\cdot)), \quad x_0 \in \mathbf{R}, \tag{4}$$

where $V(x_0)$ is set to be $+\infty$ in the case when $\mathcal{U}_{x_0}$ is empty.

Definition 2.3 Problem (LQ) is called well-posed if

$$V(x_0) > -\infty, \quad \forall x_0 \in \mathbf{R}. \tag{5}$$

It is well known that V is a continuous, though not necessarily differentiable, function when Problem (LQ) is well-posed. Also note that a well-posed problem is not necessarily attainable with respect to any x_0 (see Example 6.2 in Section 6).

3 Two Lemmas

In this section we present two lemmas that are useful in the sequel.

Lemma 3.1 (Tanaka's formula) *Let $X(t)$ be a continuous semimartingale. Then*

$$\begin{array}{rcl} dX^+(t) & = & 1_{(X(t)>0)}dX(t) + \frac{1}{2}dL(t), \\ dX^-(t) & = & -1_{(X(t)\le 0)}dX(t) + \frac{1}{2}dL(t), \end{array} \tag{6}$$

where $L(\cdot)$ is an increasing continuous process, called the local time of $X(\cdot)$ at 0, satisfying

$$\int_0^t |X(s)|dL(s) = 0, \quad \mathbf{P} - \text{a.s.} \tag{7}$$

In particular, $X^+(t)$ and $X^-(t)$ are semimartingales.

Proof : See, for example, [24, Chapter VI, Theorem 1.2 and Proposition 1.3]. □

Lemma 3.2 *Let constants $N_+, N_- \in \mathbf{R}$ be given. Then for any admissible pair $(x(\cdot), u(\cdot))$ with respect to x_0, we have, for every $t \ge 0$,*

$$\begin{array}{rl} & \mathbf{E}\int_0^t [Qx(s)^2 + u(s)'Ru(s)]ds \\ = & N_+(x_0^+)^2 + N_-(x_0^-)^2 - \mathbf{E}[N_+x^+(t)^2] - \mathbf{E}[N_-x^-(t)^2] \\ & +\mathbf{E}\int_0^t \Big\{Qx(s)^2 + (2A + \sum_{j=1}^k C_j^2)N_+x^+(s)^2 + (2A + \sum_{j=1}^k C_j^2)N_-x^-(s)^2 \\ & +u(s)'[R + 1_{(x(s)>0)}N_+ \sum_{j=1}^k D_j'D_j + 1_{(x(s)\le 0)}N_- \sum_{j=1}^k D_j'D_j]u(s) \\ & +2(B + \sum_{j=1}^k C_jD_j)u(s)N_+x^+(s) - 2(B + \sum_{j=1}^k C_jD_j)u(s)N_-x^-(s)\Big\}ds. \end{array} \tag{8}$$

Proof : Let $x(\cdot)$ be the solution of (1) under an arbitrary $u(\cdot) \in \mathcal{U}_{x_0}$. By Lemma 3.1, we have

$$dx^+(t) = 1_{(x(t)>0)}[Ax(t) + Bu(t)]dt + 1_{(x(t)>0)}\sum_{j=1}^k [C_jx(t) + D_ju(t)]dw_j(t) + \frac{1}{2}dL(t),$$
$$dx^-(t) = -1_{(x(t)\le 0)}[Ax(t) + Bu(t)]dt - 1_{(x(t)\le 0)}\sum_{j=1}^k [C_jx(t) + D_ju(t)]dw_j(t) + \frac{1}{2}dL(t).$$

In the above equations, $L(\cdot)$ is the local time as specified in Lemma 3.1. Applying Itô's formula, we get

$$
\begin{aligned}
& d[N_+x^+(t)^2] \\
= \; & 2N_+x^+(t)\{1_{(x(t)>0)}[Ax(t)+Bu(t)]dt + 1_{(x(t)>0)}\sum_{j=1}^{k}[C_jx(t)+D_ju(t)]dw_j(t) + \frac{1}{2}dL(t)\} \\
& +1_{(x(t)>0)}N_+\sum_{j=1}^{k}[C_jx(t)+u(t)'D_j'][C_jx(t)+D_ju(t)]dt \\
= \; & \{N_+[2Ax^+(t)^2+2Bu(t)x^+(t)+1_{(x(t)>0)}\sum_{j=1}^{k}(C_jx(t)+u(t)'D_j')(C_jx(t)+D_ju(t))]\}dt \\
& +N_+\sum_{j=1}^{k}[2C_jx^+(t)^2+2D_ju(t)x^+(t)]dw_j(t) \\
= \; & [(2A+\sum_{j=1}^{k}C_j^2)N_+x^+(t)^2+2Bu(t)N_+x^+(t)+2\sum_{j=1}^{k}C_jD_ju(t)N_+x^+(t) \\
& +1_{(x(t)>0)}N_+u(t)'\sum_{j=1}^{k}D_j'D_ju(t)]dt+[N_+(t)\sum_{j=1}^{k}(2C_jx^+(t)^2+2D_ju(t)x^+(t))]dw_j(t),
\end{aligned}
\tag{9}
$$

where we have used the fact that $x^+(t)dL(t)=0$ by virtue of (7). Similarly, for the constant $N_- \in \mathbf{R}$,

$$
\begin{aligned}
& d[N_-x^-(t)^2] \\
= \; & \{N_-[2Ax^-(t)^2-2Bu(t)x^-(t)+1_{(x(t)\le 0)}\sum_{j=1}^{k}(C_jx(t)+u(t)'D_j')(C_jx(t)+D_ju(t))]\}dt \\
& +N_-\sum_{j=1}^{k}[2C_jx^-(t)^2-2D_ju(t)x^-(t)]dw_j(t) \\
= \; & [(2A+\sum_{j=1}^{k}C_j^2)N_-x^-(t)^2-2Bu(t)P_-x^-(t)-2\sum_{j=1}^{k}C_jD_ju(t)N_-x^-(t) \\
& +1_{(x(t)\le 0)}N_-u(t)'\sum_{j=1}^{k}D_j'D_ju(t)]dt+[N_-(t)\sum_{j=1}^{k}(2C_jx^-(t)^2-2D_ju(t)x^-(t))]dw_j(t).
\end{aligned}
\tag{10}
$$

Fix $t \ge 0$, define a sequence of stopping times

$$
\begin{aligned}
\tau_n := \; & \inf\{r\in[0,t] : \int_0^r |N_+(s)\sum_{j=1}^{k}(2C_jx^+(s)^2+2D_ju(s)x^+(s))|^2ds \\
& +\int_0^r |N_-(s)\sum_{j=1}^{k}(2C_jx^-(s)^2-2D_ju(s)x^-(s))|^2ds \ge n\}, \quad n=1,2,...,
\end{aligned}
$$

where $\inf\emptyset := t$. It is clear that $\tau_n \uparrow t$ as $n\to+\infty$, due to $E\int_0^t(|x(s)|^2+|u(s)|^2)ds<+\infty$. Now, summing up (9) and (10), taking integration from 0 to τ_n and then taking expectation, we obtain (8) with t replaced by τ_n. Thus, (8) follows by sending $n\to+\infty$ together with Fatou's lemma. □

4 Conic Stabilizability

In this section we address the issue of conic stabilizability of system (1). Notice that the conic stabilizability is different from the usual stabilizability with unconstrained controls, for clearly the former requires more stringent conditions. Here we will give a *complete* characterization of the conic stabilizability in terms of simple conditions involving linear matrix inequalities (LMI's).

Introduce a pair of functions F_+, F_- from Γ to $\mathbf{R}$:

$$F_+(K) := 2A + \sum_{j=1}^{k} C_j^2 + 2(B + \sum_{j=1}^{k} C_j D_j)K + K' \sum_{j=1}^{k} D_j' D_j K, \tag{11}$$

and

$$F_-(K) := 2A + \sum_{j=1}^{k} C_j^2 - 2(B + \sum_{j=1}^{k} C_j D_j)K + K' \sum_{j=1}^{k} D_j' D_j K. \tag{12}$$

Theorem 4.1 *The following assertions are equivalent.*

(i) *System (1) is mean-square conic stabilizable.*

(ii) *There exist $K_+ \in \Gamma$ and $K_- \in \Gamma$ such that $F_+(K_+) < 0$ and $F_-(K_-) < 0$. In this case the feedback control $u(t) = K_+ x^+(t) + K_- x^-(t)$ is stabilizing.*

(iii) *There exist $K_+ \in \Gamma$ and $K_- \in \Gamma$ such that*

$$\begin{pmatrix} 2A + \sum_{j=1}^{k} C_j^2 + 2(B + \sum_{j=1}^{k} C_j D_j)K_+ & K_+' D' \\ DK_+ & -I \end{pmatrix} < 0, \tag{13}$$

$$\begin{pmatrix} 2A + \sum_{j=1}^{k} C_j^2 - 2(B + \sum_{j=1}^{k} C_j D_j)K_- & K_-' D' \\ DK_- & -I \end{pmatrix} < 0, \tag{14}$$

where $D \in \mathbf{R}^{m \times m}$ satisfies $D'D = \sum_{j=1}^{k} D_j' D_j$. In this case the feedback control $u(t) = K_+ x^+(t) + K_- x^-(t)$ is stabilizing.

Proof : Take a feedback control $u(t) = K_+ x^+(t) + K_- x^-(t)$ and consider the corresponding state $x(\cdot)$ with an initial state x_0. Note that by standard SDE theory (cf., e.g., [16]) such $x(\cdot)$ uniquely exists. Moreover,

$$\mathbf{E} \sup_{0 \le t \le T} |x(t)|^p dt < +\infty, \quad \forall T > 0, \quad \forall p \ge 0. \tag{15}$$

Making use of (9) with $N_+ = 1$ and $u(t) = K_+ x^+(t) + K_- x^-(t)$, we obtain

$$\begin{aligned} & dx^+(t)^2 \\ = \; & [2A + \sum_{j=1}^{k} C_j^2 + 2(B + \sum_{j=1}^{k} C_j D_j)K_+ + K_+' \sum_{j=1}^{k} D_j' D_j K_+] x^+(t)^2 dt \\ & + \sum_{j=1}^{k} (2C_j + 2D_j K_+) x^+(t)^2 dw_j(t) \\ = \; & F_+(K_+) x^+(t)^2 dt + \sum_{j=1}^{k} (2C_j + 2D_j K_+) x^+(t)^2 dw_j(t). \end{aligned}$$

Taking integration and then expectation yields (after a localization argument as in the proof of Lemma 3.2)

$$\frac{d\mathbf{E}[x^+(t)^2]}{dt} = F_+(K_+) \mathbf{E}[x^+(t)^2], \tag{16}$$

where the expectation of the Itô integral vanishes due to (15). Similarly, we have

$$\frac{d\mathbf{E}[x^-(t)^2]}{dt} = F_-(K_-)\mathbf{E}[x^-(t)^2]. \tag{17}$$

Hence, the equivalence between the assertions (i) and (ii) is evident noting that $\lim_{t\to+\infty} \mathbf{E}|x(t)|^2 = 0$ if and only if $\lim_{t\to+\infty} \mathbf{E}|x^+(t)|^2 = 0$ and $\lim_{t\to+\infty} \mathbf{E}|x^-(t)|^2 = 0$.

Finally, the equivalence between the assertions (ii) and (iii) follows from Schur's lemma. $\square$

Conditions (13) and (14) are in terms of LMI's. On the other hand, some of the conic constraint can also be expressed as LMI's, e.g., when $\Gamma = \mathbf{R}_+^m$ or when Γ is a second-order cone. Hence, Theorem 4.1-(iii) provides an easy way of numerically checking the stabilizability of system (1) due to the availability of many LMI solvers. Note that in general there may exist many feasible solutions to LMI's. Denote the two sets of feedback gains as

$$\mathcal{K}_+ := \{K \in \Gamma \big| F_+(K) < 0\}, \ \ \mathcal{K}_- := \{K \in \Gamma \big| F_-(K) < 0\}.$$

Then Theorem 4.1 implies the following

Theorem 4.2 *System (1) is conic stabilizable if and only if* $\mathcal{K}_+ \neq \emptyset$ *and* $\mathcal{K}_- \neq \emptyset$.

Corollary 4.1 *If* $2A + \sum_{j=1}^{k} C_j^2 < 0$*, then system (1) is conic stabilizable.*

Proof : In this case $0 \in \mathcal{K}_+ \cap \mathcal{K}_-$. $\square$

Proposition 4.1 *We have the following results.*

(i) $\mathcal{K}_+$ *and* $\mathcal{K}_-$ *are convex sets if* Γ *is a convex set.*

(ii) $\mathcal{K}_+$ *and* $\mathcal{K}_-$ *are bounded if* $\sum_{j=1}^{k} D_j' D_j \Big|_\Gamma > 0$.

Proof : (i) Define the following operator M_+ from Γ to $\mathbf{S}^{(m+1)\times(m+1)}$:

$$M_+(K) = \begin{pmatrix} 2A + \sum_{j=1}^{k} C_j^2 + 2(B + \sum_{j=1}^{k} C_j D_j)K & K'D' \\ DK & -I \end{pmatrix},$$

where $D'D = \sum_{j=1}^{k} D_j' D_j$. By Theorem 4.1, $\mathcal{K}_+$ can be equivalently represented as $\mathcal{K}_+ = \{K \in \Gamma \big| M_+(K) < 0\}$. Thus the convexity of $\mathcal{K}_+$ follows from that of Γ together with the fact that the operator M_+ is affine. Similarly we can show the convexity of $\mathcal{K}_-$.

(ii) If $\mathcal{K}_+$ is unbounded, then there is a sequence $\{K_n\} \subset \mathcal{K}_+$ so that $|K_n| \to +\infty$. Since $\sum_{j=1}^{k} D_j' D_j \Big|_\Gamma > 0$, we have $K_n' \sum_{j=1}^{k} D_j' D_j K_n \geq \delta |K_n|^2 \to +\infty$ as $n \to +\infty$, where $\delta > 0$ is some constant. Therefore $F_+(K_n) \to +\infty$ as $n \to +\infty$. This contradicts to $F_+(K_n) < 0$, $\forall n$. Hence $\mathcal{K}_+$ is bounded. Similarly, $\mathcal{K}_-$ is bounded. $\square$

5 Well-posedness

Since the cost weighting matrices Q and R are allowed to be indefinite, the well-posedness of the problem is no longer automatic or trivial (as opposed to the classical definite case when $Q \geq 0$ and $R > 0$). In fact, the well-posedness for an indefinite LQ control problem is a prerequisite for the optimality, and is an interesting problem in its own right. In this section we will carry out an extensive investigation on the well-posedness and some related issues, including necessary and sufficient conditions for the well-posedness in terms of the nonemptiness of certain sets.

5.1 Representation of Value Function

In this subsection we present the following representation result, which is a key to many results of this paper.

Proposition 5.1 *Problem (LQ) is well-posed if and only if the value function can be represented as*

$$V(x_0) = P_+(x_0^+)^2 + P_-(x_0^-)^2, \quad \forall x_0 \in \mathbf{R}, \tag{18}$$

for some $P_+, P_- \in \mathbf{R}$.

Proof : We prove only the "only if" part as the "if" part is evident.

Assume that Problem (LQ) is well-posed. Fix any $x > 0, y > 0$. Since $V(y) > -\infty$, for any $\varepsilon > 0$ there is $u^\varepsilon(\cdot) \in \mathcal{U}_y$ along with the corresponding state $x^\varepsilon(\cdot)$ (with the initial state y) satisfying

$$V(y) \geq J(y; u^\varepsilon(\cdot)) - \varepsilon = \mathbf{E} \int_0^{+\infty} [Qx^\varepsilon(t)^2 + u^\varepsilon(t)' R u^\varepsilon(t)] dt - \varepsilon. \tag{19}$$

Now, as $x > 0$, the linearity of the dynamics (1) and the conic control constraint ensure that $xu^\varepsilon(\cdot) \in \mathcal{U}_{xy}$ with the corresponding state $xx^\varepsilon(\cdot)$. Hence it follows from (19) that

$$V(y) \geq \frac{1}{x^2} \mathbf{E} \int_0^{+\infty} [Q|xx^\varepsilon(t)|^2 + (xu^\varepsilon(t))' R (xu^\varepsilon(t))] dt - \varepsilon \geq \frac{1}{x^2} V(xy) - \varepsilon. \tag{20}$$

Sending $\varepsilon \to 0$ we obtain

$$V(xy) \leq V(y) x^2, \quad \forall x > 0, y > 0. \tag{21}$$

Similarly, one can show that

$$V(xy) \leq V(-y) x^2, \quad \forall x < 0, y > 0. \tag{22}$$

Now for any $x > 0$, by (21) we have $V(x) \leq V(1)x^2$. On the other hand, (21) also implies $V(1) = V(x\frac{1}{x}) \leq \frac{1}{x^2} V(x)$. So we have shown that

$$V(x) = V(1)x^2, \quad \forall x > 0. \tag{23}$$

Similarly, in view of (22) we can prove that

$$V(x) = V(-1)x^2, \quad \forall x < 0. \tag{24}$$

Finally, the continuity of the value function along with (23) yields

$$V(0) = 0. \tag{25}$$

The desired result (18) thus follows from (23)–(25) with $P_+ := V(1)$ and $P_- := V(-1)$. □

Remark 5.1 The preceding proposition, which suggests the form of the value function when the underlying LQ problem is well-posed, is crucial for all the main results in this paper. In fact, the proofs for the stabilizability in the previous section, the characterization of the well-posedness in this section, and the optimality in the next section, are *all* inspired by this result. This also explains why one needs to apply Tanaka's formula to evaluate $dx^+(t)$ and $dx^-(t)$ as we have seen in the previous section and will continue to see in the subsequent sections.

Remark 5.2 We saw that the value function for the constrained LQ problem is *not* smooth.

5.2 Characterization of Well-posedness

Define the following functions from $\mathbf{R}$ to $\mathbf{R} \cup \{-\infty\}$:

$$\Phi_+(P) := \inf_{K\in\Gamma} \big[K'(R + P\sum_{j=1}^{k} D_j'D_j)K + 2(B + \sum_{j=1}^{k} C_jD_j)PK\big],$$
$$\Phi_-(P) := \inf_{K\in\Gamma} \big[K'(R + P\sum_{j=1}^{k} D_j'D_j)K - 2(B + \sum_{j=1}^{k} C_jD_j)PK\big].$$

Remark 5.3 Since $0 \in \Gamma$, we must have

$$\Phi_+(P) \le 0, \quad \Phi_-(P) \le 0, \ \ \forall P \in \mathbf{R}. \tag{26}$$

On the other hand, $\Phi_+(P)$ and $\Phi_-(P)$ have finite values if $(R + P\sum_{j=1}^{k} D_j'D_j)\Big|_{\Gamma} > 0$. Indeed, in this case there exist constants $\alpha_1 = \alpha_1(P) > 0$ and $\alpha_2 = \alpha_2(P) > 0$ such that

$$K'(R + P\sum_{j=1}^{k} D_j'D_j)K + 2(B + \sum_{j=1}^{k} C_jD_j)PK \ge \alpha_1|K|^2 - \alpha_2|K| = \alpha_1|K|(|K| - \frac{\alpha_2}{\alpha_1}).$$

If $|K| > \dfrac{\alpha_2}{\alpha_1}$, then the above expression is positive. Taking (26) into consideration we conclude

$$\Phi_+(P) = \inf_{K\in\Gamma, |K|\le\frac{\alpha_2}{\alpha_1}} \big[K'(R + P\sum_{j=1}^{k} D_j'D_j)K + 2(B + \sum_{j=1}^{k} C_jD_j)PK\big] > -\infty.$$

Hence, $\Phi_+(P)$ is finite. The same is true for $\Phi_-(P)$.

Next, we define the following two sets:

$$\mathcal{P}_+ := \{P \in \mathbf{R} \,\Big|\, (2A + \sum_{j=1}^{k} C_j^2)P + Q + \Phi_+(P) \ge 0, \ (R + P\sum_{j=1}^{k} D_j'D_j)\Big|_{\Gamma} \ge 0\},$$
$$\mathcal{P}_- := \{P \in \mathbf{R} \,\Big|\, (2A + \sum_{j=1}^{k} C_j^2)P + Q + \Phi_-(P) \ge 0, \ (R + P\sum_{j=1}^{k} D_j'D_j)\Big|_{\Gamma} \ge 0\}.$$

The following is the main result of the section, which characterizes the well-posedness of Problem (LQ) by the nonemptiness of the sets $\mathcal{P}_+$ and $\mathcal{P}_-$.

Theorem 5.1 *Assume that system (1) is conic stabilizable. Then Problem (LQ) is well-posed if and only if $\mathcal{P}_+ \ne \emptyset$ and $\mathcal{P}_- \ne \emptyset$. Moreover, in this case*

$$V(x_0) \ge P_+(x_0^+)^2 + P_-(x_0^-)^2, \quad \forall x_0 \in \mathbf{R}, \quad \forall P_+ \in \mathcal{P}_+, \ \ P_- \in \mathcal{P}_-. \tag{27}$$

Proof : First we prove the "if" part. For any $x_0 \in \mathbf{R}$ let $x(\cdot)$ be the solution of (1) under an arbitrary $u(\cdot) \in \mathcal{U}_{x_0}$. Pick any $P_+ \in \mathcal{P}_+$ and $P_- \in \mathcal{P}_-$. By Lemma 3.2, we have

$$\begin{aligned}
& \mathbf{E}\int_0^t [Qx(s)^2 + u(s)'Ru(s)]ds \\
= \; & P_+(x_0^+)^2 + P_-(x_0^-)^2 - \mathbf{E}[P_+x^+(t)^2] - \mathbf{E}[P_-x^-(t)^2] \\
& +\mathbf{E}\int_0^t \Big\{Qx(s)^2 + (2A+\sum_{j=1}^k C_j^2)P_+x^+(s)^2 + (2A+\sum_{j=1}^k C_j^2)P_-x^-(s)^2 \\
& +u(s)'[R + 1_{(x(s)>0)}P_+\sum_{j=1}^k D_j'D_j + 1_{(x(s)\le 0)}P_-\sum_{j=1}^k D_j'D_j]u(s) \\
& +2(B+\sum_{j=1}^k C_jD_j)u(s)P_+x^+(s) - 2(B+\sum_{j=1}^k C_jD_j)u(s)P_-x^-(s)\Big\}ds.
\end{aligned} \tag{28}$$

Denote by $\psi(x(s),u(s))$ the integrand on the right hand side of (28) and fix $s \in [0,t]$. If $x(s) > 0$, then write $u(s) = Kx(s)$ (note that K may depend on s). Since $u(s) \in \Gamma$ and Γ is a cone, we have $K \in \Gamma$. Hence at s, bearing in mind that $1_{(x(s)\le 0)} = 0$, we have

$$\begin{aligned}
& \psi(x(s),u(s)) \\
= \; & Qx(s)^2 + (2A+\sum_{j=1}^k C_j^2)P_+x(s)^2 + K'[R + P_+\sum_{j=1}^k D_j'D_j]Kx(s)^2 \\
& +2(B+\textstyle\sum_{j=1}^k C_jD_j)KP_+x(s)^2 \\
= \; & [(2A+\sum_{j=1}^k C_j^2)P_+ + Q + K'(R+P_+\sum_{j=1}^k D_j'D_j)K + 2(B+\sum_{j=1}^k C_jD_j)KP_+]x(s)^2 \\
\ge \; & [(2A+\sum_{j=1}^k C_j^2)P_+ + Q + \Phi_+(P_+)]x(s)^2 \\
\ge \; & 0.
\end{aligned} \tag{29}$$

If $x(s) < 0$, then write $u(s) = -Kx(s)$. Again $K \in \Gamma$. A similar argument as above yields

$$\psi(x(s),u(s)) \ge [(2A+\sum_{j=1}^k C_j^2)P_- + Q + \Phi_-(P_-)]x(s)^2 \ge 0$$

at s. Finally, if $x(s) = 0$ at s, then

$$\psi(x(s),u(s)) = u(s)'[R + P_-\sum_{j=1}^k D_j'D_j]u(s) \ge 0.$$

The preceding analysis shows that it always holds that $\psi(x(s),u(s)) \ge 0$, $\forall s \in [0,t]$. Consequently, it follows from (28) that

$$\mathbf{E}\int_0^t [Qx(s)^2 + u(s)'Ru(s)]ds \ge P_+(x_0^+)^2 + P_-(x_0^-)^2 - \mathbf{E}[P_+x^+(t)^2] - \mathbf{E}[P_-x^-(t)^2].$$

Letting $t \to +\infty$ and noting that $u(\cdot)$ is conic stabilizing, we obtain

$$\begin{aligned}
J(x_0;u(\cdot)) &= \lim_{t\to+\infty}\mathbf{E}\int_0^t [Qx(s)^2 + u(s)'Ru(s)]ds \\
&\ge P_+(x_0^+)^2 + P_-(x_0^-)^2.
\end{aligned}$$

Since $u(\cdot) \in \mathcal{U}_{x_0}$ is arbitrary, we conclude $V(x_0) \geq P_+(x_0^+)^2 + P_-(x_0^-)^2 > -\infty$. Hence Problem (LQ) is well-posed.

To prove the "only if" part, suppose that the LQ problem is well-posed. Then by Proposition 5.1 the value function has the following representation:

$$V(x) = P_+(x^+)^2 + P_-(x^-)^2, \quad \forall x \in \mathbf{R}.$$

We want to show that $P_+ \in \mathcal{P}_+$, $P_- \in \mathcal{P}_-$. To this end, applying the optimality principle of the dynamic programming and noting the time-invariance of the underlying system, we obtain

$$\begin{array}{rl} & P_+(x_0^+)^2 + P_-(x_0^-)^2 \\ \leq & \mathbf{E}\left\{\int_0^h [Qx(t)^2 + u(t)'Ru(t)]dt + P_+[x^+(h)]^2 + P_-[x^-(h)]^2\right\}, \\ & \forall h > 0, \ \forall u(\cdot) \in \mathcal{U}_{x_0}, \ \forall x_0 \in \mathbf{R}. \end{array} \tag{30}$$

Using the above and applying Lemma 3.2, we obtain

$$\mathbf{E}\int_0^h \psi(x(s), u(s))ds \geq 0, \ \forall h > 0, \ \forall u(\cdot) \in \mathcal{U}_{x_0}, \ \forall x_0 \in \mathbf{R}, \tag{31}$$

where, as before, the mapping ψ is defined via the integrand on the right hand side of (28). Set $x_0 > 0$ and take the following control

$$u_1(t) := \left\{ \begin{array}{ll} K, & 0 \leq t < 1, \\ K_+x^+(t) + K_-x^-(t), & t \geq 1, \end{array} \right.$$

where $K \in \Gamma$ is arbitrarily fixed and $K_+x^+(t) + K_-x^-(t)$ is a conic stabilizing feedback control which exists due to the stabilizability assumption. Clearly $u_1(\cdot) \in \mathcal{U}_{x_0}$ and let $x_1(\cdot)$ be the corresponding state. Since $x_1(s) \to x_0$ and $u_1(s) \to K$ as $s \to 0$, $\mathbf{P}$-a.s., we have

$$\begin{array}{rl} & \psi(x_1(s), u_1(s)) \\ \to & [(2A + \sum_{j=1}^k C_j^2)P_+ + Q]x_0^2 + 2(B + \sum_{j=1}^k C_jD_j)KP_+x_0 + K'(R + P_+\sum_{j=1}^k D_j'D_j)K, \\ & \text{as } s \to 0, \quad \mathbf{P}-\text{a.s.} \end{array}$$

Thus appealing to (31) the dominated convergence theorem yields

$$\begin{array}{rl} 0 \leq & \frac{1}{h}\mathbf{E}\int_0^h \psi(x_1(s), u_1(s))ds \\ \to & [(2A + \sum_{j=1}^k C_j^2)P_+ + Q]x_0^2 + 2(B + \sum_{j=1}^k C_jD_j)KP_+x_0 + K'(R + P_+\sum_{j=1}^k D_j'D_j)K, \\ & \text{as } h \to 0. \end{array}$$

Letting $x_0 \to 0$ we obtain $K'(R + P_+\sum_{j=1}^k D_j'D_j)K \geq 0$. The arbitrariness of $K \in \Gamma$ then implies $(R + P_+\sum_{j=1}^k D_j'D_j)\Big|_\Gamma \geq 0$. On the other hand, take $x_0 > 0$ and consider the following feedback control

$$u_2(t) := \left\{ \begin{array}{ll} Kx^+(t) + \tilde{K}x^-(t), & 0 \leq t < 1, \\ K_+x^+(t) + K_-x^-(t), & t \geq 1, \end{array} \right.$$

where $K \in \Gamma$ and $\tilde{K} \in \Gamma$ are arbitrarily fixed. Let $x_2(\cdot)$ be the state under $u_2(\cdot)$. Noting $x_2(s) \to x_0$ and $u_2(s) = Kx_2^+(s) + \tilde{K}x_2^-(s) \to Kx_0$ as $s \to 0$, $\mathbf{P}$-a.s., we obtain

$$
\begin{aligned}
&\psi(x_2(s), u_2(s)) \\
\to\ &[(2A + \sum_{j=1}^{k} C_j^2)P_+ + Q + K'(R + P_+ \sum_{j=1}^{k} D_j'D_j)K + 2(B + \sum_{j=1}^{k} C_jD_j)KP_+]x_0^2 \\
&\text{as } s \to 0, \quad \mathbf{P} - \text{a.s.}
\end{aligned}
$$

An analysis similar to the preceding one leads to

$$
(2A + \sum_{j=1}^{k} C_j^2)P_+ + Q + K'(R + P_+ \sum_{j=1}^{k} D_j'D_j)K + 2(B + \sum_{j=1}^{k} C_jD_j)KP_+ \geq 0.
$$

Since $K \in \Gamma$ is arbitrary, we arrive at $(2A + \sum_{j=1}^{k} C_j^2)P_+ + Q + \Phi_+(P_+) \geq 0$. So far we have shown $P_+ \in \mathcal{P}_+$. Similarly, we can prove $P_- \in \mathcal{P}_-$.

Finally, the inequality (27) has been proved in the proof of the "if" part. □

Remark 5.4 From the above proof we see that the stabilizability assumption is in fact *not* necessary for the "if" part of the theorem.

Remark 5.5 The above theorem tells that the positive definiteness or positive semidefiniteness of Q and R are *not* necessary for Problem (LQ) to be well-posed.

Corollary 5.1 *If $R\big|_{\Gamma} \geq 0$ and $Q \geq 0$, then Problem (LQ) is well-posed.*

Proof : In this case $0 \in \mathcal{P}_+ \cap \mathcal{P}_-$. □

Proposition 5.2 *$\mathcal{P}_+$ and $\mathcal{P}_-$ are both convex sets (hence they are both intervals). Moreover, if system (1) is stabilizable and Problem (LQ) is well-posed, then $\mathcal{P}_+$ and $\mathcal{P}_-$ each has a finite maximum element.*

Proof : The convexity of $\mathcal{P}_+$ and $\mathcal{P}_-$ are clear noticing that the functions $\Phi_+(P)$ and $\Phi_-(P)$ are concave in P. To prove the existence of the finite maximum elements, we note that Proposition 5.1 provides

$$
V(x_0) = P_+^*(x_0^+)^2 + P_-^*(x_0^-)^2, \quad \forall x_0 \in \mathbf{R}, \tag{32}
$$

for some $P_+^*, P_-^* \in \mathbf{R}$. Moreover, the proof of Theorem 5.1 implies $P_+^* \in \mathcal{P}_+$ and $P_-^* \in \mathcal{P}_-$. Hence it follows from (27) that P_+^* and P_-^* are the maximum elements of $\mathcal{P}_+$ and $\mathcal{P}_-$ respectively. □

Remark 5.6 Proposition 5.2 indicates that if system (1) is stabilizable and Problem (LQ) is well-posed, then the infimum value of problem (LQ) or, equivalently, the P_+ and P_- in the representation of the value function as stipulated in Proposition 5.1, can be obtained by solving the following two mathematical programming problems respectively:

$$
\begin{aligned}
&\text{Maximize} \quad P \\
&\text{subject to} \quad \begin{cases} (2A + \sum_{j=1}^{k} C_j^2)P + Q + \Phi_+(P) \geq 0, \\ (R + P\sum_{j=1}^{k} D_j'D_j)\Big|_{\Gamma} \geq 0, \end{cases}
\end{aligned} \tag{33}
$$

and

$$\begin{array}{ll} \text{Maximize} & P \\ \text{subject to} & \begin{cases} (2A+\sum_{j=1}^{k} C_j^2)P + Q + \Phi_-(P) \geq 0, \\ (R + P\sum_{j=1}^{k} D_j' D_j)\Big|_\Gamma \geq 0. \end{cases} \end{array} \tag{34}$$

5.3 An Algorithm

Theorem 5.1 stipulates that it suffices to check the nonemptiness of $\mathcal{P}_+$ and $\mathcal{P}_-$ or, the feasibility of the problems (33) and (34), in order to verify the well-posedness of a given LQ problem. However, it is sometimes hard to check numerically the aforementioned feasibility because, on one hand, the functions $\Phi_+(\cdot)$ and $\Phi_-(\cdot)$ in general do not have analytical forms, and on the other hand, the constraint $(R + P\sum_{j=1}^{k} D_j'D_j)\Big|_\Gamma \geq 0$ is usually very hard to verify for a general cone Γ (except second-order cones for which the constraint can be reformulated as an LMI; see [29, Theorem 1]).

In this subsection we give an algorithm that can check the well-posedness more directly. First we need a lemma.

Lemma 5.1 *Assume that Problem (LQ) is well-posed. Given $K_+ \in \mathcal{K}_+$ and $K_- \in \mathcal{K}_-$. Set $\tilde{P}_+ := -\frac{Q+K_+'RK_+}{F_+(K_+)}$ and $\tilde{P}_- := -\frac{Q+K_-'RK_-}{F_-(K_-)}$. Then*

$$P_+ \leq \tilde{P}_+,\ \forall P_+ \in \mathcal{P}_+, \quad \textit{and} \quad P_- \leq \tilde{P}_-,\ \forall P_- \in \mathcal{P}_-. \tag{35}$$

Proof : By their definitions $\tilde{P}_+$ and $\tilde{P}_-$ satisfy respectively

$$(2A+\sum_{j=1}^{k} C_j^2)\tilde{P}_+ + Q + K_+'(R+\tilde{P}_+\sum_{j=1}^{k} D_j'D_j)K_+ + 2(B+\sum_{j=1}^{k} C_jD_j)K_+\tilde{P}_+ = 0 \tag{36}$$

and

$$(2A+\sum_{j=1}^{k} C_j^2)\tilde{P}_- + Q + K_-'(R+\tilde{P}_-\sum_{j=1}^{k} D_j'D_j)K_- - 2(B+\sum_{j=1}^{k} C_jD_j)K_-\tilde{P}_- = 0. \tag{37}$$

Take a feedback control $u(t) = K_+x^+(t) + K_-x^-(t)$, which is stabilizing by Theorem 4.1 (bearing in mind the definitions of $\mathcal{K}_+$ and $\mathcal{K}_-$), and let $x(\cdot)$ be the corresponding state with $x(0) = x_0$. Then a similar calculation as in (28) yields

$$\begin{aligned} & \mathbf{E}\int_0^t [Qx(s)^2 + u(s)'Ru(s)]ds \\ = \;& \tilde{P}_+(x_0^+)^2 + \tilde{P}_-(x_0^-)^2 - \mathbf{E}[\tilde{P}_+x^+(t)^2] - \mathbf{E}[\tilde{P}_-x^-(t)^2] + \mathbf{E}\int_0^t \psi(x(s),u(s))ds, \end{aligned} \tag{38}$$

where $\psi(x(s),u(s))$ is as the integrand on the right hand side of (28) with P_+ and P_- replaced by $\tilde{P}_+$ and $\tilde{P}_-$ respectively. However, $u(s) = K_+x(s)$ whenever $x(s) > 0$; hence

$$\begin{aligned} & \psi(x(s),u(s)) \\ = \;& [(2A+\sum_{j=1}^{k} C_j^2)\tilde{P}_+ + Q + K_+'(R+\tilde{P}_+\sum_{j=1}^{k} D_j'D_j)K_+ + 2(B+\sum_{j=1}^{k} C_jD_j)K\tilde{P}_+]x(s)^2 \\ = \;& 0 \end{aligned}$$

in view of (36). Similarly based on (37) one can show that $\psi(x(s), u(s)) = 0$ whenever $x(s) \le 0$. It then follows from (38) that

$$\mathbf{E}\int_0^t [Qx(s)^2 + u(s)'Ru(s)]ds = \tilde{P}_+(x_0^+)^2 + \tilde{P}_-(x_0^-)^2 - \mathbf{E}[\tilde{P}_+ x^+(t)^2] - \mathbf{E}[\tilde{P}_- x^-(t)^2].$$

Since $u(\cdot)$ is stabilizing, we have

$$J(x_0; u(\cdot)) = \lim_{t\to+\infty} \mathbf{E}\int_0^t [Qx(s)^2 + u(s)'Ru(s)]ds = \tilde{P}_+(x_0^+)^2 + \tilde{P}_-(x_0^-)^2.$$

By virtue of Theorem 5.1, we conclude

$$\begin{aligned} \tilde{P}_+(x_0^+)^2 + \tilde{P}_-(x_0^-)^2 &= J(x_0; u(\cdot)) \\ &\ge V(x_0) \ge P_+(x_0^+)^2 + P_-(x_0^-)^2, \ \forall P_+ \in \mathcal{P}_+, \forall P_- \in \mathcal{P}_-. \end{aligned}$$

This proves (35). □

We now assume that (1) is conic stabilizable. According to Theorem 4.1 there exist $K_+ \in \Gamma$ and $K_- \in \Gamma$ such that $F_+(K_+) < 0$ and $F_-(K_-) < 0$. Take $\delta > 0$ sufficiently small with $\delta < \min\{-F_+(K_+), -F_-(K_-)\}$. Calculate

$$P_+^{(1)} := \min_{F_+(K)\le -\delta} \Big\{ -\frac{Q + K'RK}{F_+(K)} \Big\}, \ \ P_-^{(1)} := \min_{F_-(K)\le -\delta} \Big\{ -\frac{Q + K'RK}{F_-(K)} \Big\}. \tag{39}$$

In view of Lemma 5.1, $P_+^{(1)}$ (respectively $P_-^{(1)}$) is a (very tight) upper bound of $\mathcal{P}_+$ (respectively $\mathcal{P}_-$) under the well-posedness assumption. As a consequence, if Problem (LQ) is well-posed, then it is necessary that $(R + P_+^{(1)} \sum_{j=1}^{k} D_j' D_j)\big|_\Gamma \ge 0$ and $(R + P_-^{(1)} \sum_{j=1}^{k} D_j' D_j)\big|_\Gamma \ge 0$. Now, calculate

$$P^{(0)} := \min_{(R + P\sum_{j=1}^k D_j' D_j)\big|_\Gamma \ge 0} P. \tag{40}$$

Then we know that points of $\mathcal{P}_+$ (respectively $\mathcal{P}_-$), if any, must lie between $P^{(0)}$ and $P_+^{(1)}$ (respectively $P_-^{(1)}$). Inspired by the above discussion, we have the following algorithm.

Step 1. Applying Theorem 4.1-(iii) to obtain $K_+, K_- \in \Gamma$ with $F_+(K_+) < 0$ and $F_-(K_-) < 0$. Set $\delta := \min\{\varepsilon, -F_+(K_+), -F_-(K_-)\}$, where ε is a very small number allowed by the computer.

Step 2. Calculate $P_+^{(1)}$ and $P_-^{(1)}$ via (39). If either $(R+P_+^{(1)} \sum_{j=1}^k D_j' D_j)\big|_\Gamma < 0$ or $(R+P_-^{(1)} \sum_{j=1}^k D_j' D_j)\big|_\Gamma < 0$ holds, stop and Problem (LQ) is not well-posed.

Step 3. Calculate $P^{(0)}$ via (40).

Step 4. If there exists a $P \in [P^{(0)}, P_+^{(1)})$ satisfying $L_+(P) \ge 0$, then go to Step 5; otherwise, stop and Problem (LQ) is not well-posed.

Step 5. If there exists a $P \in [P^{(0)}, P_-^{(1)})$ satisfying $L_-(P) \ge 0$, then stop and Problem (LQ) is well-posed; otherwise, stop and Problem (LQ) is not well-posed.

5.4 Well-posedness Margin

In view of Remark 5.5, Problem (LQ) may still be well-posed when R is indefinite or even negative definite. That said, it is clear that R cannot be *too* negative for the well-posedness. Therefore, it is interesting to study the *range* of R over which Problem (LQ) is well-posed, given that all the other data is fixed. Specifically, define

$$r^* := \inf\{r \in \mathbf{R} \mid \text{Problem (LQ) is well-posed for any } R \in \mathbf{S}^{m\times m} \text{ with } R > rI\}, \tag{41}$$

where $\inf \emptyset := +\infty$. The value r^* is called the well-posedness margin. By its very definition, r^* has the following interpretation: Problem (LQ) is well-posed if the smallest eigenvalue of R, $\lambda_{min}(R)$, is such that $\lambda_{min}(R) > r^*$, and is not well-posed if the largest eigenvalue of R, $\lambda_{max}(R)$, is such that $\lambda_{max}(R) < r^*$.

It follows from Theorem 5.1 that, provided that system (1) is stabilizable, the well-posedness margin r^* can be obtained by solving the following nonlinear program (with P and r being the decision variables):

$$\begin{array}{ll} \text{Minimize} & r \\ \text{subject to} & \left\{ \begin{array}{l} (2A + \sum_{j=1}^{k} C_j^2)P + Q + \Phi_+(P,r) \geq 0, \\ (2A + \sum_{j=1}^{k} C_j^2)P + Q + \Phi_-(P,r) \geq 0, \\ (rI + P\sum_{j=1}^{k} D_j'D_j)\Big|_\Gamma \geq 0, \end{array} \right. \end{array} \tag{42}$$

where

$$\Phi_+(P,r) := \inf_{K\in\Gamma} \big[K'(rI + P\sum_{j=1}^{k} D_j'D_j)K + 2(B + \sum_{j=1}^{k} C_jD_j)PK\big],$$
$$\Phi_-(P,r) := \inf_{K\in\Gamma} \big[K'(rI + P\sum_{j=1}^{k} D_j'D_j)K - 2(B + \sum_{j=1}^{k} C_jD_j)PK\big].$$

Notice, again, that it is hard to solve the preceding mathematical program as, in addition to the difficulty associated with the last constraint, $\Phi_+(P,r)$ and $\Phi_+(P,r)$ in general do not have analytical forms. In the following, we provide an explicit lower bound of r^*.

Theorem 5.2 *Assume that system (1) is stabilizable. Then $\bar{r} := \max\{\bar{r}_+, \bar{r}_-\}$ is a lower bound of the well-posedness margin, where*

$$\bar{r}_+ := \left\{ \begin{array}{ll} \dfrac{\lambda Q}{\inf_{K\in\mathcal{K}_+}\{F_+(K) - \lambda K'K\}}, & \text{if } Q \geq 0, \\ \dfrac{\lambda Q}{\sup_{K\in\mathcal{K}_+}\{F_+(K) - \lambda K'K\}}, & \text{if } Q < 0, \end{array} \right. \tag{43}$$

and

$$\bar{r}_- := \left\{ \begin{array}{ll} \dfrac{\lambda Q}{\inf_{K\in\mathcal{K}_-}\{F_-(K) - \lambda K'K\}}, & \text{if } Q \geq 0, \\ \dfrac{\lambda Q}{\sup_{K\in\mathcal{K}_-}\{F_-(K) - \lambda K'K\}}, & \text{if } Q < 0, \end{array} \right. \tag{44}$$

with $\lambda := \inf_{K\in\Gamma, |K|=1} K' \sum_{j=1}^{k} D_j'D_jK \geq 0.$

Proof : Suppose Problem (LQ) is well-posed with $R = -rI,\ r \in \mathbf{R}$. Then $\mathcal{P}_+ \neq \emptyset$. Since system (1) is stabilizable, we can take a $K \in \mathcal{K}_+$. It follows from Lemma 5.1 that $P := -\dfrac{Q + rK'K}{F_+(K)}$ is an upper bound of the nonempty set $\mathcal{P}_+$. Because $(rI + P_+ \sum_{j=1}^{k} D_j' D_j)\big|_\Gamma \geq 0$, $\forall P_+ \in \mathcal{P}_+$, we conclude $(rI + P \sum_{j=1}^{k} D_j' D_j)\big|_\Gamma \geq 0$, which is equivalent to $r + PK' \sum_{j=1}^{k} D_j' D_j K \geq 0$ for any $K \in \Gamma, |K| = 1$. Hence $r + P\lambda \geq 0$. Substituting $P = -\dfrac{Q + rK'K}{F_+(K)}$ we obtain $r \geq \dfrac{\lambda Q}{F_+(K) - \lambda K'K}$. Since $K \in \mathcal{K}_+$ is arbitrary, we obtain easily that $r \geq \bar{r}_+$. Similarly, we have $r \geq \bar{r}_-$. Our analysis implies that Problem (LQ) is *not* well-posed whenever the largest eigenvalue of R, $\lambda_{max}(R)$, is such that $\lambda_{max}(R) < \bar{r}$. Hence $\bar{r}$ is a lower bound of the well-posedness margin r^*. □

6 Optimality

This section is devoted to solving the optimal LQ control problem under consideration. We will first introduce two algebraic equations, in the spirit of the classical Riccati equation (for the unconstrained LQ problem), along with the notion of the so-called stabilizing solution. Then the optimality of Problem (LQ) is addressed via the stabilizing solutions of the two algebraic equations.

We impose the following assumptions in the rest of the paper.

Assumption 6.1 *System (1) is conic stabilizable.*

Assumption 6.2 *Problem (LQ) is well-posed.*

6.1 Extended Algebraic Riccati Equations (EARE's)

In this subsection we define the two algebraic equations that play a key role in solving Problem (LQ). Denote $\mathcal{R} := \{P \in \mathbf{R} | (R + P \sum_{j=1}^{k} D_j' D_j)\big|_\Gamma > 0\}$ and consider the following two functions from $\mathcal{R}$ to $\mathbf{R}$:

$$\xi_+(P) := \arg\min_{K \in \Gamma} \big[K'(R + P\sum_{j=1}^{k} D_j' D_j)K + 2(B + \sum_{j=1}^{k} C_j D_j)PK\big],$$
$$\xi_-(P) := \arg\min_{K \in \Gamma} \big[K'(R + P\sum_{j=1}^{k} D_j' D_j)K - 2(B + \sum_{j=1}^{k} C_j D_j)PK\big].$$

Note that the minimizers above are *uniquely* achievable due to a similar argument in Remark 5.3 and the fact that Γ is closed. Moreover, it is evident that both $\xi_+(\cdot)$ and $\xi_-(\cdot)$ are continuous on $\mathcal{R}$.

Define a pair of functions L_+ and L_- from $\mathcal{R}$ to $\mathbf{R}$:

$$L_+(P) := (2A + \sum_{j=1}^{k} C_j^2)P + Q + \Phi_+(P),$$
$$L_-(P) := (2A + \sum_{j=1}^{k} C_j^2)P + Q + \Phi_-(P).$$

The following two equations

$$L_+(P) = 0, \quad (R + P\sum_{j=1}^{k} D_j' D_j)\Big|_\Gamma > 0, \tag{45}$$

$$L_-(P) = 0, \quad (R + P\sum_{j=1}^{k} D_j' D_j)\Big|_\Gamma > 0 \tag{46}$$

are called extended algebraic Riccati equations (EARE's). Note that the constraint $(R+P\sum_{j=1}^{k} D_j' D_j)\Big|_\Gamma > 0$ is *part* of each of the two equations; so the EARE's are not exactly equations in strict sense. Also, being an algebraic equation, each of them may admit more than one solutions, or may admit no solution at all. Remark that, the EARE's introduced here both reduce to the *same* stochastic algebraic Riccati equation extensively studied in [2].

Definition 6.1 A solution P of EARE (45) (respectively (46)) is called a stabilizing solution if $\xi_+(P) \in \mathcal{K}_+$ (respectively $\xi_-(P) \in \mathcal{K}_-$).

It should be noted that the EARE's may not admit any stabilizing solution (see Proposition 6.1 in Section 6).

Before we conclude this subsection, we present several lemmas.

Lemma 6.1 *We have the following inequalities*

$$(P_2 - P_1)[F_+(\xi_+(P_2)) - F_+(\xi_+(P_1))] \le 0, \tag{47}$$

$$(P_2 - P_1)[F_-(\xi_-(P_2)) - F_-(\xi_-(P_1))] \le 0, \tag{48}$$

$$L_+(P_2) - L_+(P_1) \le (P_2 - P_1)F_+(\xi_+(P_1)), \tag{49}$$

$$L_-(P_2) - L_-(P_1) \le (P_2 - P_1)F_-(\xi_-(P_1)), \tag{50}$$

for any $P_1, P_2 \in \mathcal{R}$.

Proof : Denoting $v_1 := \xi_+(P_1)$ and $v_2 := \xi_+(P_2)$, we have

$$v_2'(R + P_1\sum_{j=1}^{k} D_j' D_j)v_2 + 2(B + \sum_{j=1}^{k} C_j D_j)P_1 v_2 - \Phi_+(P_1) \ge 0,$$
$$v_1'(R + P_2\sum_{j=1}^{k} D_j' D_j)v_1 + 2(B + \sum_{j=1}^{k} C_j D_j)P_2 v_1 - \Phi_+(P_2) \ge 0.$$

Then, adding the two inequalities up, we get

$$\begin{aligned} & [v_2'(R + P_1\sum_{j=1}^{k} D_j' D_j)v_2 + 2(B + \sum_{j=1}^{k} C_j D_j)P_1 v_2] - \Phi_+(P_2) \\ \ge\ & \Phi_+(P_1) - [v_1'(R + P_2\sum_{j=1}^{k} D_j' D_j)v_1 + 2(B + \sum_{j=1}^{k} C_j D_j)P_2 v_1]. \end{aligned}$$

Recall that the infimum in $\Phi_+(P_i)$ is achieved by v_i, $i = 1, 2$; hence the above yields

$$\begin{aligned} & (P_2 - P_1)[v_2' \sum_{j=1}^{k} D_j' D_j v_2 + 2(B + \sum_{j=1}^{k} C_j D_j) v_2] \\ \leq \ & (P_2 - P_1)[v_1' \sum_{j=1}^{k} D_j' D_j v_1 + 2(B + \sum_{j=1}^{k} C_j D_j) v_1]. \end{aligned}$$

This is equivalent to (47). Similarly we can prove (48).

Next, we calculate

$$\begin{aligned} & L_+(P_2) - L_+(P_1) \\ = \ & (2A + \sum_{j=1}^{k} C_j^2)(P_2 - P_1) + \Phi_+(P_2) - \Phi_+(P_1) \\ \leq \ & (2A + \sum_{j=1}^{k} C_j^2)(P_2 - P_1) + v_1'(R + P_2 \sum_{j=1}^{k} D_j' D_j) v_1 \\ & + 2(B + \sum_{j=1}^{k} C_j D_j) P_2 v_1 - \Phi_+(P_1) \\ = \ & (P_2 - P_1)[(2A + \sum_{j=1}^{k} C_j^2) + v_1' \sum_{j=1}^{k} D_j' D_j v_1 + 2(B + \sum_{j=1}^{k} C_j D_j) v_1] \\ = \ & (P_2 - P_1) F_+(\xi_+(P_1)). \end{aligned}$$

This proves (49). Similarly we can show (50). □

Lemma 6.2 *Assume* $P_1 \in \mathcal{R}$ *and* $P_2 \geq P_1$. *If* $\xi_+(P_1) \in \mathcal{K}_+$ *(respectively* $\xi_-(P_1) \in \mathcal{K}_-$*) then* $\xi_+(P_2) \in \mathcal{K}_+$ *(respectively* $\xi_-(P_2) \in \mathcal{K}_-$*).*

Proof : Since $P_2 \geq P_1$ it follows from (47) of Lemma 6.1 that

$$F_+(\xi_+(P_2)) - F_+(\xi_+(P_1)) \leq 0.$$

As $F_+(\xi_+(P_1)) < 0$ we have $F_+(\xi_+(P_2)) < 0$ implying $\xi_+(P_2) \in \mathcal{K}_+$. Similarly we can prove the assertion for $\mathcal{K}_-$. □

Lemma 6.3 *If there exists* $P_+^{(0)} \in \mathcal{R}$ *(respectively* $P_-^0 \in \mathcal{R}$*) with* $\xi_+(P_+^{(0)}) \in \mathcal{K}_+$ *(respectively* $\xi_-(P_-^0) \in \mathcal{K}_-$*), then* $L_+(\cdot)$ *(respectively* $L_-(\cdot)$*) is strictly decreasing on* $[P_+^{(0)}, +\infty)$ *(respectively* $[P_-^0, +\infty)$*).*

Proof : Take $P_2 > P_1 \geq P_+^{(0)}$. It follows from Lemma 6.2 that $F_+(\xi_+(P_1)) < 0$. On the other hand, it is clear that $P_1, P_2 \in \mathcal{R}$. Hence Lemma 6.1 yields

$$L_+(P_2) - L_+(P_1) \leq (P_2 - P_1) F_+(\xi_+(P_1)) < 0.$$

This proves that $L_+(P_2) < L_+(P_1)$. Similarly we can prove the assertion for $L_-(P)$. □

6.2 Optimality of LQ Problem via EARE's

In this subsection we prove that stabilizing solutions of (45) and (46), if any, lead to a complete and explicit solution to Problem (LQ).

Theorem 6.1 *If EARE's (45) and (46) admit stabilizing solutions P_+^* and P_-^* respectively, then the following feedback control*

$$u^*(t) = \xi_+(P_+^*)x^+(t) + \xi_-(P_-^*)x^-(t) \tag{51}$$

is optimal for Problem (LQ) with respect to any initial state x_0. Moreover, the value function is

$$V(x_0) = P_+^*(x_0^+)^2 + P_-^*(x_0^-)^2, \quad \forall x_0 \in \mathbf{R}. \tag{52}$$

Proof : Since P_+^* solves (45), we have

$$P_+^* = -\frac{Q + \xi_+(P_+^*)'R\xi_+(P_+^*)}{F_+(\xi_+(P_+^*))}.$$

Similarly,

$$P_-^* = -\frac{Q + \xi_-(P_-^*)'R\xi_-(P_-^*)}{F_-(\xi_-(P_-^*))}.$$

Moreover, $\xi_+(P_+^*) \in \mathcal{K}_+, \xi_-(P_-^*) \in \mathcal{K}_-$ as both P_+^* and P_-^* are stabilizing solutions. Thus the proof of Lemma 5.1 yields $V(x_0) \le J(x_0; u^*(\cdot)) = P_+^*(x_0^+)^2 + P_-^*(x_0^-)^2$. On the other hand, $P_+^* \in \mathcal{P}_+$ and $P_-^* \in \mathcal{P}_-$. Hence it follows from Theorem 5.1 that $V(x_0) \ge P_+^*(x_0^+)^2 + P_-^*(x_0^-)^2$. Therefore, $V(x_0) = J(x_0; u^*(\cdot)) = P_+^*(x_0^+)^2 + P_-^*(x_0^-)^2$. □

The above proof has also shown the following result.

Corollary 6.1 *If EARE's (45) and (46) admit stabilizing solutions P_+^* and P_-^* respectively, then $P_+^* = \max\{P|P \in \mathcal{P}_+\}$ and $P_-^* = \max\{P|P \in \mathcal{P}_-\}$. As a result, (45) and (46) each has at most one stabilizing solution.*

Corollary 6.1 guarantees that any stabilizing solution is the maximal solution of the respective EARE's. This result is in parallel with the unconstrained case (see, e.g., [3, Theorem 2.3]).

Note that the converse of Theorem 6.1 is not necessarily true. The following example shows that the existence of solution to the EARE's is not *necessary* for the LQ problem to be attainable with respect to any initial state.

Example 6.1 Consider the following LQ problem

$$\begin{array}{ll} \text{Minimize} & J(x_0; u(\cdot)) = \mathbf{E} \displaystyle\int_0^{+\infty} [|x(t)|^2 - |u(t)|^2]dt \\ \text{subject to} & \begin{cases} dx(t) = [-x(t) + u(t)]dt + [-x(t) + u(t)]dw(t), \\ x(0) = x_0, \end{cases} \end{array} \tag{53}$$

where all the variables are scalar-valued and $\Gamma = \mathbf{R}$. This example was originally discussed in [33, p. 817, Example 6.1]. It was verified in [33] that the system is stabilizable, and the LQ problem is attainable with respect to any x_0 (in fact there are infinitely many optimal feedback controls). But both EARE's (45) and (46) in this case reduce to $-p + 1 = 0, \quad -1 + p > 0$, which clearly admits no solution at all.

In spite of the preceding remarks and example, the following result shows that under an additional assumption, the EARE's indeed admit stabilizing solutions if Problem (LQ) is attainable.

Theorem 6.2 *Assume that there exist $P_+ \in \mathcal{P}_+$ and $P_- \in \mathcal{P}_-$ such that $(R + P\sum_{j=1}^k D_j' D_j)\Big|_\Gamma > 0$ for $P = P_+, P_-$. If Problem (LQ) is attainable with respect to any $x_0 \in \mathbf{R}$, then EARE's (45) and (46) admit stabilizing solutions P_+^* and P_-^* respectively. Moreover, any optimal control with respect to a given x_0 must be unique and represented by the feedback control (51).*

Proof : The proof of Proposition 5.2 yields that, under Assumptions 6.1 and 6.2, the value function can be represented as (32), where P_+^* and P_-^* are the maximum elements of $\mathcal{P}_+$ and $\mathcal{P}_-$ respectively. Moreover, by the assumption we have

$$(R + P_+^* \sum_{j=1}^k D_j' D_j)\Big|_\Gamma > 0, \quad (R + P_-^* \sum_{j=1}^k D_j' D_j)\Big|_\Gamma > 0. \tag{54}$$

Now, for any $x_0 \in \mathbf{R}$ let $x^*(\cdot)$ be the solution of (1) under an optimal control $u^*(\cdot) \in \mathcal{U}_{x_0}$ which exists by the attainability assumption. Then a similar calculation to that of (28) leads to

$$\begin{array}{rl} & \mathbf{E}\int_0^t [Qx^*(s)^2 + u^*(s)'Ru^*(s)]ds \\ = & P_+^*(x_0^+)^2 + P_-^*(x_0^-)^2 - \mathbf{E}[P_+^* x^{*+}(t)^2] - \mathbf{E}[P_-^* x^{*-}(t)^2] + \mathbf{E}\int_0^t \psi(x^*(s), u^*(s))ds, \end{array} \tag{55}$$

where $\psi(x^*(s), u^*(s))$ is the same as the integrand on the right hand side of (28) with P_+ and P_- replaced by P_+^* and P_-^* respectively. Letting $t \to +\infty$ and noting that $u^*(\cdot)$ is stabilizing, we obtain

$$\begin{aligned} V(x_0) \equiv J(x_0; u^*(\cdot)) &= \lim_{t\to+\infty} \mathbf{E}\int_0^t [Qx^*(s)^2 + u^*(s)'Ru^*(s)]ds \\ &= P_+^*(x_0^+)^2 + P_-^*(x_0^-)^2 + \mathbf{E}\int_0^{+\infty} \psi(x^*(s), u^*(s))ds. \end{aligned}$$

Recalling that $V(x_0) = P_+^*(x_0^+)^2 + P_-^*(x_0^-)^2$ and that $\psi(x^*(s), u^*(s)) \geq 0$ (via the same proof of Theorem 5.1), we conclude

$$\psi(x^*(s), u^*(s)) = 0, \ \text{ a.e. } s \in [0, +\infty), \ \ \mathbf{P} - \text{a.s.}$$

Fix $s \in [0, +\infty)$ satisfying the above equality. If $x^*(s) > 0$, then we can write $u^*(s) = K(s)x^*(s)$ where $K(s) \in \Gamma$. Going through the same analysis as in (29), we obtain

$$\begin{aligned} 0 &= \psi(x^*(s), u^*(s)) \\ &= [(2A + \sum_{j=1}^k C_j^2)P_+^* + Q + K(s)'(R + P_+^*\sum_{j=1}^k D_j'D_j)K(s) + 2(B + \sum_{j=1}^k C_jD_j)K(s)P_+^*]x^*(s)^2 \\ &\geq [(2A + \sum_{j=1}^k C_j^2)P_+^* + Q + \Phi_+(P_+^*)]x^*(s)^2 \\ &\geq 0. \end{aligned}$$

Thus, all the inequalities above become equalities and, noting that $x^*(s) \neq 0$, one has $K(s) = \xi_+(P_+^*)$ and $L_+(P_+^*) = 0$. As a result, $u^*(s) \equiv K(s)x^*(s) = \xi_+(P_+^*)x^*(s)$ at s when $x^*(s) > 0$. Similarly, we can prove that $u^*(s) = -\xi_-(P_-^*)x^*(s)$ at s when $x^*(s) \leq 0$, and $L_(P_-^*) = 0$. To summarize, we have shown that any optimal control $u^*(\cdot)$ can be represented by (51), and hence the uniqueness of optimal control follows. On the other hand, we have also proved that P_+^* and P_-^* are solutions to EARE's (45) and (46) respectively. Moreover, they must be stabilizing solutions because $u^*(\cdot)$, which is now represented by (51), is a stabilizing control. □

Remark 6.1 Theorem 6.2 shows that the existence of stabilizing solutions to EARE's (45) and (46) is *almost* necessary for the attainability of Problem (LQ). The only exception, as also demonstrated by Example 6.1, is the "singular" case when $(R + P\sum_{j=1}^{k} D_j'D_j)\Big|_\Gamma = 0$ for *all* elements P in at least one of the sets $\mathcal{P}_+$ and $\mathcal{P}_-$.

6.3 Existence of Stabilizing Solutions to EARE's

Theorem 6.1 asserts that if one can find stabilizing solutions to the EARE's, then the original optimal LQ control can be solved completely and explicitly, in terms of obtaining the optimal feedback control as well as the value function. The next natural questions are, then, *when* the EARE's admit stabilizing solutions and *how* to find them. These are the issues that we are going to address in this subsection. Indeed, we will identify and discuss three cases when the EARE's do have the stabilizing solutions.

Theorem 6.3 *If there exist $P_+^{(0)}$, $P_-^{(0)}$ satisfying*

$$L_+(P_+^{(0)}) \geq 0, \quad (R + P_+^{(0)}\sum_{j=1}^{k} D_j'D_j)\Big|_\Gamma > 0, \tag{56}$$

$$L_-(P_-^{(0)}) \geq 0, \quad (R + P_-^{(0)}\sum_{j=1}^{k} D_j'D_j)\Big|_\Gamma > 0, \tag{57}$$

and

$$F_+(\xi_+(P_+^{(0)})) < 0, \quad F_-(\xi_-(P_-^{(0)})) < 0, \tag{58}$$

then EARE's (45) and (46) admit unique stabilizing solutions P_+^* and P_-^* respectively.

Proof : If $L_+(P_+^{(0)}) = 0$, then $P_+^{(0)}$ is the stabilizing solution to (45) and we are done. So let us assume that $L_+(P_+^{(0)}) \equiv (2A + \sum_{j=1}^{k} C_j^2)P_+^{(0)} + Q + \Phi_+(P_+^{(0)}) > 0$, namely,

$$F_+(\xi_+(P_+^{(0)}))P_+^{(0)} + Q + \xi_+(P_+^{(0)})'R\xi_+(P_+^{(0)}) > 0.$$

Since $F_+(\xi_+(P_+^{(0)})) < 0$, we have

$$P_+^{(0)} < -\frac{Q + \xi_+(P_+^{(0)})'R\xi_+(P_+^{(0)})}{F_+(\xi_+(P_+^{(0)}))} := P_+^{(1)}. \tag{59}$$

By Lemma 6.2, we have $\xi_+(P_+^{(1)}) \in \mathcal{K}_+$ because $P_+^{(1)} > P_+^{(0)}$. Moreover,

$$\begin{aligned} L_+(P_+^{(1)}) &\equiv (2A + \sum_{j=1}^{k} C_j^2)P_+^{(1)} + Q + \Phi_+(P_+^{(1)}) \\ &\leq (2A + \sum_{j=1}^{k} C_j^2)P_+^{(1)} + Q + \xi_+(P_+^{(0)})'(R + P_+^{(1)}\sum_{j=1}^{k} D_j'D_j)\xi_+(P_+^{(0)}) \\ &\quad +2(B + \sum_{j=1}^{k} C_jD_j)P_+^{(1)}\xi_+(P_+^{(0)}) \\ &= 0, \end{aligned} \tag{60}$$

where the last inequality was due to the very definition of $P_+^{(1)}$. Now, if $L_+(P_+^{(1)}) = 0$ then $P_+^{(1)}$ is the stabilizing solution of (45). If $L_+(P_+^{(1)}) < 0$, then, noticing that $L_+(\cdot)$ is a strictly decreasing (by Lemma 6.3) continuous function on the interval $[P_+^{(0)}, P_+^{(1)}]$ along with $L_+(P_+^{(0)} > 0$, we conclude that there exists a unique $P_+^* \in (P_+^{(0)}, P_+^{(1)})$ such that $L_+(P_+^*) = 0$. Clearly $(R+P_+^* \sum_{j=1}^k D_j' D_j)\Big|_\Gamma > 0$, and $\xi_+(P_+^*) \in \mathcal{K}_+$ thanks to Lemma 6.2. Hence P_+^* is the stabilizing solution of (45).

The proof for the existence of the stabilizing solution to (46) is completely analogous. Finally, the uniqueness of the stabilizing solutions follows from Corollary 6.1. □

Theorem 6.4 *If there exist* $P_+^{(0)}$, $P_-^{(0)}$ *satisfying*

$$L_+(P_+^{(0)}) > 0, \quad (R + P_+^{(0)} \sum_{j=1}^k D_j' D_j)\Big|_\Gamma > 0, \tag{61}$$

and

$$L_-(P_-^{(0)}) > 0, \quad (R + P_-^{(0)} \sum_{j=1}^k D_j' D_j)\Big|_\Gamma > 0, \tag{62}$$

then EARE's (45) and (46) admit unique stabilizing solutions P_+^* *and* P_-^* *respectively.*

Proof : Take $K_+ \in \mathcal{K}_+$, which exists by the stabilizability assumption. Set

$$P_+^{(1)} := -\frac{Q + K_+' R K_+}{F_+(K_+)}. \tag{63}$$

It follows from Lemma 5.1 that $P_+^{(1)} \geq P_+^{(0)}$ since $P_+^{(0)} \in \mathcal{P}_+$. Hence $(R + P_+^{(1)} \sum_{j=1}^k D_j' D_j)\Big|_\Gamma > 0$. Moreover,

$$\begin{aligned} L_+(P_+^{(1)}) &\equiv (2A + \sum_{j=1}^k C_j^2) P_+^{(1)} + Q + \Phi_+(P_+^{(1)}) \\ &\leq (2A + \sum_{j=1}^k C_j^2) P_+^{(1)} + Q + K_+'(R + P_+^{(1)} \sum_{j=1}^k D_j' D_j) K_+ \\ &\quad +2(B + \sum_{j=1}^k C_j D_j) P_+^{(1)} K_+ \\ &= 0. \end{aligned} \tag{64}$$

Applying Lemma 6.1 and noting that $L_+(P_+^{(0)}) > 0$ we get

$$0 < L_+(P_+^{(0)}) - L_+(P_+^{(1)}) \leq (P_+^{(0)} - P_+^{(1)}) F_+(\xi_+(P_+^{(1)})). \tag{65}$$

Hence $F_+(\xi_+(P_+^{(1)})) < 0$ or $\xi_+(P_+^{(1)}) \in \mathcal{K}_+$. Now set

$$P_+^{(2)} := -\frac{Q + \xi_+(P_+^{(1)})' R \xi_+(P_+^{(1)})}{F_+(\xi_+(P_+^{(1)}))}. \tag{66}$$

Again by Lemma 5.1 we obtain $P_+^{(2)} \geq P_+^{(0)}$ and, therefore, $(R + P_+^{(2)} \sum_{j=1}^k D_j' D_j)\Big|_\Gamma > 0$. On the other hand, the fact that $L_+(P_+^{(1)}) \leq 0$ can be rewritten as $P_+^{(2)} \leq P_+^{(1)}$ in view of the relation

(66). Moreover, similar analysis to that for $P_+^{(1)}$ above leads to $L_+(P_+^{(2)}) \le 0$, and $\xi_+(P_+^{(2)}) \in \mathcal{K}_+$ or $F_+(\xi_+(P_+^{(2)})) < 0$.

In general, we construct iteratively the following sequence

$$P_+^{(i+1)} := -\frac{Q + \xi_+(P_+^{(i)})' R \xi_+(P_+^{(i)})}{F_+(\xi_+(P_+^{(i)}))}, \quad i = 1, 2, \cdots . \tag{67}$$

An induction argument shows that $P_+^{(0)} \le \cdots \le P_+^{(i+1)} \le P_+^{(i)} \cdots \le P_+^{(1)}$, $(R + P_+^{(i)} \sum_{j=1}^{k} D_j' D_j)\Big|_\Gamma > 0$, $L_+(P_+^{(i)}) \le 0$, and $\xi_+(P_+^{(i)}) \in \mathcal{K}_+$, $i = 1, 2, \cdots$. Since the sequence $\{P_+^{(i)}\}$ is decreasing with a lower bound $P_+^{(0)}$, there exists $P_+^* \in \mathbf{R}$ so that $P_+^* = \lim_{i\to\infty} P_+^{(i)}$. Moreover it is clear that $(R + P_+^* \sum_{j=1}^{k} D_j' D_j)\Big|_\Gamma \ge (R + P_+^{(0)} \sum_{j=1}^{k} D_j' D_j)\Big|_\Gamma > 0$. On the other hand, $L_+(P_+^*) \le 0$ since each $L_+(P_+^{(i)}) \le 0$. Thus argument similar to (65) yields $\xi_+(P_+^*) \in \mathcal{K}_+$ or $F_+(\xi_+(P_+^*)) < 0$. As a result, we can pass limit in (67) to obtain $P_+^* = -\dfrac{Q + \xi_+(P_+^*)' R \xi_+(P_+^*)}{F_+(\xi_+(P_+^*))}$, which is equivalent to $L_+(P_+^*) = 0$. This shows that P_+^* is the stabilizing solution to (45). Similarly we can prove that (46) admits the stabilizing solution. □

Remark 6.2 Recall that Theorem 5.1 characterizes the well-posedness of Problem (LQ) by the nonemptiness of the sets $\mathcal{P}_+$ and $\mathcal{P}_-$. Theorems 6.3 and 6.4 spell out two important cases when EARE's (45) and (46) have stabilizing solutions and, therefore, Problem (LQ) can be completely solved with explicit solutions. These two cases are specified in terms of existence of certain "special elements" of the sets $\mathcal{P}_+$ and $\mathcal{P}_-$. Specifically, the case with Theorem 6.3 is one when each of $\mathcal{P}_+$ and $\mathcal{P}_-$ has a "stabilizing element" in the sense that (58) holds. On the other hand, Theorem 6.4 asserts that the nonemptiness of the *interiors* of $\mathcal{P}_+$ and $\mathcal{P}_-$ is sufficient for the existence of stabilizing solutions to the EARE's. In view of the fact that the nonemptiness of $\mathcal{P}_+$ and $\mathcal{P}_-$ is the minimum requirement for the underlying LQ problem to be meaningful, the sufficient conditions respectively given in Theorems 6.3 and 6.4 are very mild indeed.

Remark 6.3 The proof of Theorem 6.4 constitutes an algorithm for finding the stabilizing solutions to the EARE's. In fact it is given by the iterative scheme (67) with an initial point (63). On the other hand, although the proof of Theorem 6.3 has not given an explicit algorithm for computing the stabilizing solutions, one can use a middle-point algorithm to find them based on the proof. Alternatively, one may use the same iterative scheme (67) with the initial point, $P_+^{(1)}$, given by (59). It can be proved using almost the same analysis as that in the proof of Theorem 6.4 that the constructed sequence converges to the desired point, P_+^*. The only argument that needs to be modified is that for proving $\xi_+(P_+^*) \in \mathcal{K}_+$. In this case, $\xi_+(P_+^*) \in \mathcal{K}_+$ is seen from the facts that $P_+^* \ge P_+^{(0)}$ and $\xi_+(P_+^{(0)}) \in \mathcal{K}_+$ along with Lemma 6.2.

Finally we present the results on the definite case $Q \ge 0$ and $R\Big|_\Gamma \ge 0$ (including the so-called singular case when R is allowed to be *singular*).

Theorem 6.5 *Assume $Q \ge 0$ and $R\Big|_\Gamma \ge 0$. Then the EARE's (45) and (46) admit unique stabilizing solutions P_+^* and P_-^* respectively under one of the following additional conditions.*

(i) *$Q > 0$ and $R\Big|_\Gamma > 0$.*

(ii) $Q = 0,\ R\big|_\Gamma > 0,\ \textit{and}\ 2A + \sum_{j=1}^{k} C_j^2 \neq 0$

(iii) $Q > 0,\ R\big|_\Gamma \geq 0,\ \textit{and}\ \sum_{j=1}^{k} D_j' D_j\big|_\Gamma > 0.$

Proof : (i) In this case $P_+^{(0)} = P_-^{(0)} = 0$ satisfy the assumption of Theorem 6.4.

(ii) If $2A + \sum_{j=1}^{k} C_j^2 < 0$, then take $P_+^{(0)} = P_-^{(0)} = 0$. We see that $L_+(P_+^{(0)}) = L_-(P_-^{(0)}) = Q = 0$ and $F_+(\xi_+(P_+^{(0)})) = F_-(\xi_-(P_-^{(0)})) = 2A + \sum_{j=1}^{k} C_j^2 < 0$. Thus the assumption of Theorem 6.3 is satisfied.

If $2A + \sum_{j=1}^{k} C_j^2 > 0$, due to the stabilizability assumption there is $0 \neq K_+ \in \mathcal{K}_+$. Set $P_+^{(1)} := -\frac{K_+' R K_+}{F_+(K_+)}$. As $F_+(K_+) < 0$, $R\big|_\Gamma > 0$ and $K_+ \neq 0$, we have $P_+^{(1)} > 0$. Define

$$P_+^{(i+1)} := -\frac{\xi_+(P_+^{(i)})' R \xi_+(P_+^{(i)})}{F_+(\xi_+(P_+^{(i)}))}, \quad i = 1, 2, \cdots. \tag{68}$$

Then a similar analysis to that in the proof of Theorem 6.4 leads to $0 \leq \cdots \leq P_+^{(i+1)} \leq P_+^{(i)} \cdots \leq P_+^{(1)}$, $(R + P_+^{(i)} \sum_{j=1}^{k} D_j' D_j)\big|_\Gamma > 0$, $L_+(P_+^{(i)}) \leq 0$, and $\xi_+(P_+^{(i)}) \in \mathcal{K}_+$, $i = 1, 2, \cdots$. Hence there exists $P_+^* \geq 0$ so that $P_+^* = \lim_{i\to\infty} P_+^{(i)}$. Moreover $(R + P_+^* \sum_{j=1}^{k} D_j' D_j)\big|_\Gamma \geq R\big|_\Gamma > 0$. Note that at this point we can no longer apply the same argument as that used in the proof of Theorem 6.4 to conclude $F_+(\xi_+(P_+^*)) < 0$, because the element 0, which substitutes the point $P_+^{(0)}$ of Theorem 6.4, does not satisfy $L_+(0) > 0$. To get around, let us suppose $F_+(\xi_+(P_+^*)) = 0$ (recall that it always holds that $F_+(\xi_+(P_+^*)) \leq 0$ since $F_+(\xi_+(P_+^{(i)})) < 0$). Multiplying (68) by $F_+(\xi_+(P_+^{(i)}))$ and then passing to limit, we obtain $\xi_+(P_+^*)' R \xi_+(P_+^*) = 0$, resulting in $\xi_+(P_+^*) = 0$. Thus $F_+(\xi_+(P_+^*)) = 2A + \sum_{j=1}^{k} C_j^2 > 0$, which is a contradiction. This proves that $F_+(\xi_+(P_+^*)) < 0$. The rest of the proof is the same as that of Theorem 6.4.

(iii) First note for any $P > 0$,

$$\begin{aligned} 0 \geq\ & \Phi_+(P) \\ \geq\ & \inf_{K\in\Gamma} K' R K + P \inf_{K\in\Gamma} [K' \sum_{j=1}^{k} D_j' D_j K + 2(B + \sum_{j=1}^{k} C_j D_j) K] \\ =\ & P \inf_{K\in\Gamma} [K' \sum_{j=1}^{k} D_j' D_j K + 2(B + \sum_{j=1}^{k} C_j D_j) K]. \end{aligned}$$

Hence $\lim_{P\to 0+} \Phi_+(P) = 0$. Since $Q > 0$, we have by the definition of $L_+(\cdot)$ that there exists $P_+^{(0)} > 0$ so that $L_+(P_+^{(0)}) > 0$. On the other hand, it is clear that $(R + P_+^{(0)} \sum_{j=1}^{k} D_j' D_j)\big|_\Gamma \geq P_+^{(0)} \sum_{j=1}^{k} D_j' D_j\big|_\Gamma > 0$.

Consequently Theorem 6.4 applies. $\square$

One may be curious about what happens if $Q = 0$, $R\big|_\Gamma > 0$, and $2A + \sum_{j=1}^k C_j^2 = 0$ (refer to Theorem 6.5-(ii)). It turns out that in this case the EARE's *never* admit stabilizing solutions.

Proposition 6.1 *Neither (45) nor (46) admits any stabilizing solution when $Q = 0$, $R\big|_\Gamma > 0$, and $2A + \sum_{j=1}^k C_j^2 = 0$.*

Proof: By Assumption 6.1, there exists $0 \neq K_+ \in \mathcal{K}_+$ and $0 \neq K_- \in \mathcal{K}_-$ since $F_+(0) = F_-(0) = 0$. Denote $K_+^\varepsilon := \varepsilon K_+$ and $K_-^\varepsilon := \varepsilon K_-$ for $\varepsilon \in (0, 1]$. Then $K_+^\varepsilon \in \Gamma$ and $K_-^\varepsilon \in \Gamma$.

For any fixed $P_+ > 0$, set $\varepsilon := \min\left\{\frac{-P_+F_+(K_+)}{2K_+'RK_+}, 1\right\} \in (0, 1]$. Then

$$
\begin{aligned}
L_+(P_+) &= \Phi_+(P_+) \\
&\leq (K_+^\varepsilon)'(R + P_+\sum_{j=1}^k D_j'D_j)K_+^\varepsilon + 2P_+(B + \sum_{j=1}^k C_jD_j)K_+^\varepsilon \\
&\leq \varepsilon\Big\{\varepsilon K_+'RK_+ + P_+[K_+'\sum_{j=1}^k D_j'D_jK_+ + 2(B + \sum_{j=1}^k C_jD_j)K_+]\Big\} \\
&\leq \varepsilon\left[\frac{-P_+F_+(K_+)}{2K_+'RK_+}(K_+'RK_+) + P_+F_+(K_+)\right] \\
&= \frac{\varepsilon}{2}P_+F_+(K_+) \\
&< 0.
\end{aligned}
$$

This implies that there exists no *positive* solution to EARE (45).

Next, for any fixed $P_+ < 0$ with $(R + P_+\sum_{j=1}^k D_j'D_j)\big|_\Gamma > 0$, set $\varepsilon := \min\left\{\frac{-|P_+|F_-(K_-)}{2K_-'RK_-}, 1\right\} \in (0, 1]$. Then

$$
\begin{aligned}
L_+(P_+) &= \Phi_+(P_+) \\
&\leq (K_-^\varepsilon)'(R + P_+\sum_{j=1}^k D_j'D_j)K_-^\varepsilon + 2P_+(B + \sum_{j=1}^k C_jD_j)K_-^\varepsilon \\
&\leq \varepsilon\Big\{\varepsilon K_-'RK_- + |P_+|[K_-'\sum_{j=1}^k D_j'D_jK_- - 2(B + \sum_{j=1}^k C_jD_j)K_-]\Big\} \\
&\leq \varepsilon\left[\frac{-|P_+|F_-(K_-)}{2K_-'RK_-}(K_-'RK_-) + |P_+|F_-(K_-)\right] \\
&= \frac{\varepsilon}{2}|P_+|F_-(K_-) \\
&< 0.
\end{aligned}
$$

Hence there is no *negative* solution to EARE (45).

Finally, when $P_+ = 0$, we do have $L_+(P_+) = 0$ but $F_+(\xi_+(P_+)) = F_+(0) = 0$. So $P_+ = 0$ is not a stabilizing solution either.

Similarly, we can prove the non-existence of stabilizing solution to EARE (46). $\square$

Although the conclusion of Proposition 6.1 does not necessarily lead to the non-existence of optimal feedback control for the corresponding LQ problem (refer to Section 6.2), the following example shows that the latter could indeed occur.

Example 6.2 Consider the following LQ problem

$$\begin{array}{ll} \text{Minimize} & J(x_0; u(\cdot)) = \mathbf{E} \int_0^{+\infty} r|u(t)|^2 dt \\ \text{subject to} & \begin{cases} dx(t) = [ax(t) + bu(t)]dt + cx(t)dw(t), \\ x(0) = x_0, \end{cases} \end{array} \tag{69}$$

where all the variables are scalar-valued, $\Gamma = \mathbf{R}$, $2a + c^2 = 0$, $b > 0$, and $r > 0$. It is easy to verify that the problem is stabilizable and well-posed. Take a feedback control $u^\varepsilon(t) = -\varepsilon x^\varepsilon(t)$ for $\varepsilon > 0$. Under this control the state satisfies

$$\mathbf{E}|x^\varepsilon(t)|^2 = e^{(2a+c^2-2b\varepsilon)t} x_0^2 = e^{-2b\varepsilon t} x_0^2.$$

Hence u^ε is stabilizing. Moreover, the cost under this control is

$$J(x_0; u^\varepsilon(\cdot)) = \varepsilon^2 r^2 \mathbf{E} \int_0^{+\infty} |x^\varepsilon(t)|^2 dt = \frac{r^2 x_0^2}{2b} \varepsilon.$$

Letting $\varepsilon \to 0$ we see that $V(x_0) = 0$, $\forall x_0 \in \mathbf{R}$. Note that this value cannot be attained if $x_0 \neq 0$, for whenever $\mathbf{E} \int_0^{+\infty} r|u^*(t)|^2 dt = 0$ it is necessary that $u^*(t) = 0$, a.e.$t \geq 0$. However, this control, $u^*(t)$, is *not* stabilizable when $x_0 \neq 0$. In other words, the LQ problem is not attainable with respect to $x_0 \neq 0$.

Remark 6.4 Theorem 6.5-(i), (ii), Proposition 6.1 together with Example 6.2 give a complete answer to the question of optimality for Problem (LQ) in the classical definite case $Q \geq 0$ and $R\big|_\Gamma > 0$. Moreover, Theorem 6.5-(iii) addresses the case when R is possibly singular. Note that this case occurs often in financial applications (where typically $R = 0$).

Remark 6.5 In view of Theorem 6.1, under the respective assumptions of Theorems 6.3, 6.4 and 6.5, Problem (LQ) has the optimal feedback control (51) and the value function (52). Moreover, as per Remark 5.6, in these cases the stabilizing solutions P_+^* and P_-^* can also be obtained, in addition to the preceding algorithms, by solving the mathematical programs (33) and (34) if the corresponding constraints are tractable.

7 Numerical Examples

To numerically calculate optimal solution to Problem (LQ) one needs to carry out two steps: the first is to check the conic stabilizability and the well-posedness, and the second is to find the stabilizing solutions to the EARE's. The procedures for the first step are depicted in Sections 4 and 5.3, whereas that for the second part is described in Section 6. Here we give an example to illustrate the whole process (where we used the computing tool, *Scilab*, to carry out all the calculations).

Example 7.1 Consider Problem (LQ) with $m = k = 3$, $\Gamma = \mathbf{R}_+^3$, and the dynamics coefficients as follows:

$$
\begin{array}{ll}
A = 2.00, & B = (-50 \quad -100 \ \ 200), \\
C_1 = -0.84, & D_1 = (6.85 \quad -8.78 \ \ 0.68), \\
C_2 = -3.78, & D_2 = (11.22 \ \ 13.24 \ \ 14.53), \\
C_3 = 0.849, & D_3 = (-1.98 \quad -5.44 \quad -2.32).
\end{array}
$$

The eigenvalues of $\sum_{j=1}^{3} D_j' D_j$ are 1.5880509, 126.23912 and 547.84943. So $\sum_{j=1}^{3} D_j' D_j$ is positive definite in this case. The cost parameters are

$$
Q = 10, \quad R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -5.0 & 0 \\ 0 & 0 & 4.0 \end{pmatrix}.
$$

Hence this is an indefinite LQ problem.

To solve this problem, we first apply Theorem 4.1-(iii) (note that in this example the constraint $K \in \mathbf{R}_+^3$ is also an LMI) to obtain stabilizing feedback gains

$$
K_+ = \begin{pmatrix} 0.5435353 \\ 0.5307289 \\ 0.0701460 \end{pmatrix}, \quad K_- = \begin{pmatrix} 0.0816967 \\ 0.0562570 \\ 0.7185273 \end{pmatrix}.
$$

Next we use the algorithm in Section 5.3 to find out that Problem (LQ) is well-posed. Furthermore, when $P_+^{(0)} = P_-^{(0)} = 0.1$, the eigenvalues of $R + 0.1 * \sum_{j=1}^{3} D_j' D_j$ are 1.641309, 10.293806 and 54.632545; hence $R + 0.1 * \sum_{j=1}^{3} D_j' D_j > 0$. On the other hand, $L_+(0.1) = 1.6073676 > 0$ and $L_-(0.1) = 4.0651986 > 0$. According to Theorem 6.4, Problem (LQ) has an optimal feedback control. Now we use the algorithm given in the proof of Theorem 6.4 to obtain the optimal control and optimal value. First, set the initial values $P_+^{(1)}$ and $P_-^{(1)}$ by using K_+, K_- and formulas similar to (63), respectively. They are

$$
P_+^{(1)} = 1.1814412, \quad P_-^{(1)} = 13.167238.
$$

By the iterative formula (67), we obtain

$$
\begin{aligned}
P_+^* &= 0.1225762, \quad \xi_+(P_+^*) = \begin{pmatrix} 0.2889240 \\ 0.4918755 \\ 0 \end{pmatrix}, \\
P_-^* &= 0.1561608, \quad \xi_-(P_-^*) = \begin{pmatrix} 0 \\ 0 \\ 0.5875815 \end{pmatrix}.
\end{aligned}
$$

Therefore, the optimal feedback control is

$$
u^*(t) = \begin{pmatrix} 0.2889240 \\ 0.4918755 \\ 0 \end{pmatrix} x^+(t) + \begin{pmatrix} 0 \\ 0 \\ 0.5875815 \end{pmatrix} x^-(t),
$$

with the optimal cost

$$
J^*(x_0) = 0.1225762 * (x_0^+)^2 + 0.1561608 * (x_0^-)^2.
$$

In the next example we demonstrate the calculation of a lower bound of the well-posedness margin (refer to Section 5.4).

Example 7.2 With the same values of the coefficients A, B, C_j, D_j, $j = 1, 2, 3$, and Q in Example 7.1, we want to compute a lower bound of the well-posedness margin r^*.

According to Theorem 5.2, we first calculate $\lambda = 176.7313$. Next we have $\bar{r}_+ = -9.434553$ and $\bar{r}_- = -6.4967141$. Hence, the lower bound of the well-posedness margin is $\bar{r} = -6.4967141$. Note that, as seen in Example 7.1, the problem is still well-posed when one of the eigenvalues of R is -5.

8 Conclusion

In this paper, an indefinite stochastic LQ control problem in the infinite time horizon with conic control constraint is studied. Several key issues, including conic stabilizability, well-posedness and optimality are addressed with complete solutions. In particular, two algebraic equations, the EARE's, are newly introduced, in lieu of the classical algebraic Riccati equation, whose stabilizing solutions give rise to the explicit forms of the optimal feedback control and the value function. It is also seen that the representation of the value function given by Proposition 5.1 serves as the technical key to all the main results of this paper, which motivates the utilization of the cerebrated Tanaka formula.

It should be stressed again that the approach in this paper crucially depends on the special structure of the problem. One main assumption is that the state of the system is one-dimensional. While the conclusion of Proposition 5.1 appears to hold, *mutatis mutandis,* for the problem with multi-dimensional state variable, it seems that an analogy of Lemma 3.2, if any, would be far more complicated. This makes the multi-dimensional problem very challenging. Another structural property of the model is that the dynamics of the system is homogeneous (in state and control) and the cost contains no first-order term of the state variable as well as no control-state cross term. As a result, our approach will fail, say, for the case when there is no state and control dependent noise, but rather that with fixed variance. Solving these kind of problems calls for different techniques. Finally, an even more difficult problem is the stochastic LQ control with state constraint.

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