DC&

MASTER SET 1

MATHEMATICS OF OPERATIONS RESEARCH
Vol. 0, No. 0, Month 1996
Printed in U.S.A.

DETERMINISTIC NEAR-OPTIMAL CONTROLS. PART II: DYNAMIC PROGRAMMING AND VISCOSITY SOLUTION APPROACH

XUN YU ZHOU

Near-optimization is as sensible and important as optimization for both theory and applications. This paper concerns dynamic near-optimization, or near-optimal controls, for systems governed by deterministic ordinary differential equations, and uses dynamic programming to study the near-optimality. Since nonsmoothness is inherent in this subject, the viscosity solution approach is employed to investigate the problem. The dynamic programming equation is derived in terms of ε-superdifferential/subdifferential. The relationships among the adjoint functions, the value functions, and the Hamiltonian along near-optimal trajectories are revealed. Verification theorems with which near-optimal feedback controls can be constructed are obtained.

Author Query
See msp 1
O.K.

1. Introduction. By a near-optimal control we mean a control whose objective function value comes within a small neighborhood of the optimal one. Studying near-optimal controls makes as good sense as studying optimal controls for both theory and applications. See the Introduction in Part I (Zhou 1995) of this series for a detailed argument and justification of establishing a general theory of near-optimal controls.

In Part I (Zhou 1995), necessary and sufficient conditions for near-optimal controls are derived in terms of near-maximum conditions of the Hamiltonian. These results are parallel to the classical Pontryagin maximum principle in exact-optimality (Pontryagin et al. 1962). It is well known that another important approach in optimal control theory is the Bellman dynamic programming (Bellman 1957). Can we apply dynamic programming to near-optimality? Since nonsmoothness is inherent in this subject, how can we apply it?

As the second in a series of papers on near-optimal controls, this paper tries to answer the above questions. We are concerned with systems described by deterministic ordinary differential equations (ODEs). As is well known, the classical dynamic programming theory does not have a rigorous foundation, because it is based on the *smoothness* assumption of the value function, which is not satisfied even in the simplest situations. To handle such nonsmooth phenomena, people have been led to study differential properties of nondifferentiable functions, or, *nonsmooth analysis*. Beginning in the 1970s and 1980s respectively, two remarkable bodies of nonsmooth analysis, namely, Clarke's *generalized gradient* (Clarke 1973) and Crandall and Lions's *viscosity solution* (Crandall and Lions 1983) have been extensively developed, and have been found extremely powerful in dealing with optimal control problems; see Clarke (1990), Lions (1982) and Fleming and Soner (1992). In this paper, we shall employ the viscosity solution approach and derive the dynamic programming equation in terms of

Received August 12, 1994; revised April 26, 1995.
AMS 1991 subject classification. Primary: 49L05; secondary: 49L20, 90C39.
OR/MS Index 1978 subject classification. Primary: Dynamic programming/optimal control/Deterministic.
Key words. Near-optimal control, Hamiltonian, adjoint function, value function, feedback control, ε-superdifferential/subdifferential, viscosity solution.

0364-765X/96/2102/0000/$01.25

the so-called ε-superdifferential and ε-subdifferential that are particularly effective for studying near-optimal controls. Further we reveal some important relationships among the value function, the adjoint function and the Hamiltonian along any near-optimal trajectories. At last, we prove verification theorems for near-optimality, which can be used to construct near-optimal feedback controls.

The plan of the rest of the paper is as follows. In §2, we set up the problem and give some preliminary definitions and results. In §3, we introduce the ε-superdifferential/subdifferential and derive a nonsmooth version of the Hamilton-Jacobi equation. In §4, we investigate the relationship among the value function, the adjoint function, and the Hamiltonian. ~~Section~~ §5 concerns the verification theorems for near-optimality. Finally, §6 concludes the paper. §

2. Preliminaries. Let us consider the following optimal control problem. Given $(s, y) \in [0, T] \times R^d$, we are to

$$\text{(2.1)} \qquad \text{minimize } J(s, y; u(\cdot)) = \int_s^T L(t, x(t), u(t))\, dt + h(x(T)),$$

$$\text{(2.2)} \qquad \text{subject to } \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), & \text{a.e. } t \in [s, T], \\ x(s) = y, \end{cases}$$

over the set of admissible controls $U_{ad}[s, T] = \{u(\cdot) | u(\cdot)$ is a Lebesgue measurable function from $[s, T]$ to $\Gamma\}$, where Γ is an arbitrary given compact set in R^m. R^m

We denote the above problem by C_{sy} to recall the dependence on the initial time s and the initial state y. The value function is defined as

$$\text{(2.3)} \qquad V(s, y) = \inf\{J(s, y; u(\cdot)) : u(\cdot) \in U_{ad}[s, T]\}.$$

The control problem under consideration in this paper is that of finding a control in $U_{ad}[s, T]$, for any initial $(s, y) \in [0, T] \times R^d$, that minimizes or "nearly" minimizes $J(s, y; u(\cdot))$ over $U_{ad}[s, T]$. In this connection, we need the following definitions.

DEFINITION 2.1. Any pair $(x(\cdot), u(\cdot))$, where $x(\cdot)$ is the solution of (2.2) corresponding to $u(\cdot) \in U_{ad}[s, T]$, is called an *admissible pair* with respect to the initial (s, y). An admissible pair $(\hat{x}(\cdot), \hat{u}(\cdot))$, or simply $\hat{u}(\cdot)$, is called *optimal* with respect to (s, y), if $\hat{u}(\cdot)$ achieves the minimum of $J(s, y; u(\cdot))$.

DEFINITION 2.2. For a given $\varepsilon > 0$, an admissible pair $(x^\varepsilon(\cdot), u^\varepsilon(\cdot))$, or simply $u^\varepsilon(\cdot)$, is called *ε-optimal* with respect to (s, y) if

$$|J(s, y; u^\varepsilon(\cdot)) - V(s, y)| \le \varepsilon.$$

DEFINITION 2.3. Both a family of admissible pairs $\{(x^\varepsilon(\cdot), u^\varepsilon(\cdot))\}$ parameterized by $\varepsilon > 0$ and any element $(x^\varepsilon(\cdot), u^\varepsilon(\cdot))$, or simply $u^\varepsilon(\cdot)$, in the family, are called *near-optimal* with respect to (s, y) if

$$|J(s, y; u^\varepsilon(\cdot)) - V(s, y)| \le r(\varepsilon)$$

holds for sufficiently small ε, where r is a function of ε satisfying $r(\varepsilon) \to 0$ as $\varepsilon \to 0$. The estimate $r(\varepsilon)$ is called an *error bound.* If $r(\varepsilon) = c\varepsilon^\delta$ for some $\delta > 0$ independent of the constant c, then $u^\varepsilon(\cdot)$ is called *near-optimal with order ε^δ.*

In the above definitions, the terms "admissible," "optimal," "ε-optimal," and "near-optimal" are dependent of the initial time s and initial state y. In the sequel, however, the phrase "with respect to (s, y)" may be omitted if no confusion would occur.

MASTER SET

In contrast with the above defined controls, usually called *open-loop*, the definition of feedback controls are given below, following Fleming and Rishel (1975).

DEFINITION 2.4. A measurable function u from $[0,T] \times R^d$ to Γ is called an *admissible feedback control* if for any $(s, y) \in [0,T] \times R^d$ there is a unique solution $x(\cdot; s, y)$ of the following equation

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t, x(t))), & \text{a.e. } t \in [s,T], \\ x(s) = y. \end{cases} \tag{2.4}$$

An admissible feedback control u is called *optimal* (resp. *near-optimal with order* ε^δ) *if* $(x(\cdot; s, y), u(\cdot, x(\cdot; s, y)))$ *is optimal* (*resp. near-optimal with order* ε^δ) for the problem C_{sy} for *each* (s, y), where $x(\cdot; s, y)$ is the solution of (2.4) corresponding to u.

Notation. We make use of the following notation in this paper:

a^+: $= \max\{a, 0\}$ for any real-valued number a;
a^-: $= \max\{-a, 0\}$ for any real-valued number a;
$\alpha \cdot \beta$: the inner product of any two vectors α and β;
$|\alpha|$: $= |a_1| + \cdots + |a_n|$ for any vector $\alpha = (a_1, \ldots, a_n)$;
A^*: the transpose of any vector or matrix A;
$o(y)$: a function of y such that $\lim_{y \to 0} |o(y)/y| = 0$;
ρ_x: the (conventional) derivative of a function ρ with respect to the variable x;
$C([s,T]; R^d)$: the Banach space of R^d-valued continuous functions on $[s,T]$ endowed with the maximum norm;
$L^2([s,T]; R^d)$: the Hilbert space of R^d-valued square-integrable functions on $[s,T]$ endowed with the L^2-norm;
$C(X)$: the set of all continuous functions on X; scalar-valued
$C^1(X)$: the set of all continuously differentiable functions on X;
$C, C_i, i = 1, 2, \ldots$: multiplicative constants required in the analysis.

Assumptions. The following basic assumptions will be in force throughout this paper:

(A1) f: $[0,T] \times R^d \times \Gamma \to R^d$ and L: $[0,T] \times R^d \times \Gamma \to R^1$ are measurable in (t, x, u), continuously differentiable in x for each (t, u), and there exists a constant $C > 0$ such that for $\rho = f, L$,

$$|\rho(t, x, u)| \le C(1 + |x|), \tag{2.5}$$

$$\begin{aligned} &|\rho(t, x, u) - \rho(t, x', u')| + |\rho_x(t, x, u) - \rho_x(t, x', u')| \\ &\quad \le C(|x - x'| + |u - u'|). \end{aligned} \tag{2.6}$$

(A2) h: $R^d \to R^1$ is continuously differentiable, and

$$|h(x)| \le C(1 + |x|), \tag{2.7}$$

$$|h(x) - h(x')| + |h_x(x) - h_x(x')| \le C|x - x'|. \tag{2.8}$$

REMARK 2.1. By (2.5) with $\rho = f$, it is easy to see using Gronwall's inequality that for every solution $x(\cdot)$ of (2.2), we have

$$\sup_{s \le t \le T} |x(t)| \le [C(T - s) + |y|] e^{C(T-s)}. \tag{2.9}$$

Now that the control problem has been formulated, we shall proceed to recall some definitions and results that are useful for our subsequent analysis. First we state Ekeland's variational principle (Ekeland 1974).

LEMMA 2.1. *Let (S, d) be a complete metric space, and $\rho(\cdot)$: $S \to R^1$ be lower-semi-continuous and bounded from below. For $\varepsilon \geq 0$, suppose $u^\varepsilon \in S$ satisfies*

$$\rho(u^\varepsilon) \leq \inf_{u \in S} \rho(u) + \varepsilon.$$

Then for any $\lambda > 0$, there exists $u^\lambda \in S$ such that

$$\rho(u^\lambda) \leq \rho(u^\varepsilon),$$

$$d(u^\lambda, u^\varepsilon) \leq \lambda, \quad \text{and}$$

$$\rho(u^\lambda) \leq \rho(u) + \frac{\varepsilon}{\lambda} d(u, u^\lambda), \quad \text{for all } u \in S.$$

From now on in this section, let us assume that the initial time s and initial state y are fixed. Since the control region Γ is closed, $U_{ad}[s, T]$ becomes a complete metric space when endowed with the metric

$$d(u(\cdot), u'(\cdot)) = \int_s^T |u(t) - u'(t)| \, dt. \tag{2.10}$$

For any $u(\cdot) \in U_{ad}[s, T]$ and the corresponding state trajectory $x(\cdot)$, we define the adjoint function $\psi(\cdot)$ as the one satisfying the adjoint equation

$$\begin{cases} \dot{\psi}(t) = -f_x^*(t, x(t), u(t))\psi(t) - L_x(t, x(t), u(t)), \\ \psi(T) = h_x(x(T)). \end{cases} \tag{2.11}$$

Since f_x, L_x and h_x are bounded by C by Assumptions (A1) and (A2), there exists a constant $C_1 > 0$ that is independent of $u(\cdot)$ such that

$$\sup_{s \leq t \leq T} |\psi(t)| \leq C_1. \tag{2.12}$$

Moreover, it is clear that $\psi(\cdot) \in C([s, T]; R^d)$, and we have the following estimate:

$$|\psi(t) - \psi(t')| \leq C(C_1 + 1)|t - t'|. \tag{2.13}$$

The above definition of adjoint functions leads to an operator Ψ from $U_{ad}[s, T]$ to $C([s, T]; R^d)$ as follows:

$$\psi(\cdot) = \Psi(u(\cdot)).$$

The following result, sometimes called the adjoint equality, is well known.

LEMMA 2.2. *For any $t \in [s, T]$, $g \in L^2(s, T; R^d)$, and $\alpha \in R^d$, denote by $z_{t,g,\alpha}(\cdot)$ the solution of the following linear equation;*

$$\begin{cases} \dot{z}_{t,g,\alpha}(r) = f_x(r, x(r), u(r)) z_{t,g,\alpha}(r) + g(r), \quad a.e.\ r \in [t, T], \\ z_{t,g,\alpha}(t) = \alpha. \end{cases}$$

MASTER SET

Then for $\psi(\cdot) = \Psi(u(\cdot))$, *we have*

$$\int_t^T L_x(r, x(r), u(r)) \cdot z_{t,g,\alpha}(r)\, dr + h_x(x(T)) \cdot z_{t,g,\alpha}(T)$$

$$= \psi(t) \cdot \alpha + \int_t^T \psi(r) \cdot g(r)\, dr.$$

PROOF. The result is readily seen by differentiating $\psi(r) \cdot z_{t,g,\alpha}(r)$ and then integrating from t to T. □

3. Dynamic programming. Optimal control has been extensively studied via dynamic programming and the viscosity solution approach. The so-called superdifferential and subdifferential, which are nonsmooth versions of the conventional derivative, play a central role in the approach. However, in order to study near-optimality, we may utilize some "perturbed" version of the superdifferential/subdifferential, precisely defined as follows.

Author Query See ms p 7 (E.K.)

Let Q be an open subset of some $\cancel{R^n}$, and $v: \overline{Q} \to R$ be a continuous function. R^k/

DEFINITION 3.1. The *ε-superdifferential* (resp. *ε-subdifferential*) of v at $\hat{x} \in Q$, denoted by $D^+_{\varepsilon,x}v(\hat{x})$ (resp. $D^-_{\varepsilon,x}v(\hat{x})$), is a *set* defined by

$$D^+_{\varepsilon,x}v(\hat{x}) = \left\{ p \in \cancel{R^n} \,\middle|\, \limsup_{x \in Q,\, x \to \hat{x}} \frac{v(x) - v(\hat{x}) - p \cdot (x - \hat{x})}{|x - \hat{x}|} \le \varepsilon \right\}.$$

R^k/

(resp.

$$D^-_{\varepsilon,x}v(\hat{x}) = \{ p \in \cancel{R^n} | \liminf\{ \cdots \} \ge -\varepsilon \}.)$$

R^k/

REMARK 3.1. The idea of ε-superdifferential/subdifferential originates in Ekeland and Lebourg (1976), while the definitions were given explicitly by Crandall and Lions (1985), although those authors used the notion to study optimality.

REMARK 3.2. $D^+_x v(\hat{x})$ (resp. $D^-_x v(\hat{x})$) is a subset of $D^+_{\varepsilon,x}v(\hat{x})$ (resp. $D^-_{\varepsilon,x}v(\hat{x})$). More precisely, we have

$$D^+_x v(\hat{x}) = \bigcap_{\varepsilon > 0} D^+_{\varepsilon,x}v(\hat{x}) \quad \text{and} \quad D^-_x v(\hat{x}) = \bigcap_{\varepsilon > 0} D^-_{\varepsilon,x}v(\hat{x}). \tag{3.1}$$

Here $D^+_x v$ and $D^-_x v$ denote the superdifferential and subdifferential, respectively.

LEMMA 3.1. *A vector* $p \in D^+_{\varepsilon,x}v(\hat{x})$ (*resp.* $p \in D^-_{\varepsilon,x}v(\hat{x})$) *if and only if there exists* $\phi \in C(\overline{Q}) \cap C^1(Q)$ *such that*

$$\phi(\hat{x}) = v(\hat{x}),$$

$$\dot{\phi}(\hat{x}) = p, \quad \textit{and}$$

$$\phi(x) > (\textit{resp.} <)\ v(x) - \varepsilon|x - \hat{x}|, \quad \textit{for any}\ \hat{x} \neq x \in \overline{Q}.$$

PROOF. We only prove the statement for the ε-superdifferential, the sub case being similar.

First, note that the "if" part is clear. We only prove the "only if" part. Suppose $p \in D^+_{\varepsilon,x}v(\hat{x})$. Define a function w on $\overline{Q}$ as follows:

$$w(x) = \begin{cases} \max\left\{ \dfrac{v(x) - v(\hat{x}) - \cancel{p(x-\hat{x})}}{|x-\hat{x}|}, \varepsilon \right\}, & x \neq \hat{x}, \\ \varepsilon, & x = \hat{x}. \end{cases}$$

$p \cdot (x-\hat{x})$

It is obvious that $\liminf_{x \to \hat{x}} w(x) \geq \varepsilon$. On the other hand, due to the fact that $p \in D^+_{\varepsilon,x}v(\hat{x})$, we have

$$\limsup_{x \to \hat{x}} w(x) \leq \max\left\{ \limsup_{x \to \hat{x}} \frac{v(x) - v(\hat{x}) - \cancel{p(x-\hat{x})}}{|x-\hat{x}|}, \varepsilon \right\} = \varepsilon.$$

It follows that $w \in C(\overline{Q})$ and $\lim_{x \to \hat{x}} w(x) = \varepsilon$. Define

$$\rho(r) = \sup_{|x-\hat{x}| \leq r} (w(x) - \varepsilon) \geq 0, \quad \text{for } r \geq 0, \text{ and}$$

$$\beta(x) = \int_0^{2|x-\hat{x}|} \rho(r)\,dr + |x-\hat{x}|^2.$$

It is easy to see that $\beta \in C(\overline{Q}) \cap C^1(Q)$ and

$$\beta(\hat{x}) = 0 \quad \text{and} \quad \dot{\beta}(\hat{x}) = 0.$$

Moreover, noting that ρ is nonnegative and nondecreasing, we have

$$\begin{aligned} \beta(x) &\geq \int_{|x-\hat{x}|}^{2|x-\hat{x}|} \rho(r)\,dr + |x-\hat{x}|^2 \\ &\geq \rho(|x-\hat{x}|)|x-\hat{x}| + |x-\hat{x}|^2 \\ &\geq (w(x) - \varepsilon)|x-\hat{x}| + |x-\hat{x}|^2 \\ &\geq v(x) - v(\hat{x}) - \cancel{p(x-\hat{x})} - \varepsilon|x-\hat{x}| + |x-\hat{x}|^2. \end{aligned}$$

Let $\phi(x) = \beta(x) + v(\hat{x}) + \cancel{p(x-\hat{x})}$. Then ϕ is a function that satisfies the desired properties of the lemma. □

R^k /

LEMMA 3.2. *If $p \in D^+_{\varepsilon,x}v(\hat{x})$ (resp. $p \in D^-_{\varepsilon,x}v(\hat{x})$), then there is a sequence $(x_n, p_n) \in Q \times \cancel{R^n}$ such that $p_n \in D^+_x v(x_n)$ (resp. $p_n \in D^-_x v(x_n)$), $x_n \to \hat{x}$, and for any $\alpha > 0$, it holds that $|p_n - p| \leq (1+\alpha)\varepsilon$ for sufficiently large $\cancel{n}$.*

Printer: The above is the only one to be changed in Lemma 3.2 and its proof.

PROOF. We only prove the superdifferential case.

For $p \in D^+_{\varepsilon,x}v(\hat{x})$, take ϕ as determined by Lemma 3.1. Since $\phi(x) - v(x) + \varepsilon|x - \hat{x}|$ attains a strict minimum at $\hat{x}$, the function $\phi(x) - v(x) + \varepsilon\sqrt{|x-\hat{x}|^2 + 1/\cancel{n}^2}$ attains a minimum at some $x_n \in Q$ for sufficiently large $\cancel{n}$ with $x_n \to \hat{x}$. Let p_n be the gradient of $\phi(x) + \varepsilon\sqrt{|x-\hat{x}|^2 + 1/\cancel{n}^2}$ at x_n, namely,

$$p_n = \dot{\phi}(x_n) + \varepsilon \frac{x_n - \hat{x}}{\sqrt{|x_n - \hat{x}|^2 + 1/\cancel{n}^2}}.$$

~~Printer: In Lemma and its proof Change the subscript n to k. Don't change the supersript (i.e, R^n will keep the same).~~

~~Change x_n to x_k~~
~~Change p_n to p_k~~

Ignore these !!!

Then $p_n \in D_x^+ v(x_n)$ and $|p_n - p| \le |\dot\phi(x_n) - \dot\phi(\hat{x})| + \varepsilon$. The result then follows from the fact that $\phi \in C^1(Q)$. □

Before presenting the main result in this section, let us introduce an additional assumption, which will be effective from now on.

(A3) f and L are continuous in (t, x, u).

REMARK 3.3. The value function V is globally Lipschitz continuous in (t, x) under (A1)–(A3); see Lions (1982).

In what follows, we shall use $D^+_{\varepsilon,t,x}V(t,x)$ to denote the ε-superdifferential with respect to (t, x), and $D^+_{\varepsilon,x}V(t,x)$ the (partial) ε-superdifferential with respect to x, etc.

Define the Hamiltonian

$$H(t,x,u,\psi) = -\psi\cdot f(t,x,u) - L(t,x,u),$$

for $(t, x, u, \psi) \in [0,T] \times R^d \times \Gamma \times R^d$.

THEOREM 3.1. *For any* $(t,x) \in (0,T) \times R^d$ *and any* $\varepsilon > 0$,

$$\begin{aligned} &-p + \sup_{u\in\Gamma} H(t,x,u,q) \le [C + 1 + C|x|]\varepsilon, \quad \text{for any } (p,q) \in D^+_{\varepsilon,t,x}V(t,x), \\ (3.2)\qquad &-p + \sup_{u\in\Gamma} H(t,x,u,q) \ge -[C + 1 + C|x|]\varepsilon, \quad \text{for any } (p,q) \in D^-_{\varepsilon,t,x}V(t,x), \end{aligned}$$

where C *is the same constant as in Assumption* (A1).

PROOF. Suppose $(p,q) \in D^+_{\varepsilon,t,x}V(t,x)$. By Lemma 3.2, there exists (t_n, x_n, p_n, q_n) such that $(p_n, q_n) \in D^+_{t,x}V(t_n, x_n)$, $(t_n, x_n) \to (t,x)$ and for any $\alpha > 0$, it holds that $|p_n - p| + |q_n - q| \le (1+\alpha)\varepsilon$ for sufficiently large n. Since V is a viscosity solution to the following Hamilton-Jacobi equation (cf. Lions 1982):

$$-V_t(t,x) + \sup_{u\in\Gamma} H(t,x,u,V_x(t,x)) = 0, \tag{3.3}$$

we have

$$-p_n + \sup_{u\in\Gamma} H(t_n, x_n, u, q_n) \le 0.$$

Since $|f(t,x,u)| \le C(1 + |x|)$, simple calculation leads to

$$-p + \sup_{u\in\Gamma} H(t,x,u,q) \le (1+\alpha)[C + 1 + C|x|]\varepsilon.$$

The desired inequality follows by letting $\alpha \to 0$.

Similarly we can prove the sub case. □

REMARK 3.4. By (3.1), the inequalities (3.2) imply that V is a viscosity solution to the HJ equation (3.3). The above proof shows that *vice versa*. Therefore, Theorem 3.1 is actually *equivalent* to the fact that V is a viscosity solution to the HJ equation. in

4. Value function and adjoint function. In optimal control theory, the maximum principle and the dynamic programming are deeply related via the adjoint function, the value function and the Hamiltonian along the optimal trajectories (cf. Berkovitz 1974 and Fleming and Rishel 1975). Such relationships for deterministic controls have been further investigated by using viscosity solution theory (Barron and Jensen 1986, Zhou 1990 and Cannarsa and Frankowska 1991) as well as Clarke's generalized gradient (Clarke and Vinter 1987).

As for near-optimality, various ε-maximum principles for *some selected* near-optimal controls were obtained by Ekeland (1974) and Gabasov, Kirillova and Mordukhovich (1983). Recently, an ε-maximum principle for *any* near-optimal control has been derived in Part I of this series (Zhou 1995) in some integral form of the Hamiltonian. In this section, we shall study the corresponding relationships between the ε-maximum principle derived in Zhou (1995) and the dynamic programming derived in §3 along any near-optimal trajectory.

First, let us prove the following lemma that may be called *Principle of ε-Optimality.* Note that throughout this section, the initial time s and the initial state y are fixed.

LEMMA 4.1. *If $(x^\varepsilon(\cdot), u^\varepsilon(\cdot))$ is an ε-optimal pair for the problem C_{sy}, then for any $t \in [s, T]$,*

$$V(t, x^\varepsilon(t)) \geq \int_t^T L(r, x^\varepsilon(r), u^\varepsilon(r))\, dr + h(x^\varepsilon(T)) - \varepsilon.$$

PROOF. We first prove that for any fixed $\alpha > 0$,

$$V(t, x^\varepsilon(t)) \geq \int_t^T L(r, x^\varepsilon(r), u^\varepsilon(r))\, dr + h(x^\varepsilon(T)) - (1+\alpha)\varepsilon. \tag{4.1}$$

Suppose (4.1) is not true. Then there is $u(\cdot) \in U_{ad}[t, T]$ such that

$$J(t, x^\varepsilon(t); u(\cdot)) \leq V(t, x^\varepsilon(t)) + \alpha\varepsilon < \int_t^T L(r, x^\varepsilon(r), u^\varepsilon(r))\, dr + h(x^\varepsilon(T)) - \varepsilon.$$

Construct $\bar{u}(\cdot) \in U_{ad}[s, T]$ as follows:

$$\bar{u}(r) = \begin{cases} u^\varepsilon(r), & r \in [s, t), \\ u(r), & r \in [t, T]. \end{cases}$$

Then

$$\begin{aligned} J(s, y; \bar{u}(\cdot)) &= \int_s^t L(r, x^\varepsilon(r), u^\varepsilon(r))\, dr + J(t, x^\varepsilon(t); u(\cdot)) \\ &< \int_s^T L(r, x^\varepsilon(r), u^\varepsilon(r))\, dr + h(x^\varepsilon(T)) - \varepsilon \\ &= J(s, y; u^\varepsilon(\cdot)) - \varepsilon \leq V(s, y), \end{aligned}$$

contradicting the definition of the value function V. Therefore, (4.1) is true. Since $\alpha > 0$ is arbitrary, we get the desired result by letting $\alpha \to 0$. □

Author Query
See msp 12
O.K.

THEOREM 4.1. *Suppose $(x^\varepsilon(\cdot), u^\varepsilon(\cdot))$ is an ε-optimal pair for the problem C_{sy}. Then there exists a constant $C_2 > 0$, depending only on the constant C in Assumptions (A1)–(A2), such that for any $t \in [s, T]$, there is $z = z(t) \in R^d$ satisfying*

$$|z - x^\varepsilon(t)| \leq \sqrt{\varepsilon} \quad \text{and} \quad \psi^\varepsilon(t) \in D^+_{C_2\sqrt{\varepsilon}, x} V(t, z),$$

where $\psi^\varepsilon(\cdot) = \Psi(u^\varepsilon(\cdot))$ is the corresponding adjoint function.

MASTER SET

PROOF. For simplicity, we assume that $h \equiv 0$. For $x \in R^d$, let $x(\cdot; x)$ be the solution of (2.2) with initial time t, initial state x, and control $u^\varepsilon(\cdot)|_{[t,T]}$. Then, for any $r \in [t, T]$,

$$
\begin{aligned}
x(r;x) - x^\varepsilon(r) &= x - x^\varepsilon(t) + \int_t^r [f(\theta, x(\theta;x), u^\varepsilon(\theta)) - f(\theta, x^\varepsilon(\theta), u^\varepsilon(\theta))]\, d\theta \\
&= x - x^\varepsilon(t) + \int_t^r f_x(\theta, x^\varepsilon(\theta), u^\varepsilon(\theta))(x(\theta;x) - x^\varepsilon(\theta))\, d\theta \\
&\quad + \int_t^r \int_0^1 [f_x(\theta, x^\varepsilon(\theta) + \alpha(x(\theta;x) - x^\varepsilon(\theta)), u^\varepsilon(\theta)) \\
&\qquad - f_x(\theta, x^\varepsilon(\theta), u^\varepsilon(\theta))]\, d\alpha (x(\theta;x) - x^\varepsilon(\theta))\, d\theta.
\end{aligned}
$$

Author Query
See msp 12

On the other hand, by the Principle of ε-Optimality (Lemma 4.1), we have

$$
\begin{aligned}
V(t,x) - V(t, x^\varepsilon(t)) &\le \int_t^T [L(r, x(r;x), u^\varepsilon(r)) - L(r, x^\varepsilon(r), u^\varepsilon(r))]\, dr + \varepsilon \\
&= \int_t^T L_x(r, x^\varepsilon(r), u^\varepsilon(r)) \cdot (x(r;x) - x^\varepsilon(r))\, dr \\
&\quad + \int_t^T \int_0^1 [L_x(r, x^\varepsilon(r) + \alpha(x(r;x) - x^\varepsilon(r)), u^\varepsilon(r)) \\
&\qquad - L_x(r, x^\varepsilon(r), u^\varepsilon(r))]\, d\alpha \cdot (x(r;x) - x^\varepsilon(r))\, dr + \varepsilon.
\end{aligned}
$$

O.K. by the mathematical context (sometimes it requires [illegible] & sometimes it does..

By the adjoint equality (Lemma 2.2), we may rewrite the above inequality as

$$
V(t,x) - V(t, x^\varepsilon(t)) \le \psi^\varepsilon(t) \cdot (x - x^\varepsilon(t)) + \eta(t,x) + \varepsilon, \tag{4.2}
$$

where

$$
\eta(t,x) = \int_t^T [\eta_1(r;x) + \eta_2(r;x)]\, dr, \quad \text{with}
$$

Author Query
See msp 13

$$
\begin{aligned}
\eta_1(r;x) &= \psi^\varepsilon(r) \cdot \int_0^1 [f_x(r, x^\varepsilon(r) + \alpha(x(r;x) - x^\varepsilon(r)), u^\varepsilon(r)) \\
&\qquad - f_x(r, x^\varepsilon(r), u^\varepsilon(r))]\, d\alpha (x(r;x) - x^\varepsilon(r)),
\end{aligned}
$$

O.K.
(See above)

$$
\begin{aligned}
\eta_2(r;x) &= \int_0^1 [L_x(r, x^\varepsilon(r) + \alpha(x(r;x) - x^\varepsilon(r)), u^\varepsilon(r)) \\
&\qquad - L_x(r, x^\varepsilon(r), u^\varepsilon(r))]\, d\alpha \cdot (x(r;x) - x^\varepsilon(r)).
\end{aligned}
$$

Note that by Assumptions (A1) and (A2), $\eta(t, x)$ is continuous in x and $\eta(t, x^\varepsilon(t)) = 0$. Now fix t and let

$$\rho(x) = -V(t,x) + \psi^\varepsilon(t)\cdot(x - x^\varepsilon(t)) + \eta(t,x).$$

Then (4.2) is equivalent to

$$\rho(x^\varepsilon(t)) \le \rho(x) + \varepsilon, \quad \text{for any } x \in R^d.$$

By Ekeland's principle (Lemma 2.1), there exists $z = z(t) \in R^d$ such that

$$|z - x^\varepsilon(t)| \le \sqrt{\varepsilon}, \tag{4.3}$$

and

$$\rho(z) \le \rho(x) + \sqrt{\varepsilon}|x - z|, \quad \text{for any } x \in R^d. \tag{4.4}$$

We can rewrite (4.4) as

$$V(t,x) - V(t,z) \le \psi^\varepsilon(t)\cdot(x - z) + \eta(t,x) - \eta(t,z) + \sqrt{\varepsilon}|x - z|. \tag{4.5}$$

Now let us estimate

$$\begin{aligned}
&\eta_2(r;x) - \eta_2(r;z)\\
&\quad = \int_0^1 \big[L_x(r, x^\varepsilon(r) + \alpha(x(r;x) - x^\varepsilon(r)), u^\varepsilon(r))\\
&\qquad\qquad - L_x(r, x^\varepsilon(r), u^\varepsilon(r))\big]\, d\alpha\cdot(x(r;x) - x(r;z))\\
&\qquad + \int_0^1 \big[L_x(r, x^\varepsilon(r) + \alpha(x(r;x) - x^\varepsilon(r)), u^\varepsilon(r))\\
&\qquad\qquad - L_x(r, x^\varepsilon(r) + \alpha(x(r;z) - x^\varepsilon(r)), u^\varepsilon(r))\big]\, d\alpha\cdot(x(r;z) - x^\varepsilon(r))\\
&\quad = \beta_1 + \beta_2, \quad \text{say.}
\end{aligned}$$

By Assumptions (A1)–(A2), we have

$$\begin{aligned}
|\beta_1| &\le C_3|x - x^\varepsilon(t)|\,|x - z|\\
&\le C_3(|x - z| + |z - x^\varepsilon(t)|)|x - z|\\
&\le C_3|x - z|^2 + C_3\sqrt{\varepsilon}|x - z|,
\end{aligned}$$

and

$$|\beta_2| \le C_4|x(r;z) - x(r;x)|\,|x(r;z) - x^\varepsilon(r)| \le C_5\sqrt{\varepsilon}|x - z|.$$

Thus we obtain

$$|\eta_2(r;x) - \eta_2(r;z)| \le C_3|x - z|^2 + (C_3 + C_5)\sqrt{\varepsilon}|x - z|.$$

Similar inequality holds for $|\eta_1(r;x) - \eta_1(r;z)|$ and, therefore, $|\eta(t,x) - \eta(t,z)|$. So (4.5) gives

$$V(t,x) - V(t,z) \le \psi^\varepsilon(t) \cdot (x - z) + C_2\sqrt{\varepsilon}|x - z| + o(|x - z|),$$

which implies that $\psi^\varepsilon(t) \in D^+_{C_2\sqrt{\varepsilon},x}V(t,z)$. The proof is now completed. □

THEOREM 4.2. *Suppose $(x^\varepsilon(\cdot), u^\varepsilon(\cdot))$ is an ε-optimal pair for the problem C_{sy}. Then there exists a constant $C_6 > 0$, depending only on the constant C in Assumptions (A1)–(A2), such that for almost all $t \in [s,T]$, there is $(\gamma, z) \in [s,T] \times R^d$ satisfying*

$$(4.6)\quad |\gamma - t| + |z - x^\varepsilon(t)| \le \sqrt{\varepsilon} \quad \text{and}$$
$$(H(\gamma, x^\varepsilon(\gamma), u^\varepsilon(\gamma), \psi^\varepsilon(\gamma)), \psi^\varepsilon(\gamma)) \in D^+_{C_6\sqrt{\varepsilon},t,x}V(\gamma, z),$$

where H is the Hamiltonian.

PROOF. Once again, we assume that $h \equiv 0$. For $(\tau, x) \in [t,T] \times R^d$, let $x(\cdot;\tau,x)$ be the solution of (2.2) with initial time τ, initial state x, and control $u^\varepsilon(\cdot)|_{[\tau,T]}$. Then, for any $r \in [\tau, T]$,

$$\begin{aligned}
x(r;\tau,x) - x^\varepsilon(r)
&= \left[x - x^\varepsilon(t) - \int_t^\tau f(\theta, x^\varepsilon(\theta), u^\varepsilon(\theta))\,d\theta\right] \\
&\quad + \int_\tau^r [f(\theta, x(\theta;\tau,x), u^\varepsilon(\theta)) - f(\theta, x^\varepsilon(\theta), u^\varepsilon(\theta))]\,d\theta \\
&= \left[x - x^\varepsilon(t) - \int_t^\tau f(\theta, x^\varepsilon(\theta), u^\varepsilon(\theta))\,d\theta\right] \\
&\quad + \int_\tau^r f_x(\theta, x^\varepsilon(\theta), u^\varepsilon(\theta))(x(\theta;\tau,x) - x^\varepsilon(\theta))\,d\theta \\
&\quad + \int_\tau^r \int_0^1 [f_x(\theta, x^\varepsilon(\theta) + \alpha(x(\theta;\tau,x) - x^\varepsilon(\theta)), u^\varepsilon(\theta)) \\
&\qquad - f_x(\theta, x^\varepsilon(\theta), u^\varepsilon(\theta))]\,d\alpha\,(x(\theta;\tau,x) - x^\varepsilon(\theta))\,d\theta.
\end{aligned}$$

On the other hand, by Lemma 4.1, we have

$$\begin{aligned}
V(\tau,x) - V(t, x^\varepsilon(t))
&\le -\int_t^\tau L(r, x^\varepsilon(r), u^\varepsilon(r))\,dr \\
&\quad + \int_\tau^T [L(r, x(r;\tau,x), u^\varepsilon(r)) - L(r, x^\varepsilon(r), u^\varepsilon(r))]\,dr + \varepsilon \\
&= -\int_t^\tau L(r, x^\varepsilon(r), u^\varepsilon(r))\,dr + \int_\tau^T L_x(r, x^\varepsilon(r), u^\varepsilon(r)) \cdot (x(r;\tau,x) - x^\varepsilon(r))\,dr \\
&\quad + \int_\tau^T \int_0^1 [L_x(r, x^\varepsilon(r) + \alpha(x(r;\tau,x) - x^\varepsilon(r)), u^\varepsilon(r)) \\
&\qquad - L_x(r, x^\varepsilon(r), u^\varepsilon(r))]\,d\alpha \cdot (x(r;\tau,x) - x^\varepsilon(r))\,dr + \varepsilon.
\end{aligned}$$

By the adjoint equality, we have

$$V(\tau, x) - V(t, x^\varepsilon(t)) \tag{4.7}$$
$$\leq \psi^\varepsilon(\tau) \cdot \left[x - x^\varepsilon(t) - \int_t^\tau L(r, x^\varepsilon(r), u^\varepsilon(r))\, dr\right] + \bar{\eta}(\tau, x) + \varepsilon,$$

where

$\bar{\eta}(\tau, x) = \int_\tau^T [\bar{\eta}_1(r; x) + \bar{\eta}_2(r; x)]\, dr$, with

$$\bar{\eta}_1(r; x) = \psi^\varepsilon(r) \cdot \int_0^1 [f_x(r, x^\varepsilon(r) + \alpha(x(r; \tau, x) - x^\varepsilon(r)), u^\varepsilon(r))$$
$$- f_x(r, x^\varepsilon(r), u^\varepsilon(r))]\, d\alpha (x(r; \tau, x) - x^\varepsilon(r)),$$

$$\bar{\eta}_2(r; x) = \int_0^1 [L_x(r, x^\varepsilon(r) + \alpha(x(r; \tau, x) - x^\varepsilon(r)), u^\varepsilon(r))$$
$$- L_x(r, x^\varepsilon(r), u^\varepsilon(r))]\, d\alpha \cdot (x(r; \tau, x) - x^\varepsilon(r)).$$

It is easy to see that $\bar{\eta}(\tau, x)$ is continuous in (τ, x) and $\bar{\eta}(t, x^\varepsilon(t)) = 0$. Define

$$\rho(\tau, x) = -V(\tau, x) + \psi^\varepsilon(\tau) \cdot \left[x - x^\varepsilon(t) - \int_t^\tau f(r, x^\varepsilon(r), u^\varepsilon(r))\, dr\right]$$
$$- \int_t^\tau L(r, x^\varepsilon(r), u^\varepsilon(r))\, dr + \bar{\eta}(\tau, x).$$

Author Query
See msp 15
O.K.

Then (4.7) is equivalent to

$$\rho(t, x^\varepsilon(t)) \leq \rho(\tau, x) + \varepsilon, \quad \text{for any } (\tau, x) \in [s, T] \times R^d.$$

By Ekeland's principle, there exists $(\gamma, z) \in [s, T] \times R^d$ such that

$$|\gamma - t| + |z - x^\varepsilon(t)| \leq \sqrt{\varepsilon}, \tag{4.8}$$

and

$$\rho(\gamma, z) \leq \rho(\tau, x) + \sqrt{\varepsilon}(|\tau - \gamma| + |x - z|), \quad \text{for any } (\tau, x) \in [s, T] \times R^d. \tag{4.9}$$

We can then rewrite (4.9) as

$$\begin{aligned} V(\tau, x) - V(\gamma, z) \leq & -\psi^\varepsilon(\gamma) \cdot \int_\gamma^\tau f(r, x^\varepsilon(r), u^\varepsilon(r))\, dr \\ & - \int_\gamma^\tau L(r, x^\varepsilon(r), u^\varepsilon(r))\, dr + \psi^\varepsilon(\gamma) \cdot (x - z) \\ & + (\psi^\varepsilon(\tau) - \psi^\varepsilon(\gamma)) \cdot \left(x - x^\varepsilon(t) - \int_t^\tau f(r, x^\varepsilon(r), u^\varepsilon(r))\, dr\right) \\ & + \bar{\eta}(\tau, x) - \bar{\eta}(\gamma, z) + \sqrt{\varepsilon}(|\tau - \gamma| + |x - z|). \end{aligned} \tag{4.10}$$

Now let us estimate

$$
\begin{aligned}
&|(\psi^\varepsilon(\tau) - \psi^\varepsilon(\gamma)) \cdot (x - x^\varepsilon(t))| \\
&\quad \le |\psi^\varepsilon(\tau) - \psi^\varepsilon(\gamma)||x - z| + |\psi^\varepsilon(\tau) - \psi^\varepsilon(\gamma)||z - x^\varepsilon(t)| \\
&\quad \le C(C_1 + 1)|\tau - \gamma||x - z| + C(C_1 + 1)\sqrt{\varepsilon}|\tau - \gamma|, \quad \text{(by (2.13))} \\
&\quad \le o(|\tau - \gamma| + |x - z|) + C(C_1 + 1)\sqrt{\varepsilon}|\tau - \gamma|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|(\psi^\varepsilon(\tau) - \psi^\varepsilon(\gamma)) \cdot \int_t^\tau f(r, x^\varepsilon(r), u^\varepsilon(r))\, dr\right| \\
&\quad \le |\psi^\varepsilon(\tau) - \psi^\varepsilon(\gamma)| \left|\int_t^\gamma f(r, x^\varepsilon(r), u^\varepsilon(r))\, dr\right| \\
&\qquad + |\psi^\varepsilon(\tau) - \psi^\varepsilon(\gamma)| \left|\int_\gamma^\tau f(r, x^\varepsilon(r), u^\varepsilon(r))\, dr\right| \\
&\quad \le C_7\sqrt{\varepsilon}|\tau - \gamma| + C_7|\tau - \gamma|^2.
\end{aligned}
$$

Similar to the proof of Theorem 4.1, one can also prove that

$$
|\bar{\eta}(\tau, x) - \bar{\eta}(\gamma, z)| \le C_8\sqrt{\varepsilon}(|\tau - \gamma| + |x - z|) + o(|\tau - \gamma| + |x - z|).
$$

Therefore, (4.10) gives

$$
\begin{aligned}
V(\tau, x) - V(\gamma, z) \le{}& H(\gamma, x^\varepsilon(\gamma), u^\varepsilon(\gamma), \psi^\varepsilon(\gamma))(\tau - \gamma) + \psi^\varepsilon(\gamma) \cdot (x - z) \\
&+ C_6\sqrt{\varepsilon}(|\tau - \gamma| + |x - z|) + o(|\tau - \gamma| + |x - z|),
\end{aligned}
$$

which implies $(H(\gamma, x^\varepsilon(\gamma), u^\varepsilon(\gamma), \psi^\varepsilon(\gamma)), \psi^\varepsilon(\gamma)) \in D^+_{C_6\sqrt{\varepsilon}, t, x} V(\gamma, z)$. The proof is now completed. □

REMARK 4.1. Results similar to Theorems 4.1 and 4.2 in the context of optimal controls have been obtained in Zhou (1990), which basically concerns the relationship between the maximum principle and the dynamic programming along optimal trajectories. The results in this section give the corresponding relationship in some neighborhood of near-optimal trajectories. It should be noted that due to (3.1), the results here cover those in Zhou (1990).

Let us now give an application of Theorem 4.2.

THEOREM 4.3. *Suppose that $(x^\varepsilon(\cdot), u^\varepsilon(\cdot))$ is an ε-optimal pair for the problem C_{sy}, and that $u^\varepsilon(\cdot)$ is Lipschitz continuous. Then there exists a constant $C_9 > 0$ (possibly dependent of the Lipschitz bound of $u^\varepsilon(\cdot)$) such that for almost all $t \in [s, T]$,*

$$
H(t, x^\varepsilon(t), u^\varepsilon(t), \psi^\varepsilon(t)) \ge \max_{u \in \Gamma} H(t, x^\varepsilon(t), u, \psi^\varepsilon(t)) - C_9\sqrt{\varepsilon}. \tag{4.11}
$$

MASTER SET

PROOF. By Theorem 4.2, for a.e. $t \in [s, T]$, there is (γ, z) such that (4.6) holds. Applying Theorem 3.1, we have

$$-H(\gamma, x^\varepsilon(\gamma), u^\varepsilon(\gamma), \psi^\varepsilon(\gamma)) + \sup_{u \in \Gamma} H(\gamma, z, u, \psi^\varepsilon(\gamma)) \le [C + 1 + C|z|]C_6\sqrt{\varepsilon}.$$

It is then easy to get the desired result by virtue of the estimate in (4.6) as well as the Lipschitz continuity of $\psi^\varepsilon(\cdot)$ and $u^\varepsilon(\cdot)$. □

REMARK 4.2. Theorem 4.3 is a necessary condition of near-optimality in a pointwise form, which is better than Theorem 3.1 in Zhou (1995), as the condition there is in an integral form. Note that the assumption that $u^\varepsilon(\cdot)$ is Lipschitz (in time t) is not very restrictive. Unlike with optimal controls, which is bang-bang for many linear systems and therefore not continuous in time, Lipschitz near-optimal controls exist in general; see Example 5.2 in §5. On the other hand, although the constant C_9 may depend on the Lipschitz bound of $u^\varepsilon(\cdot)$, one may have a priori information about the bound for some *selected* controls. A good example is the so-called L-admissible control obtained by Gabasov, Kirillova and Mordukhovich (1983) which turns out to be piecewise constant in our case.

5. Verification theorems. In optimal control theory, the verification theorem plays an important role in testing for the optimality of a given control, and in constructing optimal feedback controls. Nonsmooth versions of the verification theorem within the framework of viscosity solution have also been obtained by Zhou (1993). In this section, we shall see that verification technique applies to near-optimality as well.

Author Query See msp O.K.

R^k / R^k /

Let $Q \subset R^n$ be an open set. For $\hat{x} \in Q$ and $\xi \in R^n$, we denote by $v'(\hat{x}; \xi)$ the (one-sided) directional gradient (along ξ) of v at $\hat{x}$, namely

$$v'(\hat{x}; \xi) = \lim_{h \to 0+} \frac{v(\hat{x} + h\xi) - v(\hat{x})}{h},$$

whenever the right-hand side limit exists.

R^k /

LEMMA 5.1. *Suppose $v'(\hat{x}; \xi)$ exists for a given $\hat{x} \in Q$ and $\xi \in R^n$. Then,*

$$\sup_{p \in D^-_{\varepsilon,x}v(\hat{x})} p \cdot \xi - \varepsilon|\xi| \le v'(\hat{x}; \xi) \le \inf_{p \in D^+_{\varepsilon,x}v(\hat{x})} p \cdot \xi + \varepsilon|\xi|, \tag{5.1}$$

where $\sup\{\emptyset\} = -\infty$, $\inf\{\emptyset\} = +\infty$.

delete "{ and

PROOF. The result is clear if $\xi = 0$. So we assume $\xi \neq 0$. For any $p \in D^+_{\varepsilon,x}v(\hat{x})$,

$$\limsup_{h \to 0+} \frac{v(\hat{x} + h\xi) - v(\hat{x}) - h\xi \cdot p}{h|\xi|} \le \varepsilon,$$

hence $v'(\hat{x}; \xi) \le p \cdot \xi + \varepsilon|\xi|$. This implies the right-hand side of (5.1). Similarly for the left-hand side. □

THEOREM 5.1. *Let $(x^\varepsilon(\cdot), u^\varepsilon(\cdot))$ be a given admissible pair for the problem C_{sy}. Suppose that there is a constant C_{10} such that for a.e. $t \in [s, T]$, there exists $(p^\varepsilon(t), q^\varepsilon(t)) \in D^+_{\varepsilon,t,x}V(t, x^\varepsilon(t))$ satisfying*

$$p^\varepsilon(t) \le H(t, x^\varepsilon(t), u^\varepsilon(t), q^\varepsilon(t)) + C_{10}\varepsilon, \tag{5.2}$$

then $(x^\varepsilon(\cdot), u^\varepsilon(\cdot))$ is a near-optimal pair with order ε.

PROOF. We shall set $f^\varepsilon(t) = f(t, x^\varepsilon(t), u^\varepsilon(t))$, etc. to simplify the notation. Since both V and x^ε are Lipschitz, $t \mapsto V(t, x^\varepsilon(t))$ is differentiable almost everywhere. Fix $r \in [s,T]$ such that $dV(t, x^\varepsilon(t))/dt|_{t=r}$ exists, that $\lim_{h\to 0+} \int_r^{r+h} f^\varepsilon(t)\,dt = f^\varepsilon(r)$, and that (5.2) holds. Then,

$$
\begin{aligned}
\frac{d}{dt}V(t,x^\varepsilon(t))|_{t=r} &= \lim_{h\to 0+} \frac{V(r+h, x^\varepsilon(r+h)) - V(r, x^\varepsilon(r))}{h} \\
&= \lim_{h\to 0+} \frac{V\left(r+h, x^\varepsilon(r) + \int_r^{r+h} f^\varepsilon(r)\,dr\right) - V(r, x^\varepsilon(r))}{h} \\
&= \lim_{h\to 0+} \frac{V(r+h, x^\varepsilon(r) + hf^\varepsilon(r) + o(h)) - V(r, x^\varepsilon(r))}{h} \\
&= \lim_{h\to 0+} \frac{V(r+h, x^\varepsilon(r) + hf^\varepsilon(r)) - V(r, x^\varepsilon(r))}{h} \qquad (5.3)\\
&\qquad\qquad \text{(by Lipschitz property of } V\text{)} \\
&= V'((r, x^\varepsilon(r)); (1, f^\varepsilon(r))) \\
&\le p^\varepsilon(r) + q^\varepsilon(r)\cdot f^\varepsilon(r) + \varepsilon(1 + |f^\varepsilon(r)|) \quad \text{(by Lemma 5.1)} \\
&\le p^\varepsilon(r) + q^\varepsilon(r)\cdot f^\varepsilon(r) + \varepsilon(1 + C + C|x^\varepsilon(r)|) \\
&\le -L^\varepsilon(r) + C_{11}\varepsilon \quad \text{(by (5.2))}.
\end{aligned}
$$

It follows that

$$V(T, x^\varepsilon(T)) - V(s,y) = \int_s^T \frac{d}{dt}V(t, x^\varepsilon(t))|_{t=r}\,dr \le -\int_s^T L^\varepsilon(r)\,dr + C_{11}\varepsilon(T-s),$$

which implies

$$J(s,y;u^\varepsilon(\cdot)) = \int_s^T L^\varepsilon(r)\,dr + h(x^\varepsilon(T)) \le V(s,y) + C_{11}\varepsilon(T-s).$$

This proves the theorem. □

REMARK 5.1. The condition (5.2) implies that

$$H(t, x^\varepsilon(t), u^\varepsilon(t), q^\varepsilon(t)) \ge \max_{u\in\Gamma} H(t, x^\varepsilon(t), u, q^\varepsilon(t)) - C_{12}\varepsilon. \tag{5.4}$$

This is easily seen by Theorem 3.1:

$$-p^\varepsilon(t) + \sup_{u\in\Gamma} H(t, x^\varepsilon(t), u, q^\varepsilon(t)) \le (C + 1 + C|x^\varepsilon(t)|)\varepsilon \le C_{13}\varepsilon,$$

which yields (5.4) under (5.2).

Now let us give a parallel result in terms of $D^-_{\varepsilon,t,x}V$.

THEOREM 5.2. *Let $(x^\varepsilon(\cdot), u^\varepsilon(\cdot))$ be a given admissible pair for the problem C_{sy}. Suppose that there is a constant C_{14} such that for a.e. $t \in [s,T]$, there exists*

MASTER SET

$(p^{\varepsilon}(t), q^{\varepsilon}(t)) \in D^{-}_{\varepsilon, t, x}V(t, x^{\varepsilon}(t))$ satisfying

$$p^{\varepsilon}(t) \le H(t, x^{\varepsilon}(t), u^{\varepsilon}(t), q^{\varepsilon}(t)) + C_{14}\varepsilon, \tag{5.5}$$

then $(x^{\varepsilon}(\cdot), u^{\varepsilon}(\cdot))$ is a near-optimal pair with order ε.

PROOF. Fix $r \in [s, T]$ such that $dV(t, x^{\varepsilon}(t))/dt|_{t=r}$ exists, that $\lim_{h\to 0+} \int_{r-h}^{r} f^{\varepsilon}(t)\,dt = f^{\varepsilon}(r)$, and that (5.5) holds. Then,

Possible to reword to avoid letter spacing? Comp

The present setting looks o.k.

$$\begin{aligned}
\frac{d}{dt}V(t, x^{\varepsilon}(t))|_{t=r} &= -\lim_{h\to 0+} \frac{V(r-h, x^{\varepsilon}(r-h)) - V(r, x^{\varepsilon}(r))}{h} \\
&= -\lim_{h\to 0+} \frac{V(r-h, x^{\varepsilon}(r) - \int_{r-h}^{r} f^{\varepsilon}(r)\,dr) - V(r, x^{\varepsilon}(r))}{h} \\
&= -\lim_{h\to 0+} \frac{V(r-h, x^{\varepsilon}(r) - hf^{\varepsilon}(r)) - V(r, x^{\varepsilon}(r))}{h} \\
&= -V'((r, x^{\varepsilon}(r)); (-1, -f^{\varepsilon}(r))) \qquad (5.6) \\
&\le -\max_{(p,q)\in D^{-}_{t,x}V(r, x^{\varepsilon}(r))} [-p - q\cdot f^{\varepsilon}(r)] + \varepsilon(1 + |f^{\varepsilon}(r)|) \\
&= \min_{(p,q)\in D^{-}_{t,x}V(r, x^{\varepsilon}(r))} [p + q\cdot f^{\varepsilon}(r)] + \varepsilon(1 + |f^{\varepsilon}(r)|) \\
&\le p^{\varepsilon}(r) + q^{\varepsilon}(r)\cdot f^{\varepsilon}(r) + \varepsilon(1 + C + C|x^{\varepsilon}(r)|) \\
&\le -L^{\varepsilon}(r) + C_{15}\varepsilon.
\end{aligned}$$

The desired result therefore follows exactly as in the proof of Theorem 5.1. □

EXAMPLE 5.1. Consider a deterministic production planning problem with a single machine producing a single type of products. Let $x(t)$ and $u(t)$ denote the surplus (state variable) and the production rate (control variable) at time $t \in [0, T]$, respectively. Suppose that the demand rate is a constant z. The dynamic is

Author Query. See msp 20

should be "dyna

$$\dot{x}(t) = u(t) - z, \qquad x(s) = y.$$

The cost function is

$$J(s, y; u(\cdot)) = \int_{s}^{T} (ax^{+}(t) + bx^{-}(t))\,dt,$$

where $a, b \ge 0$, representing the inventory cost and the backlog penalty, respectively. The machine has a maximum production rate k, which is assumed to be larger than z. It is well known that the optimal policy for such a problem is the so-called zero-inventory policy, namely, the feedback control given as follows:

$$u(t, x) = \begin{cases} 0, & \text{if } x > 0, \\ k, & \text{if } x < 0, \\ z, & \text{if } x = 0. \end{cases}$$

The value function can therefore be easily calculated to be

$$V(t,x) = \begin{cases} \dfrac{ax^2}{2z}, & \text{if } x \geq 0, \dfrac{x}{z} + t \leq T, \\ ax(T-t) - \frac{1}{2}az(T-t)^2, & \text{if } x > 0, \dfrac{x}{z} + t > T, \\ \dfrac{bx^2}{2(k-z)}, & \text{if } x \leq 0, \dfrac{-x}{k-z} + t \leq T, \\ -bx(T-t) - \frac{1}{2}b(k-z)(T-t)^2, & \text{if } x < 0, \dfrac{-x}{k-z} + t > T. \end{cases}$$

Now let us consider an ε-inventory policy, ε being a sufficiently small positive number, defined as follows:

$$u^{\varepsilon}(t,x) = \begin{cases} 0, & \text{if } x > \varepsilon, \\ k, & \text{if } x < \varepsilon, \\ z, & \text{if } x = \varepsilon. \end{cases}$$

Intuitively, this policy should be near-optimal. Now let us verify this by Theorem 5.1. For simplicity, consider the problem with initial time 0 and initial state 0. Under the feedback $u^{\varepsilon}(t,x)$, the state and the control trajectories are

$$(x^{\varepsilon}(t), u^{\varepsilon}(t)) = \begin{cases} ((k-z)t, k), & \text{if } 0 \leq t \leq \dfrac{\varepsilon}{k-z}, \\ (\varepsilon, z), & \text{if } \dfrac{\varepsilon}{k-z} \leq t \leq T. \end{cases}$$ <

For $0 \leq t < \varepsilon/(k-z)$, take $(p^{\varepsilon}(t), q^{\varepsilon}(t)) = (V_t(t, x^{\varepsilon}(t)), V_x(t, x^{\varepsilon}(t))) = (0, a(k-z)t/z)$, then it is easy to obtain that

$$0 = p^{\varepsilon}(t) < H(t, x^{\varepsilon}(t), u^{\varepsilon}(t), q^{\varepsilon}(t)) + \frac{ak}{z}\varepsilon.$$

Similarly, one can show that

$$p^{\varepsilon}(t) \leq H(t, x^{\varepsilon}(t), u^{\varepsilon}(t), q^{\varepsilon}(t)) + a\varepsilon, \quad \text{for } \frac{\varepsilon}{k-z} \leq t \leq T.$$

Thus the ε-inventory policy for the problem C_{00} is indeed near-optimal with order ε.

Now let us describe how to construct near-optimal feedback controls by the verification theorems obtained.

THEOREM 5.3. *Let u^{ε} be an admissible feedback control, and p^{ε} and q^{ε} be two measurable functions of (t,x) satisfying*

$$(p^{\varepsilon}(t,x), q^{\varepsilon}(t,x)) \in D^{+}_{\varepsilon,t,x}V(t,x) \cup D^{-}_{\varepsilon,t,x}V(t,x) \text{ for all } (t,x).$$

Displayed) to avoid poor break in math OK? Comp

If

$$(5.7) \qquad p^{\varepsilon}(t,x) \leq H(t,x,u^{\varepsilon}(t,x), q^{\varepsilon}(t,x)) + C_{16}\varepsilon$$

for all $(t,x) \in [0,T] \times R^d$, then u^{ε} is near-optimal with order ε.

O.K.

PROOF. Let $x^\varepsilon(\cdot)$ be the state corresponding to u^ε with a fixed initial (s, y), and let $p^\varepsilon(t) = p^\varepsilon(t, x^\varepsilon(t))$ and $q^\varepsilon(t) = q^\varepsilon(t, x^\varepsilon(t))$. Since $(p^\varepsilon(t), q^\varepsilon(t)) \in D^+_{\varepsilon,t,x}V(t, x^\varepsilon(t)) \cup D^-_{\varepsilon,t,x}V(t, x^\varepsilon(t))$ and (5.7) holds, we can obtain either (5.3) or (5.6) for a.e. t, depending on whether $(p^\varepsilon(t), q^\varepsilon(t)) \in D^+_{\varepsilon,t,x}V(t, x^\varepsilon(t))$ or $(p^\varepsilon(t), q^\varepsilon(t)) \in D^-_{\varepsilon,t,x}V(t, x^\varepsilon(t))$. Therefore, the result follows from the proofs of Theorems 5.1 and 5.2. □

By Theorem 5.3, one can obtain a near-optimal feedback control by choosing appropriate $u(t, x)$ that satisfies (5.7) for each (t, x). The following is an example.

EXAMPLE 5.2. For the production planning model in Example 5.1, an ε-inventory policy has been shown to be near-optimal. Note it is still a bang-bang type control, just like the optimal zero-inventory policy. Let us now try to select a near-optimal feedback control that is Lipschitz continuous in the state. To this end, we note that

$$V_t(t,x) - H(t,x,u,V_x(t,x)) = \begin{cases} \dfrac{axu}{z}, & \text{if } x \ge 0,\ \dfrac{x}{z} + t \le T, \\ a(T-t)u, & \text{if } x > 0,\ \dfrac{x}{z} + t > T, \\ \dfrac{bx(u-k)}{k-z}, & \text{if } x \le 0,\ \dfrac{-x}{k-z} + t \le T, \\ b(T-t)(k-u), & \text{if } x < 0,\ \dfrac{-x}{k-z} + t > T. \end{cases}$$

Therefore the minimum value of $V_t(t, x) - H(t, x, u, V_x(t, x))$ over $u \in [0, k]$ is 0. By Theorem 5.3, any feedback control that attains a near-minimum value of $V_t(t, x) - H(t, x, u, V_x(t, x))$ for each (t, x) is near-optimal. The following is one such control with order ε:

$$u^\varepsilon(t,x) = \begin{cases} 0, & \text{if } x > \varepsilon, \\ -\dfrac{z}{\varepsilon}x + z, & \text{if } -\dfrac{k-z}{z}\varepsilon \le x \le \varepsilon, \\ k, & \text{if } x \overset{<}{\cancel{\le}} -\dfrac{k-z}{z}\varepsilon. \end{cases}$$

This control is in fact a "Lipschitz modification" of the zero-inventory control.

6. Concluding remarks. By studying near-optimality, it is possible to greatly simplify the optimization process with only a small loss in the objective of the decision-makers. In this paper, we applied the dynamic programming approach to deal with near-optimal controls in terms of a small parameter ε. Here ε may appear in two different situations. First, it may reflect the loss in the objective value allowed by the decision-maker, who may have set this "tolerance level" before he or she started to seek a near-optimal policy. Second, ε may be a parameter representing the complexity of the original decision problem that can be approximated by ~~simple~~ simpler models as ε is sufficiently small. For the second situation, two good examples are hierarchical production models studied by Sethi, Zhang and Zhou (1994) and Sethi and Zhou (1994), and discrete approximation studied by Gabasov, Kirillova and Mordukhovich (1983) and Mordukhovich (1988).

In this paper, we employed viscosity solution theory to handle the inherent nonsmoothness occurring in the dynamic programming approach. As mentioned in the Introduction, another important framework of nonsmooth analysis is Clarke's generalized gradient, which is indeed where the name of "nonsmooth analysis" originates. The theory of generalized gradient has proved very useful in treating *deterministic* controls with *state constraints*. The nonsmooth versions of the maximum principle, dynamic programming, and their relationships for optimal controls have been well established within this framework (Clarke 1990 and Clarke and Vinter 1987). We believe that the theory can also be applied to near-optimal control problems, particularly those with state constraints. This remains an interesting and challenging research problem.

The reason we choose the framework of viscosity solution to study near-optimality in this series of papers is twofold. First, "viscosity solution" and "generalized gradient" are two *different* frameworks of nonsmooth analysis (see discussions in Frankowska 1989 and Zhou 1993), each having its own advantages and disadvantages. It is certainly desirable to derive results in one consistent framework. Second, we intend to study *stochastic* near-optimal controls in this series of papers. In order to apply dynamic programming to stochastic controls of diffusion type, one has to work with *second-order* derivatives. Note that the extension from first-order to second-order superdifferential/subdifferential is obvious and straightforward (Lions 1983), while it is not the case for generalized gradient. Indeed, there have been many different versions of generalized second-order derivatives introduced for different purposes. Recently, Haussmann (1992) proposed a generalized Hessian based on Brownian motion. Clearly this notion targeted solving the stochastic control problem. Later, Haussmann (1994) developed a kind of generalized solution of the HJB equation within his framework of generalized Hessian. However, the study of this new notion and its applications is yet to be comprehensive. On the other hand, there has been a rather extensive study on unifying the maximum principle and dynamic programming in *stochastic* controls by using the viscosity solution approach (Zhou 1991), but there is no similar study in the language of generalized gradient according to our knowledge. Consequently, we think that viscosity solution may be as well a better framework to deal with stochastic near-optimal controls, which will be a subject of our forthcoming papers.

Acknowledgements. This work was supported by the RGC Earmarked Grant CUHK 249/94E. The author is grateful to the referees for their constructive comments and suggestions, which resulted in a much improved version of the paper.

References

Barron, E. N., R. Jensen (1986). The Pontryagin maximum principle from dynamic programming and viscosity solutions to first-order partial differential equations. *Trans. Amer. Math. Soc.* **298** 635–641.

Bellman, R. (1957). *Dynamic Programming*, Princeton University Press, Princeton, New Jersey.

Berkovitz, L. D. (1974). *Optimal Control Theory*, Springer-Verlag, New York.

Cannarsa, P., H. Frankowska (1991). Some characterizations of optimal trajectories in control theory. *SIAM J. Control Optim.* **29** 1322–1347.

Clarke, F. H. (1973). *Necessary conditions for nonsmooth problems in optimal control and the calculus of variations*, Ph.D. Thesis, University of Washington, St. Louis.

_____ (1990). *Optimization and Nonsmooth Analysis*, SIAM, Philadelphia, Pennsylvania.

_____, R. B. Vinter (1987). The relationship between the maximum principle and dynamic programming. *SIAM J. Control Optim.* **25** 1291–1311.

Crandall, M. G., P. L. Lions (1983). Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.* **277** 1–42.

_____, _____ (1985). Hamilton-Jacobi equations in infinite dimensions. I. Uniqueness of viscosity solutions. *J. Funct. Anal.* **62** 379–396.

Ekeland, I. (1974). On the variational principle. *J. Math. Anal. Appl.* **47** 324–353.

_____, G. Lebourg (1976). Generic Frechet differentiability and perturbed optimization in Banach space. *Trans. Amer. Math. Soc.* **224** 193–216.

Fleming, W. H., R. W. Rishel (1975). *Deterministic and Stochastic Optimal Controls*, Springer-Verlag, New York.

_____, H. M. Soner (1992). *Controlled Markov Processes and Viscosity Solution*, Springer-Verlag, New York.

Author Quer See msp 25

O.K.

Frankowska, H. (1989). Hamilton-Jacobi equations: Viscosity solutions and generalized gradients. *J. Math. Anal. Appl.* **141** 21–26.

Gabasov, R., F. M. Kirillova, B. Sh. Mordukhovich (1983). The ε-maximum principle for suboptimal controls. *Soviet Math. Dokl.* **27** 95–99.

Haussmann, U. G. (1992). A probabilistic approach to the generalized Hessian. *Math. Oper. Res.* **17** 411–443.

_____ (1994). Generalized solutions of the Hamilton-Jacobi equation of stochastic control. *SIAM J. Control Optim.* **32** 728–743.

Lions, P. L. (1982). *Generalized Solutions of Hamilton-Jacobi Equations*, Pitman, London, England.

_____ (1983). Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations, Part 2: Viscosity solutions and uniqueness. *Comm. Partial Differential Equations* **11** 1229–1276.

Mordukhovich, B. Sh. (1988). *Approximation Methods in Problems of Optimization and Control*, Nauka, Moscow.

Pontryagin, M. L., V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mischenko (1962). *The Mathematics Theory of Optimal Processes*, Interscience, New York.

Sethi, S. P., Q. Zhang, X. Y. Zhou (1994). Hierarchical controls in stochastic manufacturing systems with convex costs. *J. Optim. Theory Appl.* **80** 299–318.

_____, X. Y. Zhou (1994). Dynamic stochastic jobshops and hierarchical controls. *IEEE Trans. Automat. Control* **AC-39** 2061–2076.

Zhou, X. Y. (1990). Maximum principle, dynamic programming, and their connection in deterministic control. *J. Optim. Theory Appl.* **65** 363–373.

_____ (1991). A unified treatment of maximum principle and dynamic programming in stochastic controls. *Stochastics Stochastics Rep.* **36** 137–161.

_____ (1993). Verification theorems within the framework of viscosity solutions. *J. Math. Anal. Appl.* **177** 208–225.

_____ (1995). Deterministic near-optimal controls. Part I: Necessary and sufficient conditions for near-optimality. *J. Optim. Theory Appl.* **85** 473–488.

X. Y. Zhou: Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong; e-mail: xyzhou@se.cuhk.hk