Stochastic Control for Linear Systems Driven by Fractional Noises

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March 23, 2006

Abstract

This paper is concerned with optimal control of stochastic linear systems involving fractional Brownian motion (FBM). First, as a prerequisite for studying the underlying control problems, some new results on stochastic integrals and stochastic differential equations associated with FBM are established. Then, three control models are formulated and studied. In the first two models, the state is scalar-valued and the control is taken as Markovian. The problems are either completely solved based on a Riccati equation (for Model 1 where the cost is a quadratic functional on state and control variables) or optimality is characterized (for Model 2 where the cost is a power functional). The last control model under investigation is a general one where the system involves the Stratonovich integral with respect to FBM, the state is multi-dimensional and the admissible controls are not limited to Markovian. A new Riccati-type equation, which is a backward stochastic differential equation involving both FBM and normal Brownian motion, is introduced. Optimal control and optimal value of the model are explicitly obtained based on the solution to this Riccati-type equation.

Key Words. Fractional Brownian motion (FBM), stochastic linear–quadratic (LQ) control, Itô integral, Stratonovich integral, Hu–Meyer formula, multiple integral, Riccati equation, Malliavin derivative.

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1 Introduction

Stochastic processes with long memory have been studied in the literature. One important class of such processes is the fractional Brownian motion (FBM). In recent years there has been considerable research interest on stochastic calculus for FBM. A central issue is to define a proper stochastic integral with respect to the FBM. Attempts were made in Lin [25] and Dai and Heyde [6]; however the stochastic integrals defined in [25], [6] do not satisfy the familiar zero-mean property as opposed to the normal Brownian motion counterpart. More recently, Duncan, Hu and Pasik-Duncan [9] employed the Wick calculus to define a fractional stochastic integral whose mean is indeed zero. This property is very convenient for both theoretical development and practical applications. A substantial stochastic calculus was established for this new integral in [9].

It is natural and interesting to study the optimal control of systems driven by fractional noises, namely, systems described by stochastic differential equations involving FBM. Some such control problems as well as their applications in mathematical finance have been studied in [3], [15], [20], [22], within the framework of the fractional calculus of [9].

On the other hand, a classically important class of stochastic control problems is the so-called linear–quadratic (LQ) control, where the system is linear with respect to the state and control variables and the cost functional is quadratic in the two variable. Such a problem can often be solved explicitly by solving a relevant Riccati equation. While stochastic LQ control for systems driven by the (normal) Brownian motion (i.e., the systems are perturbed by the usual "white noise") is a classical problem on which a vast amount of research works have been documented, there have recently been renewed interest on the problem, in the form of the so-called "indefinite stochastic LQ control". See [1], [4], [5], [23], [28], [29] and the references therein.

This paper represents the first attempt, to our best knowledge, in systematically solving some optimal control problems on linear systems driven by FBM, including LQ control with fractional noises. The controlled systems under consideration include

$$dx_t = (A_t x_t + B_t u_t)dt + (C_t x_t + D_t u_t)dW_t^H\,, \quad x_0 = x \quad \text{is given}\,, \tag{1.1}$$

and

$$dx_t = (A_t x_t + B_t u_t)dt + (C_t x_t + D_t u_t)\circ dW_t^H\,, \quad x_0 = x \quad \text{is given}\,, \tag{1.2}$$

where A_t, B_t, C_t and D_t are given (matrix-valued) stochastic processes, $\{W_t^H, t \geq 0\}$ is an FBM of Hurst parameter $H \in [1/2, 1)$, and dW_t^H and $\circ dW_t^H$ are the Itô and Stratonovich integrals, respectively, as defined in [9]. Our problem is to minimize a cost functional under the constraint (1.1) or (1.2).

The above problem, in its greatest generality, seems to be too difficult to solve at the moment owing to the long range dependence of the FBM. In this paper we propose and study three special models. The first model is for system (1.1) where the state x_t is one dimensional, D_t is absent, and the cost functional is quadratic. Moreover, we consider optimal control among the class of linear

Markovian controls, namely, the ones of the form

$$u_t = K_t x_t \,, \tag{1.3}$$

where K_t is a deterministic (matrix-valued) function of t. (Notice that linear Markovian control constitutes an important control class due to its mathematical simplicity and ease of implementation. Also it is well known that the optimal stochastic LQ control with normal Brownian motion turns out to be a linear Markovian control.) In this case, the optimal control (or equivalently the optimal K_t) is found based on the solution to a Riccati equation. This Riccati equation is solvable by using the classical theory. Thus for this special yet interesting control model we obtain a complete solution to the LQ problem with fractional noises. The second model is more general than the first one in that $D_t \not\equiv 0$ and the cost functional is of a power form. Remark that the presence of D_t is a crucially different and difficult feature even in the normal stochastic LQ control (see [29] for detailed discussions). For this second model we use the technique of calculus of variation to obtain an integral equation that the optimal K_t must satisfy. We then demonstrate how to solve this integral equation in several special cases. The last control model under study is an LQ problem where the system is governed by (1.2), the state variable is multi-dimensional and the admissible controls are not limited to being Markovian. However D_t is assumed to be identically zero. We introduce a Riccati-type equation, which is a backward stochastic differential equation involving integrals with respect to both FBM and normal Brownian motion. Assuming the solvability of this equation along with some technical conditions, we derive an optimal control along with the optimal value based on the completion-of-square technique. As a by-product the optimal control is in the form of linear Markovian type. We also illustrate how we obtain such a Riccati-type equation heuristically using the method of approximation.

It is worth mentioning that study on stochastic control problems with

fractional noises will inevitably involve stochastic calculus on FBM, the associated stochastic integrals and differential equations. However, a rich, complete theory on such calculus is still lacking due to the fact that the very definition of the fractional stochastic integral we are employing was introduced only very recently [9]. Therefore, in this paper we spend some effort in deriving results on fractional stochastic calculus that are necessary for studying the subsequent control models. Some of the results are new and interesting in its own right.

The remainder of the paper is organized as follows. Section 2 is devoted to fractional stochastic calculus useful for the sequel. Three different control models are treated in Sections 3,4 and 5 respectively. Finally, Section 6 concludes the paper with some remarks.

2 Fractional Stochastic Calculus

This section presents stochastic calculus on fractional Brownian motion that is necessary for the control problems under investigation in this paper. Most of the results are new and interesting in its own right from the point of view of fractional calculus.

2.1 Fractional Brownian motion

In this subsection we outline some of the notation and results on fractional Brownian motion. The primary reference in this regard is [9]. Let $\Omega = C_0(0,T;\mathbb{R})$ be the Banach space of real-valued continuous function on $[0,T]$ with the initial value zero and the super norm. There is a probability measure P on $(\Omega\,,\mathcal{F})$, where $\mathcal{F}$ is the Borel σ-algebra on Ω, such that on the probability space $(\Omega\,,\mathcal{F}\,,P)$ the process $\{W_t^H\,,0\le t\le T\}$ defined as

$$W_t^H(\omega)=\omega(t),\ \ \forall\omega\in\Omega$$

is a (one-dimensional) Gaussian process with mean

$$\mathbb{E}\ W_t^H = W_0^H = 0,\ \ \forall t\in[0,T],$$

and covariance

$$\mathbb{E}\ \left(W_t^H W_s^H\right) = \frac{1}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right)\ ,\ \forall t,s\in[0,T].$$

This canonical process $\{W_t^H\,,0\le t\le T\}$ is called a (standard) *fractional Brownian motion* (FBM) of *Hurst parameter* H. In this paper $H\in[\frac{1}{2},1)$ is fixed. An FBM with $H=\frac{1}{2}$ reduces to the (normal) Brownian motion.

An FBM generates a filtration $\{\mathcal{F}_t,\ 0\le t\le T\}$ with $\mathcal{F}_t=\sigma\{W_s^H,0\le s\le t\}$. Throughout this paper, only the canonical process and the associated filtered probability space $(\Omega\,,\mathcal{F}\,,P;\mathcal{F}_t)$, as described above, are used. Also notice that the FBM is assumed to be one-dimensional only for notational simplicity; there is no essential difficulty with a multi-dimensional FBM where each component is a one-dimensional FBM with the *same* Hurst parameter H.

The following lemma is useful (see [7], [13], [26]).

Lemma 2.1 *An FBM W_t^H with $H\in(\frac{1}{2},1)$ can be represented as*

$$W_t^H = \int_0^t Z(t,s)dW_s\,, \tag{2.1}$$

where $\{W_t\,,0\le t\le T\}$ is a one-dimensional Brownian motion, and

$$Z(t,s):=(H-\frac{1}{2})c_H s^{\frac{1}{2}-H}\int_s^t r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}}dr,$$

with $c_H=\sqrt{\frac{H(2H-1)\Gamma(\frac{3}{2}-H)}{\Gamma(H-\frac{1}{2})\Gamma(2-2H)}}$ ($\Gamma(\cdot)$ is the gamma function). Moreover, W_t and W_t^H generate the same filtrations.

Similar to the Brownian motion, the FBM is not differentiable almost surely. As the "derivative" of a Brownian motion is usually refereed to as white noise, that of an FBM is refereed to as *fractional noise*.

For a fixed $H \in (\frac{1}{2}, 1)$, denote a function $\phi : [0,T] \times [0,T] \to \mathbb{R}_+$ by

$$\phi(s,t) := H(2H-1)|s-t|^{2H-2}, \quad \forall s,t \in [0,T], \tag{2.2}$$

and let $L^2_\phi([0,T])$ be the Hilbert space of Borel measurable, scalar-valued functions f such that

$$|f|^2_\phi := \int_0^T\int_0^T \phi(s,t) f_s f_t ds dt < \infty, \tag{2.3}$$

with the norm $|f|_\phi$; see [11], [12], [13] for a detailed discussion on this space. The inner product on $L^2_\phi([0,T])$ is denoted by $\langle \cdot, \cdot \rangle_\phi$. Define a mapping Φ on $L^2_\phi([0,T])$:

$$(\Phi g)_t := \int_0^T \phi(t,s) g_s ds, \quad \forall g \in L^2_\phi([0,T]). \tag{2.4}$$

Then one can define the ϕ-derivative of a random variable $F \in L^p(\Omega, \mathcal{F}, P)$ $(p \geq 1)$ in the direction of Φg, where $g \in L^2_\phi([0,T])$, by

$$D_{\Phi g} F(\omega) := \lim_{\delta \to 0} \frac{1}{\delta} \left\{ F(\omega + \delta \int_0^\cdot (\Phi g)_t dt) - F(\omega) \right\} \tag{2.5}$$

if the limit exists in $L^p(\Omega, \mathcal{F}, P)$. Moreover, if there is a process $\{D^H_s F,\ 0 \leq s \leq T\}$ satisfying

$$D_{\Phi g} F = \int_0^T [D^H_s F] g_s ds \quad \text{a.s.}, \quad \forall g \in L^2_\phi([0,T]), \tag{2.6}$$

then F is said to be *ϕ-differentiable* and $D^H_s F$ is termed as the *Malliavin derivative* of F. Refer to [9],[19] for more information about the Malliavin derivative.

Finally, we remark that if $H = 1/2$, then the function $\phi(\cdot, \cdot)$ should be replaced by the Delta function $\delta(\cdot, \cdot)$ satisfying

$$\int_0^T \int_0^T \delta(s,s') f(s,s') ds ds' = \int_0^T f(s,s) ds, \quad \forall f(\cdot,\cdot) \in L^1([0,T] \times [0,T]).$$

2.2 Stratonovich integrals

Next, let us discuss on the Stratonovich stochastic integral, which will be dealt with later in this paper. First of all, let X_t be a semimartingale of the form

$$X_t = X_0 + \int_0^t f_s ds + \int_0^t g_s dW_s\,,$$

where $\{W_s\,, 0 \leq s \leq T\}$ is the Brownian motion given by Lemma 2.1 and X_0 a deterministic constant. If $f, g \in L^2_{\mathcal{F}}(0,T)$, where $L^2_{\mathcal{F}}(0,T)$ denotes the set of scalar-valued, $\mathcal{F}_t$-adapted, square integrable processes on $[0,T]$, then it is well known that (see, e.g., [21, Chapter III])

$$\int_0^t X_s \circ dW_s = \int_0^t X_s dW_s + \frac{1}{2}\int_0^t g_s ds, \quad \forall t \in [0,T], \tag{2.7}$$

where $\int_0^t X_s \circ dW_s$ denotes the Stratonovich (symmetric) integral with respect to the Brownian motion W_s. As a result,

$$\mathbb{E}\left(\int_0^T X_s \circ dW_s\right) = \frac{1}{2}\mathbb{E}\int_0^T g_s ds\,. \tag{2.8}$$

However, if a process Y_t is given as

$$Y_t = Y_0 + \int_0^t f_s ds + \int_0^t g_s dW_s^H\,, \tag{2.9}$$

where the stochastic integral with respect to the FBM W_s^H is defined as in [9], then we have a quite different result.

Theorem 2.2 *Let f_s and g_s satisfy the following conditions:*

$$\mathbb{E}\int_0^T |f_s|^2 ds < \infty, \quad \sup_{0\le s\le T}\mathbb{E}\,|g_s|^2 ds < \infty, \quad \text{and} \quad \mathbb{E}\int_0^T |D_s^H g_s|^2 ds < \infty\,, \tag{2.10}$$

and Y_t be given by (2.9), where $H \in (\frac{1}{2}, 1)$. Then

$$\int_0^t Y_s \circ dW_s = \int_0^t Y_s dW_s\,,\quad \forall t \in [0,T]. \tag{2.11}$$

As a consequence, we have

$$\mathbb{E}\left[\int_0^t Y_s \circ dW_s\right] = 0\,. \tag{2.12}$$

Proof Under the condition (2.10) on f and g we see that Y_t is well-defined (see [9]). Let $\pi : 0 = t_0 < t_1 < t_2 < \cdots < t_n = t$, be a partition of the interval $[0,t]$. By definition,

$$\begin{aligned}
\int_0^t Y_s \circ dW_s &= \lim_{|\pi|\to 0}\sum_{k=0}^{n-1}\frac{1}{2}\left(Y_{t_k} + Y_{t_{k+1}}\right)\left(W_{t_{k+1}} - W_{t_k}\right)\\
&= \lim_{|\pi|\to 0}\sum_{k=0}^{n-1} Y_{t_k}\left(W_{t_{k+1}} - W_{t_k}\right) + \lim_{|\pi|\to 0}\sum_{k=0}^{n-1}\frac{1}{2}\left(Y_{t_{k+1}} - Y_{t_k}\right)\left(W_{t_{k+1}} - W_{t_k}\right)\\
&:= \lim_{|\pi|\to 0} I_1 + \lim_{|\pi|\to 0} I_2\,,
\end{aligned}$$

where the limit is taken in the sense of "in probability" and $|\pi| := \max_{0\le k\le n-1}(t_{k+1} - t_k)$. It is easy to see by the definition of the Itô integral that the first term I_1 converges to $\int_0^t Y_s dW_s$.

To consider the second term I_2 we estimate $\mathbb{E}\,|Y_{t_{k+1}} - Y_{t_k}|^2$. From the definition of Y_t it follows

$$\begin{aligned}
\mathbb{E}\,\left|Y_{t_{k+1}} - Y_{t_k}\right|^2 &\le 2\mathbb{E}\left|\int_{t_k}^{t_{k+1}} f_s ds\right|^2 + 2\mathbb{E}\left|\int_{t_k}^{t_{k+1}} g_s dW_s^H\right|^2\\
&:= 2(I_3 + I_4)\,.
\end{aligned}$$

It follows from (2.10) that

$$I_3 \le \mathbb{E}\int_{t_k}^{t_{k+1}} |f_s|^2 ds \cdot |t_{k+1} - t_k|\,. \tag{2.13}$$

Moreover, by virtue of the Itô isometry ([9, Theorem 3.9]), we have

$$\begin{aligned} I_4 &= \int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}} \phi(t,s)\mathbb{E}\ (g_s g_t)\, dsdt + \mathbb{E}\ \left[\int_{t_k}^{t_{k+1}} D_s^H g_s ds\right]^2 \\ &:= I_5 + I_6\,. \end{aligned}$$

The assumption of the theorem and the definition of ϕ yield

$$\begin{aligned} I_5 &\le C\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}} \phi(t,s) dsdt \\ &\le C|t_{k+1}-t_k|^{2H}\,, \end{aligned}$$

where (and elsewhere) C is a generic positive constant that may vary from place to place. On the other hand,

$$I_6 \le \mathbb{E}\ \int_{t_k}^{t_{k+1}} |D_s^H g_s|^2 ds \cdot |t_{k+1}-t_k|\,.$$

Combining the above computations, we conclude that there is a constant C such that

$$\mathbb{E}\ |Y_{t_{k+1}} - Y_{t_k}|^2 \le C|t_{k+1}-t_k|^{2H} + U_k|t_{k+1}-t_k|, \tag{2.14}$$

where

$$U_k := \mathbb{E}\ \int_{t_k}^{t_{k+1}} (|f_s|^2 + |D_s^H g_s|^2) ds\,.$$

Therefore,

$$\begin{aligned} 2\mathbb{E}\ I_2 &\le \sum_{k=0}^{n-1} \mathbb{E}\ |Y_{t_{k+1}} - Y_{t_k}|\,|W_{t_{k+1}} - W_{t_k}| \\ &\le \sum_{k=0}^{n-1} \left[\mathbb{E}\ |Y_{t_{k+1}} - Y_{t_k}|^2\right]^{1/2} \left[\mathbb{E}\ |W_{t_{k+1}} - W_{t_k}|^2\right]^{1/2} \\ &\le C\sum_{k=0}^{n-1} (t_{k+1}-t_k)^{\frac{1}{2}+H} + C\sum_{k=0}^{n-1} [U_k(t_{k+1}-t_k)]^{1/2} \left[\mathbb{E}\ |W_{t_{k+1}} - W_{t_k}|^2\right]^{1/2} \\ &:= I_7 + I_8\,. \end{aligned}$$

It is easy to check that

$$I_7 \le C\max_{0\le k\le n-1} (t_{k+1}-t_k)^{H-\frac{1}{2}} \sum_{k=0}^{n-1} (t_{k+1}-t_k) \to 0$$

and

$$\begin{aligned} I_8 &\le C\sum_{k=0}^{n-1} U_k^{1/2} (t_{k+1}-t_k) \\ &\le C\left(\sum_{k=0}^{n-1} U_k\right)^{1/2} \left(\sum_{k=0}^{n-1} (t_{k+1}-t_k)^2\right)^{1/2} \\ &\le C[\mathbb{E}\ \int_0^T (|f_s|^2 + |D_s^H g_s|^2) ds]^{1/2} |\pi|^{1/2} \to 0\,. \end{aligned}$$

This implies that I_2 converges to 0 in $L^1(\Omega, \mathcal{F}, P)$ and, consequently, in probability. This completes the proof. □

Remark 2.3 Whereas Theorem 2.2 is useful in Section 5, it is interesting in its own right. In particular, we see that (2.11) is different from the case when the integrand is a semimartingale; see (2.7). On the other hand, one has a different formula than (2.11) if the integration there is replaced by one with respect to the FBM W_s^H (under different conditions on the integrand though); for details see [9, Theorem 3.12].

Remark 2.4 Since the Stratonovich type integral can be represented by using the Itô type integral ([9]), equations (2.9) and (2.10) are still true if Y_t is represented as Stratonovich integral

$$Y_t = Y_0 + \int_0^t f_s ds + \int_0^t g_s \circ dW_s^H$$

under some mild conditions on f and g. Details are left to the interested readers.

2.3 Multiple integrals

In this section we recall some results from [9] on multiple integrals. These results are needed in this paper and we refer to [9] for more detail.

A function $f : [0,T]^n \to \mathbb{R}$ is called symmetric on $[0,T]^n$ if

$$f(s_{i_1}, \cdots, s_{i_n}) = f(s_1, \cdots, s_n), \quad (s_1, \cdots, s_n) \in [0,T]^n,$$

for any permutation $(i_1, i_2, \cdots, i_n)$ of $(1, 2, \cdots, n)$. Denote

$$\begin{aligned} L^2_\phi([0,T]^n) \quad &:= \quad \{f : [0,T]^n \to \mathbb{R} \text{ is measurable and symmetric in its arguments}\,, \\ &\qquad |f|^2_\phi := \langle f, f\rangle_\phi < \infty\Big\}\,, \end{aligned}$$

where

$$\begin{aligned} \langle f, g\rangle_\phi \quad &:= \quad \int_{[0,T]^{2n}} \phi(u_1, v_1)\phi(u_2, v_2) \cdots \phi(u_n, v_n) f(u_1, u_2, \cdots, u_n) \\ &\qquad g(v_1, v_2, \cdots, v_n) du_1 du_2 \cdots du_n dv_1 dv_2 \cdots dv_n\,. \end{aligned}$$

If $f \in L^2_\phi([0,T]^n)$, then the multiple Itô integral

$$\begin{aligned} I_n(f) \equiv I_{n,T}(f) \quad &= \quad \int_{0 \le t_1, \cdots, t_n \le T} f(t_1, \cdots, t_n) dW_{t_1}^H \cdots dW_{t_n}^H \\ &= \quad n! \int_{0 \le t_1 < \cdots < t_n \le T} f(t_1, \cdots, t_n) dW_{t_1}^H \cdots dW_{t_n}^H \end{aligned}$$

is well-defined, and (see [9, Lemma 6.6][1])

$$\mathbb{E}\ (I_n(f)I_m(g)) = \begin{cases} n!\langle f,g\rangle_\phi & \text{if } n=m \\ 0 & \text{if } n\neq m. \end{cases} \tag{2.15}$$

The following result concerns the Malliavin derivative of a multiple integral.

Theorem 2.5 *We have*

$$D_s^H I_n(f_n) = nI_{n-1}\left(\int_0^t \phi(s,r)f_{n-1}(r)dr\right),\quad \forall f_n\in L_\phi^2([0,T]^n), \tag{2.16}$$

where for any $0\le r\le T$, $f_{n-1}(r)$ denotes a function of $n-1$ variables given by

$$f_{n-1}(r)(s_1,\cdots,s_{n-1}) := f_n(s_1,\cdots,s_{n-1},r)\,.$$

Proof For all $\delta\in\mathbb{R}$ and $h\in L_\phi^2([0,T])$, one has

$$\begin{aligned} & I_n(f_n)(W^H+\delta\int_0^\cdot h(s)ds) \\ = \;& I_n(f_n)(W^H)+\delta\sum_{k=1}^n\int_{0<s_1,\cdots,s_n<T} f_n(s_1,\cdots,s_n)dW_{s_1}^H\cdots\widehat{dW_{s_k}}\cdots dW_{s_n}^H h(s_k)ds_k+o(\delta) \\ = \;& I_n(f_n)(W^H)+\delta n\int_0^T I_{n-1}(f_{n-1}(r))h(r)dr+o(\delta)\,, \end{aligned}$$

where $\widehat{dW_{s_k}}$ means that dW_{s_k} term is taken off the product. Thus

$$\begin{aligned} D_{\Phi g}F &= n\int_0^T I_{n-1}(f_{n-1}(r))\Phi g(r)dr \\ &= n\int_0^T I_{n-1}(f_{n-1}(r))\int_0^T \phi(s,r)g(s)dsdr. \end{aligned}$$

By the definition of D_s^H (see (2.6)) we have

$$\begin{aligned} D_s^H I_n(f_n) &= n\int_0^T \phi(s,r)I_{n-1}(f_{n-1}(r))dr \\ &= nI_{n-1}\left(\int_0^T \phi(s,r)f_{n-1}(r)dr\right)\,. \end{aligned} \tag{2.17}$$

□

We also need to employ the multiple Stratonovich integrals. For this we first recall the definition of the k-trace Tr_ϕ^k. Let $f\in L_\phi^2([0,T]^n)$. For any positive integer $k\le n/2$, the k-trace of f is defined as a function of $n-2k$ variables:

$$\begin{aligned} \mathrm{Tr}_\phi^k f(t_1,\cdots,t_{n-2k}) \;:=\; & \int_{[0,T]^{2k}} f(s_1,s_2,\cdots,s_{2k-1},s_{2k},t_1,\cdots,t_{n-2k}) \\ & \phi(s_1,s_2)\phi(s_3,s_4)\cdots\phi(s_{2k-1},s_{2k})ds_1\cdots ds_{2k} \end{aligned}$$

[1]There is a typo in [9, Lemma 6.6]: the factorial $n!$ is missing from the term $\langle f,g\rangle_\phi$ on the right hand side of (6.5) there.

if the right hand side is integrable.

The multiple Stratonovich integral

$$
\begin{aligned}
S_n(f) \equiv S_{n,T}(f) &= \int_{0\le t_1,\cdots,t_n\le T} f(t_1,\cdots,t_n)\circ dW^H_{t_1}\cdots\circ dW^H_{t_n} \\
&= n!\int_{0\le t_1<\cdots<t_n\le T} f(t_1,\cdots,t_n)\circ dW^H_{t_1}\cdots\circ dW^H_{t_n}
\end{aligned}
$$

is well defined, and one has the following Hu–Meyer formula (see [16]-[18] and [9, Equation (6.10)])

$$
S_n(f) = \sum_{k\le[\frac{n}{2}]} \frac{n!}{2^k k!(n-2k)!} I_{n-2k}(\mathrm{Tr}^k_\phi f)\,. \tag{2.18}
$$

2.4 Stochastic differential equations with FBM

As the dynamics of the control problems under consideration in this paper is described by stochastic differential equations (SDEs) with FBM, we need first to study linear fractional SDEs in this section.

Introduce the following two linear equations

$$
\begin{cases} dx_t = A_t x_t dt + C_t x_t dW^H_t \\ x_0\in\mathbb{R}^n \qquad \text{be given and deterministic,}\end{cases} \tag{2.19}
$$

and

$$
\begin{cases} dx_t = A_t x_t dt + C_t x_t \circ dW^H_t \\ x_0\in\mathbb{R}^n \qquad \text{be given and deterministic,}\end{cases} \tag{2.20}
$$

where dW^H_t and $\circ dW^H_t$ denote respectively the Itô type and Stratonovich type differentials, in the sense of [9].

Theorem 2.6 *If A_t and C_t are measurable and essentially bounded deterministic functions in t, then equation (2.19) admits a unique solution. Moreover, the solution satisfies*

$$
\sup_{0\le t\le T} \mathbb{E}\,|x_t|^p < \infty\,, \quad \forall\, p\ge 1\,. \tag{2.21}
$$

Furthermore, the solution $x.$ has a continuous modification.

Proof The results when $H=\frac{1}{2}$ are clearly true by virtue of the classical SDE theory for normal Brownian motion. So we assume $H>\frac{1}{2}$. Let $\Psi(t,s)$ denote the fundamental solution associated with the deterministic part of equation (2.19), namely, $\Psi(t,s)$ satisfies

$$
\frac{d}{dt}\Psi(t,s) = A_t\Psi(t,s)\,,\quad 0\le s<t;\quad \Psi(s,s)=I\,,
$$

where I denotes the identity matrix. Since A_t is uniformly bounded in t, it is routine to prove, via Gronwall's inequality, that $\Psi(t,s)$ is uniformly bounded in (t,s). This in turn implies, via the above equation, that $\Psi(t,s)$ is Lipschitz in t with a Lipschitz constant independent of (t,s).

The equation (2.19) is thus equivalent to the following equation

$$x_t = \Psi(t,0)x_0 + \int_0^t \Psi(t,s)C_s x_s dW_s^H . \tag{2.22}$$

Replacing the x_s in the integral by this equation we have

$$\begin{aligned} x_t &= \Psi(t,0)x_0 + \int_0^t \Psi(t,s)C_s\Psi(s,0)x_0 dW_s^H \\ &\quad + \int_0^t \Psi(t,s_2)C_{s_2}\int_0^{s_2} \Psi(s_2,s_1)C_{s_1}x_{s_1}dW_{s_1}^H dW_{s_2}^H . \end{aligned}$$

Repeatedly applying this procedure, we obtain

$$\begin{aligned} x_t &= \Psi(t,0)x_0 + \int_0^t \Psi(t,s)C_s\Psi(s,0)x_0 dW_s^H \\ &\quad + \sum_{n=2}^{\infty}\int_{0<s_1<\cdots<s_n<t} \Psi(t,s_n)C_{s_n}\Psi(s_n,s_{n-1})C_{s_{n-1}}\cdots\Psi(s_2,s_1)C_{s_1}\Psi(s_1,0)x_0 dW_{s_1}^H\cdots dW_{s_n}^H. \end{aligned} \tag{2.23}$$

Now we are going to show that (2.23) is convergent in $L^2(\Omega,\mathcal{F},P)$ for any fixed t. Denote

$$f_{n,t}(s_1,\cdots,s_n) := \Psi(t,s_n)C_{s_n}\Psi(s_n,s_{n-1})C_{s_{n-1}}\cdots C_{s_2}\Psi(s_2,s_1)C_{s_1}\Psi(s_1,0)x_0 , \tag{2.24}$$

where $0 < s_1 < \cdots < s_n < t$. We also use the notation $f_n^t \equiv f_n^t(\cdot,\ldots,\cdot)$ to denote the symmetric extension of $f_{n,t}$ to the domain $[0,t]^n$, namely,

$$f_n^t(s_1,\cdots,s_n) := \Psi(t,s_{i_n})C_{s_{i_n}}\Psi(s_{i_n},s_{i_{n-1}})C_{s_{i_{n-1}}}\cdots C_{s_{i_2}}\Psi(s_{i_2},s_{i_1})C_{s_{i_1}}\Psi(s_{i_1},0)x_0$$

if $s_{i_1} \le s_{i_2} \le \cdots \le s_{i_n}$, where $i_1,\cdots,i_n$ is a permutation of $1,2,\cdots,n$.

By virtue of the underlying assumption it is easy to see that there is a positive constant K_t, depending on t, such that

$$\sup_{0\le s_1,\cdots,s_n\le t} |f_n^t(s_1,\cdots,s_n)| \le K_t^n .$$

The right hand side of (2.23) can be rewritten as

$$Z_t := \sum_{n=0}^{\infty}\frac{1}{n!}I_{n,t}(f_n^t) ,$$

where

$$I_{n,t}(f_n^t) := \int_{0<s_1,\cdots,s_n<t} f_n^t(s_1,\cdots,s_n)dW_{s_1}^H\cdots dW_{s_n}^H$$

is the multiple Itô integral defined in Section 2.3. It follows from (2.15) that

$$\begin{aligned} \mathbb{E}\,|Z_t|^2 &= \sum_{n=0}^{\infty}\frac{1}{(n!)^2}\mathbb{E}\,\left[I_{n,t}(f_n^t)\right]^2 \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\langle f_n^t, f_n^t\rangle_\phi . \end{aligned}$$

By its definition (see Section 2.3), we have

$$\begin{aligned}\langle f_n^t, f_n^t\rangle_\phi &= \int_{[0,t]^{2n}} \phi(s_1,t_1)\cdots\phi(s_n,t_n) f_n^t(s_1,\cdots,s_n) f_n^t(t_1,\cdots,t_n) ds_1\cdots ds_n dt_1\cdots dt_n \\ &\le K_t^{2n}\int_{[0,t]^{2n}} \phi(s_1,t_1)\cdots\phi(s_n,t_n) ds_1\cdots ds_n dt_1\cdots dt_n \\ &\le K_t^{2n},\end{aligned}$$

where K_t is a generic constant (depending on t) whose values may vary. This leads to

$$\mathbb{E}\,|Z_t|^2 \le \sum_{n=0}^\infty \frac{1}{n!} K_t^{2n} < \infty .$$

Hence the right hand side of (2.23) converges in $L^2(\Omega,\mathcal{F},P)$, and x_t is well defined.

Next we show that $\int_0^t C_s x_s dW_s^H$ is well defined for each $t\in[0,T]$, where $x.$ is defined by (2.23). Let us prove this for $t=T$, the other case being similar. From [9, p.591] there are several conditions to define $\int_0^T C_t x_t dW_t^H$. Let us check

$$\mathbb{E}\int_0^T |D_s^H(C_s x_s)|^2 ds < \infty .$$

Other conditions are routine and easier to check. Since C_t is a bounded deterministic function of t, it suffices to verify

$$\mathbb{E}\int_0^T |D_s^H x_s|^2 dt < \infty . \tag{2.25}$$

To this end, recall that f_n^t is the symmetric extension of $f_n(t;\cdot,\ldots,\cdot)$ defined by (2.20). With this notation we have

$$x_t = \sum_{n=0}^\infty \frac{1}{n!} I_{n,t}(f_n^t) .$$

Its Malliavin derivative is given by, appealing to Theorem 2.5,

$$D_s^H x_t = \sum_{n=1}^\infty \frac{1}{(n-1)!} I_{n-1,t}\left(\int_0^t \phi(s,r) f_{n-1}^t(r) dr\right) .$$

Therefore we have

$$\mathbb{E}\,|D_s^H x_t|^2 = \sum_{n=1}^\infty \frac{1}{(n-1)!}\langle \int_0^t \phi(s,r) f_{n-1}^t(r) dr , \int_0^t \phi(s,r) f_{n-1}^t(r) dr\rangle_\phi .$$

Since f_n^t is bounded on $[0,T]$, i.e.,

$$|f_n^t| \le K_T^n$$

we have

$$\left|\langle \int_0^t \phi(s,r) f_{n-1}^t(r) dr , \int_0^t \phi(s,r) f_{n-1}^t(r) dr\rangle_\phi\right|$$

$$
\begin{aligned}
\leq \ & \left| \int_0^t \int_0^t \int_{\substack{0<s_1,\cdots,s_{n-1}<t \\ 0<t_1,\cdots,t_{n-1}<t}} f_n^t(s_1,\cdots,s_{n-1},r_1) f_n^t(t_1,\cdots,t_{n-1},r_2) \right. \\
& \left. \phi(s_1,t_1)\cdots\phi(s_{n-1},t_{n-1}) ds_1 dt_1 \cdots ds_{n-1} dt_{n-1} dr_1 dr_2 \right| \\
\leq \ & K_T^n
\end{aligned}
$$

where and in what follows we shall use K_T to denote a generic constant which depends only on T and may be different in different places. Consequently we have

$$
\sup_{0\leq s,t\leq T} \mathbb{E}\, |D_s^H x_t|^2 \leq K_T\,. \tag{2.26}
$$

This implies easily (2.25).

Next, to obtain the estimate (2.21) we use a hypercontractivity inequality. Define the number operator $\Gamma(\alpha)$, where $\alpha \in \mathbb{R}$, as

$$
\Gamma(\alpha)F := \sum_{n=0}^{\infty} \alpha^n I_{n,t}(g_n)\,, \qquad \text{if} \quad F = \sum_{n=0}^{\infty} I_{n,t}(g_n)\,,
$$

where $g_n \in L_\phi^2([0,t]^n)$ with $\sum_{n=0}^\infty I_{n,t}(g_n)$ convergent in $L^2(\Omega,\mathcal{F},P)$. Nelson's hypercontractivity theorem (see [8], [14]) yields

$$
\left(\mathbb{E}\left|\Gamma\left(\frac{1}{p-1}\right)F\right|^p\right)^{1/p} \leq \left(\mathbb{E}\,|F|^2\right)^{1/2}\,, \quad \forall p\geq 2.
$$

Now for the f_n^t defined earlier, we set

$$
Z_{t,p} := \sum_{n=0}^{\infty} \frac{(p-1)^n}{n!} I_{n,t}(f_n^t)\,.
$$

By the same estimate for Z_t we obtain

$$
\mathbb{E}\,|Z_{t,p}|^2 < \infty\,.
$$

However, noting that $\Gamma(\frac{1}{p-1})Z_{t,p} = Z_t$ and applying the hypercontractivity inequality to $F = Z_{t,p}$, we have

$$
(\mathbb{E}\,|x_t|^p)^{1/p} \equiv (\mathbb{E}\,|Z_t|^p)^{1/p} \leq \left(\mathbb{E}\,|Z_{t,p}|^2\right)^{1/2} < \infty, \ \ \forall p \geq 2.
$$

The inequality (2.21) when $1 \leq p < 2$ follows from the Cauchy-Schwartz inequality.

Finally, to see that $x.$ has a continuous version, we estimate $\mathbb{E}\,|x_t - x_s|^2$. We shall need the uniform Lipschitz condition of $\Psi(t,s)$ in t which was proved earlier. From the expression (2.23) it follows that

$$
x_t - x_s = \Psi(t,0)x_0 - \Psi(s,0)x_0 + \sum_{n=1}^{\infty} \frac{1}{n!}\left[I_{n,t}(f_n^t) - I_{n,s}(f_n^s)\right]\,.
$$

By the orthogonality of different chaos we have

$$
\begin{aligned}
\mathbb{E}\,|x_t - x_s|^2 &= [\Psi(t,0)x_0 - \Psi(s,0)x_0]^2 + \sum_{n=1}^{\infty} \frac{1}{(n!)^2}\mathbb{E}\,\left[I_{n,t}(f_n^t) - I_{n,s}(f_n^s)\right]^2 \\
&\le [\Psi(t,0)x_0 - \Psi(s,0)x_0]^2 + \sum_{n=1}^{\infty} \frac{1}{(n!)^2}\mathbb{E}\,\left[I_{n,s}(f_n^t) - I_{n,s}(f_n^s)\right]^2 \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{(n!)^2}\mathbb{E}\,\left[I_{n,t}(f_n^t) - I_{n,s}(f_n^t)\right]^2 \\
&=: I_1 + \sum_{n=1}^{\infty} \frac{1}{(n!)^2} I_{2,n} + \sum_{n=1}^{\infty} \frac{1}{(n!)^2} I_{3,n}\,.
\end{aligned}
$$

where

$$
\begin{aligned}
I_{2,n} &= \mathbb{E}\,\left[I_{n,s}(f_n^t) - I_{n,s}(f_n^s)\right]^2 \\
&= \mathbb{E}\,\left[I_{n,s}(f_n^t - f_n^s)\right]^2 \\
&= \langle f_n^t - f_n^s, f_n^t - f_n^s\rangle_\phi\,.
\end{aligned}
$$

Now it is straightforward to see that

$$
I_{2,n} \le n! K_T^n |t-s|^2\,.
$$

Denote by $T_n(s,t)$ the region

$$
\begin{aligned}
T_n(s,t) &:= \{0 < s_1 < \cdots < s_n < t\} \setminus \{0 < s_1 < \cdots < s_n < s\} \\
&= \cup_{k=0}^{n-1} \{0 < s_1 < \cdots < s_k < s, s_{k+1} < \cdots < s_n < t\} \\
&=: \cup_{k=0}^{n-1} T_{n,k}(s,t)
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{1}{(n!)^2} I_{3,n} &= \frac{1}{(n!)^2}\mathbb{E}\,\left[I_{n,t}(f_n^t) - I_{n,s}(f_n^t)\right]^2 \\
&= \mathbb{E}\,\left[\int_{T_n(s,t)} f_n(t,s_1,\cdots,s_n) dW_{s_1}^H \cdots dW_{s_2}^H\right]^2 \\
&= \int_{T_n(s,t)}\int_{T_n(s,t)} f_n(t,s_1,\cdots,s_n) f_n(t,t_1,\cdots,t_n)\phi(s_1,t_1)\cdots\phi(s_n,t_n) ds_1 dt_1 \cdots ds_n dt_n \\
&\le \sum_{k,j=0}^{n-1} \int_{T_{n,k}(s,t)}\int_{T_{n,j}(s,t)} f_n(t,s_1,\cdots,s_n) f_n(t,t_1,\cdots,t_n)\phi(s_1,t_1)\cdots\phi(s_n,t_n) ds_1 dt_1 \cdots ds_n dt_n \\
&\le \sum_{k,j=0}^{n-1} \sqrt{\int_{T_{n,k}(s,t)}\int_{T_{n,k}(s,t)} f_n(t,s_1,\cdots,s_n) f_n(t,t_1,\cdots,t_n)\phi(s_1,t_1)\cdots\phi(s_n,t_n) ds_1 dt_1 \cdots ds_n dt_n} \\
&\quad \cdot \sqrt{\int_{T_{n,j}(s,t)}\int_{T_{n,j}(s,t)} f_n(t,s_1,\cdots,s_n) f_n(t,t_1,\cdots,t_n)\phi(s_1,t_1)\cdots\phi(s_n,t_n) ds_1 dt_1 \cdots ds_n dt_n}\,.
\end{aligned}
$$

It is easy to check that for all $0 \le k \le n-1$, we have

$$
\begin{aligned}
&\int_{T_{n,k}(s,t)}\int_{T_{n,k}(s,t)} f_n(t,s_1,\cdots,s_n)f_n(t,t_1,\cdots,t_n)\phi(s_1,t_1)\cdots\phi(s_n,t_n)ds_1dt_1\cdots ds_ndt_n\\
\le\ & \int_{\substack{0<s_1<\cdots<s_k<T\\ 0<t_1<\cdots<t_k<T}}\int_{\substack{s<s_{k+1}<\cdots<s_n<t\\ s<t_{k+1}<\cdots<t_n<t}} f_n(t,s_1,\cdots,s_n)f_n(t,t_1,\cdots,t_n)\phi(s_1,t_1)\cdots\phi(s_n,t_n)ds_1dt_1\cdots ds_ndt_n\\
\le\ & \frac{1}{k!(n-k)!}K_T^n|t-s|^{2H}\\
\le\ & |t-s|^{2H}K_T^n/n!\,.
\end{aligned}
$$

This concludes that

$$
\frac{1}{(n!)^2}I_{3,n} \le \frac{1}{n!}K_T^n|t-s|^{2H}\,.
$$

Observing that

$$
I_1 \le K_T|t-s|^2\,,
$$

we obtain

$$
\mathbb{E}\,|x_t-x_s|^2 \le K_T|t-s|^{2H}\,.
$$

Since $2H>1$ we see that x_t has a continuous version by the Kolmogorov lemma. This completes the proof of the theorem. □

Remark 2.7 There has been some study on fractional stochastic differential equations (see for example [27], [24] and the references therein). However, we could not find any result on the type of equation (2.19) in the literature. When the equation is one-dimensional, explicit representation of the solution was obtained in [3], [19]; see also Theorem 2.11 below. Here we have used the Wiener chaos expansion approach to handle the multi-dimensional case. Note that the L^p estimate of the solution (2.21) is also new.

Theorem 2.8 *If A_t and C_t are measurable and essentially bounded deterministic functions in t, then equations (2.20) admits a unique solution.*

Proof Similar to the argument in proving Theorem 2.6, a solution candidate to equation (2.20) is

$$
Y_t := \sum_{n=0}^{\infty}\int_{0<s_1<\cdots<s_n<t}\Psi(t,s_n)C_{s_n}\Psi(s_n,s_{n-1})C_{s_{n-1}}\cdots\Psi(s_2,s_1)C_{s_1}\Psi(s_1,0)x_0\circ dW_{s_1}^H\cdots\circ dW_{s_n}^H\,. \quad (2.27)
$$

Fix t. Since

$$
S_n := \int_{0<s_1<\cdots<s_n<t}\Psi(t,s_n)C_{s_n}\Psi(s_n,s_{n-1})C_{s_{n-1}}\cdots\Psi(s_2,s_1)C_{s_1}\Psi(s_1,0)x_0\circ dW_{s_1}^H\cdots\circ dW_{s_n}^H,
$$

$n=0,1,2,\cdots$, are not orthogonal, we consider the convergence in $L_1(\Omega,\mathcal{F},P)$. We have

$$
\mathbb{E}\,|Y_t| \le \sum_{n=0}^{\infty}\mathbb{E}\,|S_n| \le \sum_{n=0}^{\infty}\left(\mathbb{E}\,|S_n|^2\right)^{1/2}.
$$

Defining $f_{n,t}$ as in (2.24), we can write

$$S_n = \int_{0<s_1<\cdots<s_n<t} f_{n,t}(s_1,\cdots,s_n)\circ dW^H_{s_1}\cdots\circ dW^H_{s_n}.$$

As before let f^t_n be the symmetric extension of $f_{n,t}$ to $[0,t]^n$. Since A and C are bounded, we have $|\tilde{f}_{n,t}(s_1,\cdots,s_n)| \le K_T^n$ for some constant K_T depending only on T.

With these notation we have

$$S_n = \frac{1}{n!} S_{n,t}(f^t_n)\,,$$

where

$$S_{n,t}(f^t_n) := \int_{0<s_1,\cdots,s_n<t} f^t_n(s_1,\cdots,s_n)\circ dW^H_{s_1}\cdots\circ dW^H_{s_n}\,.$$

By the Hu–Meyer formula (see (2.18)) we have

$$S_{n,t}(f^t_n) = \sum_{k\le n/2} \frac{n!}{2^k k!(n-2k)!} I_{n-2k,t}\left(\operatorname{Tr}^k_\phi f^t_n\right)\,.$$

Therefore

$$\mathbb{E}\,|S_{n,t}(f^t_n)|^2 = \sum_{k\le n/2} \frac{(n!)^2}{2^{2k}(k!)^2(n-2k)!} |\operatorname{Tr}^k_\phi f^t_n|^2_\phi\,. \tag{2.28}$$

But

$$\begin{aligned}
|\operatorname{Tr}^k_\phi f^t_n|^2_\phi \quad &= \quad \int_{\substack{0<s_1,\cdots,s_{2k}<t\\ 0<t_1,\cdots,t_{2k}<t}} \int_{\substack{0<t_1,\cdots,t_{n-2k}<t\\ 0<t'_1,\cdots,t'_{n-2k}<t}} f^t_n(s_1,s_2,\cdots,s_{2k},t_1,\cdots,t_{n-2k})\\
&\qquad f^t_n(r_1,r_2,\cdots,r_{2k},t'_1,\cdots,t'_{n-2k})\phi(s_1,s_2)\cdots\phi(s_{2k-1},s_{2k})\\
&\qquad \phi(r_1,r_2)\cdots\phi(r_{2k-1},r_{2k})\phi(t_1,t'_1)\cdots\\
&\qquad \phi(t_{n-2k},t'_{n-2k})ds_1dr_1\cdots ds_{2k}dr_{2k}dt_1dt'_1\cdots dt_{n-2k}dt'_{n-2k}\\
&\le \quad K_T^n \int_{\substack{0<s_1,\cdots,s_{2k}<t\\ 0<t_1,\cdots,t_{2k}<t}} \int_{\substack{0<t_1,\cdots,t_{n-2k}<t\\ 0<t'_1,\cdots,t'_{n-2k}<t}} \phi(s_1,s_2)\cdots\phi(s_{2k-1},s_{2k})\\
&\qquad \phi(r_1,r_2)\cdots\phi(r_{2k-1},r_{2k})\phi(t_1,t'_1)\cdots\\
&\qquad \phi(t_{n-2k},t'_{n-2k})ds_1dr_1\cdots ds_{2k}dr_{2k}dt_1dt'_1\cdots dt_{n-2k}dt'_{n-2k}\\
&\le K_T^n\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbb{E}\,|S_n|^2 \quad &\le \quad \sum_{k\le n/2}\frac{K_T^n}{(k!)^2(n-2k)!}\\
&\le \quad \frac{K_T^n}{n!}\,.
\end{aligned}$$

Thus Y_t is well-defined. It is now routine to verify that Y_t is the solution to (2.20). □

Remark 2.9 Unlike with the case of Itô integral (see Theorem 2.6), it remains an open problem whether the solution in Theorem 2.8 has a continuous modification.

Remark 2.10 The nonhomogeneous versions of (2.19) and (2.20) are

$$\begin{cases} dx_t = (A_t x_t + f_t)dt + (C_t x_t + g_t)dW_t^H \\ x_0 \in \mathbb{R}^n \qquad \text{be given and deterministic,} \end{cases} \tag{2.29}$$

and

$$\begin{cases} dx_t = (A_t x_t + f_t)dt + (C_t x_t + g_t) \circ dW_t^H \\ x_0 \in \mathbb{R}^n \qquad \text{be given and deterministic,} \end{cases} \tag{2.30}$$

respectively, where f_t and g_t are generally random processes. One may also use the Wiener chaos expansion technique to study the solvability of these equations, taking the "fundamental matrix" associated with the homogeneous versions (2.19) or (2.20) as the starting point. However, much more involved technicalities will have to be gone through. Details are left to the interested readers. Note that for studying the first and second control models in the sequel, equations (2.19) and (2.20) suffice since only Markovian type of controls are to be considered.

The following theorem gives an explicit expression of the solution to (2.19) in the one-dimensional case. Note that the result was proved in [3] using the Wick product. Here we give a different yet simple proof.

Theorem 2.11 *Let $n = 1$. If A_t and C_t are measurable and essentially bounded deterministic functions in t, then the solution of (2.19) can be represented as*

$$x_t = x_0 \exp\left[\int_0^t C_s dW_s^H + \int_0^t A_s ds - \frac{1}{2}\int_0^t \int_0^t \phi(s,s') C_s C_{s'} ds ds'\right]. \tag{2.31}$$

Proof Define

$$\begin{aligned} y_t :=&\ \exp(\textstyle\int_0^t C_s dW_s^H), \\ z_t :=&\ x_0 \exp\{\textstyle\int_0^t A_s ds - \frac{1}{2}\int_0^t\int_0^t \phi(s,s') C_s C_{s'} ds ds'\} \\ \equiv&\ x_0 \exp\{\textstyle\int_0^t A_s ds - \int_0^t \int_0^s \phi(s,s') C_s C_{s'} ds' ds\}, \\ \bar{x}_t :=&\ y_t z_t. \end{aligned}$$

It suffices to prove that $\bar{x}_t$ satisfies (2.19). To this end, noting that C_t is deterministic, we can apply a simplified Itô's formula ([9, Corollary 4.4][2]) to get

$$dy_t = C_t y_t dW_t^H + C_t y_t \int_0^t \phi(t,s) C_s ds\ dt. \tag{2.32}$$

[2]There is a typo in [9, Corollary 4.4]. The last expression on page 599 of [9] should be $\int_0^t \frac{\partial^2 f}{\partial x^2}(s,\eta_s) a_s \int_0^s \phi(s,v) a_v dv ds$.

Hence,

$$\begin{aligned} d\bar{x}_t &= d(y_t z_t) = y_t z_t[A_t dt - \textstyle\int_0^t \phi(t,s')C_t C_{s'} ds'\ dt] \\ &\quad + y_t z_t[C_t dW_t^H + C_t \textstyle\int_0^t \phi(t,s)C_s ds\ dt] \\ &= A_t \bar{x}_t dt + C_t \bar{x}_t dW_t^H . \end{aligned}$$

This completes the proof. □

The following two lemmas are useful in the sequel. In particular, the first one represents the Malliavin derivative of the solution (2.31).

Lemma 2.12 *Let x_t be the solution of (2.19) under the assumptions of Theorem 2.11. Then*

$$D_t^H x_t = x_t \int_0^t \phi(t,s) C_s ds\,, \quad \forall t \in [0,T], \quad \text{a.s.} \tag{2.33}$$

Proof Equation (2.19) is understood as

$$x_t = x_0 + \int_0^t A_s x_s ds + \int_0^t C_s x_s dW_s^H\,.$$

From the explicit expression (2.31) of the solution we could compute its Malliavin derivative. However, here we supply a simpler method. By [9, Theorem 4.2], we have

$$D_r^H x_t = \int_0^t A_s D_r^H x_s ds + \int_0^t C_s D_r^H x_s dW_s^H + \int_0^t \phi(r,s) C_s x_s ds\,, \quad \forall r,t \in [0,T], \quad \text{a.s.}$$

Fix r. Denote $z_t := D_r^H x_t$. Then the above equation can be written as

$$dz_t = A_t z_t dt + C_t z_t dW_t^H + \phi(r,t) C_t x_t dt \tag{2.34}$$

with $z_0 = 0$. We want to solve this equation to find $D_r^H x_t$. Try the solution of the form $\tilde{z}_t := \rho_t x_t$ where ρ_t is deterministic and differentiable with $\rho_0 = 0$. Then

$$\begin{aligned} d\tilde{z}_t &= \dot{\rho}_t x_t dt + \rho_t A_t x_t dt + \rho_t C_t x_t dW_t^H \\ &= \dot{\rho}_t x_t dt + A_t \tilde{z}_t dt + C_t \tilde{z}_t dW_t^H\,. \end{aligned} \tag{2.35}$$

Comparing equation (2.35) with (2.34) we conclude that if $\dot{\rho}_t = \phi(r,t) C_t$, then z_t would be equal to $\rho_t x_t$ owing to the uniqueness of solution to (2.34). This implies that

$$\rho_t = \int_0^t \phi(r,s) C_s ds\,.$$

The proof is completed. □

Lemma 2.13 *Let x_t be the solution of (2.19) under the assumptions of Theorem 2.11, and p_t and C_t be continuously differentiable deterministic functions in t. Then*

$$d(p_t x_t^2) = x_t^2 \left[\dot{p}_t dt + 2A_t p_t dt + 2p_t C_t \int_0^t \phi(t,s) C_s ds dt + 2C_t p_t dW_t^H \right]\,. \tag{2.36}$$

Proof Again it suffices to consider the case when $H > \frac{1}{2}$. We apply Ito's formula ([9, Theorems 4.5]) to $p_t x_t^2$. In order to do so we need to check the following conditions:

1) $\mathbb{E} \ \sup_{0 \le s \le T} |A_t x_t| < \infty$;

2) $\mathbb{E} \ \int_0^T |C_t x_t D_t^H x_t|^2 dt < \infty$;

3) There is $\alpha > 1 - H$ such that $\mathbb{E} \ |C_t x_t - C_s x_s|^2 \le K|t-s|^{2\alpha}$ where $|t-s| \le \delta$ for some $\delta > 0$ and $\lim_{|t-s| \to 0} \mathbb{E} \ |D_t^H (C_t x_t - C_s x_s)|^2 = 0$;

4) The function $f(t,x) := p_t x^2$ has bounded derivatives.

Remark that the condition 4) can be replaced by the following (see [13])

4') The function $f(t,x)$ has polynomial growth in x.

To prove 1), we use (2.31) to conclude

$$\sup_{0 \le t \le T} |A_t x_t| \le K_T \exp\left\{ \sup_{0 \le t \le T} | \int_0^t C_s dW_s^H | \right\}.$$

The condition 1) thus follows easily ([10]). Next, combining (2.33) and (2.21) we derive the condition 2).

To prove 3), since C_t is continuously differentiable, it suffices to show that

$$\mathbb{E} \ |x_t - x_s|^2 \le K|t-s|^{2\alpha}$$

and

$$\lim_{|t-s| \to 0} \mathbb{E} \ |D_t^H (x_t - x_s)|^2 = 0 \,.$$

The first inequality (with $\alpha = H > 1 - H$) was proved in the proof of Theorem 2.6, and the second equality is seen immediately by the fact that $D_r^H x_t = \int_0^t \phi(r,s) C_s ds$ which was proved in the proof of Lemma 2.12.

Finally, the condition 4') is obvious. Now applying [9, Theorem 4.5] we conclude

$$\begin{aligned} d(p_t x_t^2) &= \dot{p}_t x_t^2 dt + 2p_t x_t dx_t + 2p_t C_t x_t D_t^H x_t dt \\ &= \dot{p}_t x_t^2 dt + 2A_t p_t x_t^2 dt + 2C_t p_t x_t^2 dW_t^H + 2p_t C_t x_t^2 \int_0^t \phi(t,s) C_s ds dt \\ &= x_t^2 \left[\dot{p}_t dt + 2A_t p_t dt + 2p_t C_t \int_0^t \phi(t,s) C_s ds dt + 2C_t p_t dW_t^H \right] , \end{aligned} \tag{2.37}$$

where the second equality is due to Lemma 2.12. This proves the lemma. □

3 Control Model 1: Scalar State, Quadratic Cost, and Markovian Control

We start with our first control model in this section. The controlled dynamics is given by the following Itô type SDE where the state is scalar-valued:

$$\begin{cases} dx_t = (A_t x_t + B_t u_t)dt + (C_t x_t + D_t u_t)dW_t^H \\ x_0 \in \mathbb{R} \qquad \text{be given and deterministic,} \end{cases} \tag{3.1}$$

where A_t, $B_t := (B_t^1, \cdots, B_t^m)$, C_t, and $D_t := (D_t^1, \cdots, D_t^m)$, $0 \le t \le T$, are given essentially bounded deterministic functions of t. A control, $u_t = (u_t^1, \cdots, u_t^m)^*$, $0 \le t \le T$, where * denotes the matrix transpose, under consideration in this section is taken to be of the Markovian linear feedback type, namely,

$$u_t = K_t x_t\,, \tag{3.2}$$

where $K_t := (K_t^1, \cdots, K_t^m)^*$ is an essentially bounded deterministic function of t. Such a control is also referred to as an *admissible (Markovian linear feedback) control* in this section.

Under each admissible control $u_t = K_t x_t$, the system (3.1) reduces to the following linear SDE

$$\begin{cases} dx_t = (A_t + B_t K_t)x_t dt + (C_t + D_t K_t)x_t dW_t^H \\ x_0 \in \mathbb{R} \qquad \text{be given and deterministic.} \end{cases} \tag{3.3}$$

Hence $K.$ itself, also termed as the *feedback gain*, can be regarded as a control.

For every initial state x_0 and admissible control $u_t = K_t x_t$, there is an associated cost

$$J(x_0, u.) \equiv J(x_0, K.) := \mathbb{E}\left[\int_0^T (Q_t x_t^2 + u_t^* R_t u_t)dt + G x_T^2\right], \tag{3.4}$$

where $x.$ is the solution of (3.3) under the control $u.$ or equivalently $K.$ (note that the unique solvability of (3.3) under a given $K.$ is ensured by Theorem 2.6), Q_t and R_t are given essentially bounded deterministic functions in t, and G is a given deterministic scalar. Our optimal stochastic control problem is to minimize the cost functional (3.4), for each given x_0, over the set of all admissible Markovian linear feedback controls.

Theorem 3.1 *Assume that for a.e. $t \in [0, T]$, $D_t = 0$, $Q_t \ge 0$, and $R_t \succ \delta I$ for some given $\delta > 0$, and $G \ge 0$. Then the following Riccati equation*

$$\begin{cases} \dot{p}_t + 2p_t[A_t + C_t \int_0^t \phi(t,s) C_s ds] + Q_t - B_t R_t^{-1} B_t^* p_t^2 = 0 \\ p_T = G \end{cases} \tag{3.5}$$

admits a unique solution $p.$ over $[0, T]$ with $p_t \ge 0$ $\forall t \in [0, T]$. Moreover, the optimal Markovian linear feedback control for the problem (3.3)–(3.4) is given by

$$\hat{u}_t = \hat{K}_t x_t\,, \quad \text{with} \quad \hat{K}_t = -R_t^{-1} B_t^* p_t\,. \tag{3.6}$$

Finally, the optimal value of (3.4) is $p_0 x_0^2$.

Proof The unique solvability of the (classical) Riccati equation (3.5) was proved in, e.g., [29, p. 297, Corollary 2.10]. Next, for any admissible control $u_t = K_t x_t$, applying Lemma 2.13 to the equation (3.3) with $D_t = 0$, we get

$$d(p_t x_t^2) = x_t^2 \left[\dot{p}_t + 2p_t(A_t + B_t K_t) + 2p_t C_t \int_0^t \phi(t,s) C_s ds\right] dt + 2x_t^2 C_t p_t dW_t^H .$$

Taking integration from 0 to T we get

$$p_T x_T^2 = p_0 x_0^2 + \int_0^T x_t^2 \left[\dot{p}_t + 2p_t(A_t + B_t K_t) + 2p_t C_t \int_0^t \phi(t,s) C_s ds\right] dt + 2\int_0^T x_t^2 C_t p_t dW_t^H .$$

Denote $f_t := x_t^2 C_t p_t$. It follows from (2.21) that

$$\int_0^T \int_0^T \phi(s,t) \mathbb{E}\, |f_s f_t| ds dt < \infty.$$

On the other hand, $D_t^H x_t = x_t \int_0^t \phi(t,s) C_s ds$ by virtue of Lemma 2.12. So again by (2.21) we have

$$\sup_{0 \le t \le T} \mathbb{E}\, |D_t^H x_t|^p < \infty \quad \forall\, p \ge 1 .$$

This implies that f_t satisfies the condition of [9, Theorem 3.9] leading to $\mathbb{E} \int_0^T f_t dW_t^H = 0$. Hence

$$\mathbb{E}\, \left[p_T x_T^2\right] = p_0 x_0^2 + \mathbb{E} \int_0^T x_t^2 \left[\dot{p}_t + 2p_t(A_t + B_t K_t) + 2p_t C_t \int_0^t \phi(t,s) C_s ds\right] dt .$$

Since $p_T = G$, we obtain

$$\begin{aligned} J(x_0, K.) &= p_0 x_0^2 + \mathbb{E} \int_0^T x_t^2 \Big[\dot{p}_t + 2p_t(A_t + B_t K_t) + (Q_t + K_t^* R_t K_t) \\ &\qquad + 2p_t C_t \int_0^t \phi(t,s) C_s ds\Big] dt \\ &= p_0 x_0^2 + \mathbb{E} \int_0^T x_t^2 \Big[\dot{p}_t + 2p_t A_t + 2p_t C_t \int_0^t \phi(t,s) C_s ds + Q_t \qquad (3.7) \\ &\qquad + (K_t + R_t^{-1} B_t^* p_t)^* R_t (K_t + R_t^{-1} B_t^* p_t) - B_t R_t^{-1} B_t^* p_t^2\Big] dt \\ &= p_0 x_0^2 + \mathbb{E} \int_0^T (K_t + R_t^{-1} B_t^* p_t)^* R_t (K_t + R_t^{-1} B_t^* p_t) dt , \qquad (3.8) \end{aligned}$$

where the last equality was due to the Riccati equation (3.5). Equation (3.8) shows that the cost function achieves its minimum when $\hat{K}_t = -R_t^{-1} B_t^* p_t$, with the minimum value being $p_0 x_0^2$. This proves the theorem. □

Remark 3.2 It is interesting to note that the Riccati equation (3.5) corresponds to the following LQ control problem with (normal) Brownian motion:

$$\begin{array}{ll} \text{Minimize} & (3.4) \\ \text{subject to} & \begin{cases} dx_t = (A_t x_t + B_t u_t) dt + \tilde{C}_t x_t dW_t \\ x_0 \in \mathbb{R} \quad \text{be given and deterministic,} \end{cases} \end{array} \qquad (3.9)$$

where $\tilde{C}_t = \sqrt{2C_t \int_0^t \phi(t,s)C_s ds}$ assuming that $C_t \int_0^t \phi(t,s)C_s ds \geq 0$ for all $t \geq 0$; see [4]. This suggests that, in the current setting, the LQ control problem with FBM is equivalent (in the sense of sharing the same optimal feedback control and optimal value) to an LQ control problem with Brownian motion where the diffusion coefficient of the state is properly modified.

4 Control Model 2: Scalar State, Power Cost, and Markovian Control

In this section we consider a more general model (than the one tackled in the previous section) where the dynamics is given by (3.1) but the cost functional is of the following form

$$J(x_0, u.) \equiv J(x_0, u(\cdot)) = \mathbb{E}\left\{\int_0^T [Q_t x_t^\alpha + \sum_{i,j=1}^m R_{ij}(t) u_i(t)^{\beta_i} u_j(t)^{\beta_j}]dt + G x_T^\gamma\right\}, \tag{4.1}$$

where α, β_i $(i = 1, 2, \cdots, m)$, $\gamma \in \mathbb{R}$ are given, Q_t and $R_t \equiv R(t) = (R_{ij}(t))_{1\leq i,j\leq m}$ are given essentially bounded deterministic functions in t, and G is a given deterministic scalar. Moreover, we assume that $(R_{ij}(t))_{1\leq i,j\leq m}$ is positive definite for each $t \in [0, T]$. Here we interchangeably denote a control variable as

$$u_t \equiv u(t) = (u_t^1, \cdots, u_t^m)^* \equiv (u_1(t), \cdots, u_m(t))^*$$

for notational convenience. The problem is to find the *extreme* or *optimum* (i.e., minimum or maximum) of $J(x_0, u.)$, for each fixed $x_0 \in \mathbb{R}$, over the set of all admissible Markovian linear feedback controls of the form $u_t = K_t x_t$, subject to (3.1). With this type of controls we can write $J(x_0, u.) = J(x_0, K.)$ as in the previous section. Note that whether the cost functional (4.1) assumes a minimum or maximum (or both) depends on the problem data, in particular the ranges of α, β and γ. The objective of this section is to derive *necessary conditions* for a solution to the extreme problem.

Lemma 4.1 *For any admissible Markovian control $u_t = K_t x_t$, where $K_t \equiv K(t) = (K_1(t), \cdots, K_m(t))^*$, the corresponding cost has the following representation:*

$$J(x_0, K.) = \int_0^T [Q_t \Theta_{\alpha,t} + \sum_{i,j=1}^m R_{ij}(t) K_i(t)^{\beta_i} K_j(t)^{\beta_j} \Theta_{\beta_i+\beta_j,t}]dt + G\Theta_{\gamma,T}, \tag{4.2}$$

where for any $\kappa \in \mathbb{R}$,

$$\begin{aligned}\Theta_{\kappa,t} \equiv\ & \Theta_{\kappa,t}(x_0, K.) \\ :=\ & x_0^\kappa \mathbb{E}\ \exp\left\{\kappa \int_0^t (A_s + B_s K_s)ds + \tfrac{\kappa^2-\kappa}{2}\int_0^t\int_0^t \phi(s,s')(C_s + D_s K_s)(C_{s'} + D_{s'}K_{s'})dsds'\right\}.\end{aligned} \tag{4.3}$$

Proof Substituting the feedback control $u_t = K_t x_t$ into (3.1) we obtain

$$dx_t = (A_t + B_t K_t)x_t dt + (C_t + D_t K_t)x_t dW_t^H . \tag{4.4}$$

By Theorem 2.11, this equation can be solved explicitly as follows

$$\begin{aligned} x_t &= x_0 \exp\left\{ \int_0^t (C_s + D_s K_s) dW_s^H + \int_0^t (A_s + B_s K_s) ds \right. \\ &\left. -\frac{1}{2}\int_0^t \int_0^t \phi(s,s')(C_s + D_s K_s)(C_{s'} + D_{s'} K_{s'}) ds ds' \right\} . \end{aligned} \tag{4.5}$$

Let $\kappa \in \mathbb{R}$. Since $\kappa \int_0^t (C_s + D_s K_s) dW_s^H$ is Gaussian, we have

$$\begin{aligned} \mathbb{E}\ \exp\{\kappa \textstyle\int_0^t (C_s + D_s K_s) dW_s^H\} &= \exp\{\tfrac{1}{2}\kappa^2 \mathbb{E} \mid \textstyle\int_0^t (C_s + D_s K_s) dW_s^H |^2\} \\ &= \exp\{\tfrac{1}{2}\kappa^2 \textstyle\int_0^t \int_0^t \phi(s,s')(C_s + D_s K_s)(C_{s'} + D_{s'} K_{s'}) ds ds'\} \end{aligned}$$

where the last equality is due to [9, Theorem 3.10]. It then follows from (4.5) that

$$\begin{aligned} \mathbb{E}\ x_t^\kappa &= x_0^\kappa \mathbb{E}\ \exp\left\{ \kappa \int_0^t (A_s + B_s K_s) ds + \frac{\kappa^2 - \kappa}{2} \int_0^t \int_0^t \phi(s,s')(C_s + D_s K_s)(C_{s'} + D_{s'} K_{s'}) ds ds' \right\} \\ &= \Theta_{\kappa,t}(x_0, K_\cdot) \equiv \Theta_{\kappa,t} . \end{aligned} \tag{4.6}$$

The desired representation (4.2) thus follows immediately. □

Theorem 4.2 *If $K_t \equiv K(t) = (K_1(t), \cdots, K_m(t))^*$ achieves an extreme for the problem (3.1) and (4.1), then it must satisfy the following equation*

$$\begin{aligned} &\alpha B_s \int_s^T Q_t \Theta_{\alpha,t} dt + (\alpha^2 - \alpha) \int_s^T \int_0^t Q_t \Theta_{\alpha,t} \phi(s',s) C_{s'} D_s ds' dt \\ &+(\alpha^2 - \alpha) \int_s^T \int_0^t Q_t \Theta_{\alpha,t} \phi(s',s) D_{s'} K_{s'} D_s ds' dt + 2\tilde{R}(s) \\ &+ \sum_{i,j=1}^m \int_s^T R_{ij}(t) K_i(t)^{\beta_i} K_j(t)^{\beta_j} \Theta_{\beta_{ij},t} \Big[\beta_{ij} B_s + (\beta_{ij}^2 - \beta_{ij}) \int_0^t \phi(s',s) C_{s'} D_s ds' \\ &+(\beta_{ij}^2 - \beta_{ij}) \int_0^t \phi(s',s) D_{s'} K_{s'} D_s ds' \Big] dt \\ &+\gamma G \Theta_{\gamma,T} B_s + (\gamma^2 - \gamma) G \Theta_{\gamma,T} \int_0^T \phi(s',s) C_{s'} D_s ds' \\ &+(\gamma^2 - \gamma) G \Theta_{\gamma,T} \int_0^T \phi(s',s) D_{s'} K_{s'} D_s ds' = 0 , \quad \text{a.e. } s \in [0,T], \end{aligned} \tag{4.7}$$

where

$$\beta_{ij} := \beta_i + \beta_j \tag{4.8}$$

and

$$\tilde{R}(t) := (\sum_{j=1}^m \beta_1 R_{1j}(t) K_1(t)^{\beta_1 - 1} K_j(t)^{\beta_j} \Theta_{\beta_{1j},t} , \cdots , \sum_{j=1}^m \beta_m R_{mj}(t) K_m(t)^{\beta_m - 1} K_j(t)^{\beta_j} \Theta_{\beta_{mj},t}) . \tag{4.9}$$

Proof Since $K.$ attains the extreme value, it necessarily holds

$$\frac{d}{d\varepsilon} J(K. + \varepsilon\xi.)\bigg|_{\varepsilon=0} = 0 \tag{4.10}$$

for all $\xi. \in C^\infty(0,T;\mathbb{R}^m)$, where $C^\infty(0,T;\mathbb{R}^m)$ denotes the set of all smooth ($\mathbb{R}^m$-valued) functions on $[0,T]$.

To compute the above derivative (4.10) we need to first compute $\frac{d}{d\varepsilon}\Theta_{\alpha,t}(x_0, K. + \varepsilon\xi.)\Big|_{\varepsilon=0}$ which is given by

$$\begin{aligned} \frac{d}{d\varepsilon}\Theta_{\alpha,t}(x_0, K. + \varepsilon\xi.)\bigg|_{\varepsilon=0} &= \Theta_{\alpha,t}\left\{\alpha \int_0^t B_s\xi_s ds + (\alpha^2 - \alpha)\int_0^t\int_0^t \phi(s,s')D_sK_sD_{s'}\xi_{s'}dsds' \right. \\ &\quad \left. +(\alpha^2-\alpha)\int_0^t\int_0^t \phi(s,s')C_sD_{s'}\xi_{s'}dsds'\right\}. \end{aligned} \tag{4.11}$$

Consequently,

$$\begin{aligned} \frac{d}{d\varepsilon} J(\hat{k}. + \varepsilon\xi.)\bigg|_{\varepsilon=0} &= \int_0^T Q_t\Theta_{\alpha,t}\left\{\alpha\int_0^t B_s\xi_s ds + (\alpha^2-\alpha)\int_0^t\int_0^t \phi(s,s')C_sD_{s'}\xi_{s'}dsds'\right. \\ &\quad \left. +(\alpha^2-\alpha)\int_0^t\int_0^t \phi(s,s')D_sK_sD_{s'}\xi_{s'}dsds'\right\}dt \\ &\quad +2\sum_{i,j=1}^m \int_0^T \beta_i R_{ij}(t)K_i(t)^{\beta_i-1}K_j(t)^{\beta_j}\xi_i(t)\Theta_{\beta_{ij},t}dt \qquad (4.12) \\ &\quad +\sum_{i,j=1}^m \int_0^T R_{ij}(t)K_i(t)^{\beta_i}K_j(t)^{\beta_j}\Theta_{\beta_{ij},t}\left\{\beta_{ij}\int_0^t B_s\xi_s ds\right. \\ &\quad +(\beta_{ij}^2-\beta_{ij})\int_0^t\int_0^t \phi(s,s')C_sD_{s'}\xi_{s'}dsds' \\ &\quad \left. +(\beta_{ij}^2-\beta_{ij})\int_0^t\int_0^t \phi(s,s')D_sK_sD_{s'}\xi_{s'}dsds'\right\}dt \\ &\quad +H\Theta_{\gamma,T}\left\{\gamma\int_0^T B_s\xi_s ds + (\gamma^2-\gamma)\int_0^T\int_0^T \phi(s,s')C_sD_{s'}\xi_{s'}dsds'\right. \\ &\quad \left. +(\gamma^2-\gamma)\int_0^T\int_0^T \phi(s,s')D_sK_sD_{s'}\xi_{s'}dsds'\right\} \qquad (4.13) \\ &= \int_0^T Q_t\Theta_{\alpha,t}\left\{\alpha\int_0^t B_s\xi_s ds + (\alpha^2-\alpha)\int_0^t\int_0^t \phi(s,s')C_sD_{s'}\xi_{s'}dsds'\right. \\ &\quad \left. +(\alpha^2-\alpha)\int_0^t\int_0^t \phi(s,s')D_sK_sD_{s'}\xi_{s'}dsds'\right\}dt \\ &\quad +2\int_0^T \tilde{R}(t)\xi_t dt \qquad (4.14) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i,j=1}^{m}\int_0^T R_{ij}(t)K_i(t)^{\beta_i}K_j(t)^{\beta_j}\Theta_{\beta_{ij},t}\Big\{\beta_{ij}\int_0^t B_s\xi_s ds \\
&+(\beta_{ij}^2-\beta_{ij})\int_0^t\int_0^t \phi(s,s')C_sD_{s'}\xi_{s'}dsds' \\
&+(\beta_{ij}^2-\beta_{ij})\int_0^t\int_0^t \phi(s,s')D_sK_sD_{s'}\xi_{s'}dsds'\Big\}dt \\
&+G\Theta_{\gamma,T}\Big\{\gamma\int_0^T B_s\xi_s ds+(\gamma^2-\gamma)\int_0^T\int_0^T\phi(s,s')C_sD_{s'}\xi_{s'}dsds' \\
&+(\gamma^2-\gamma)\int_0^T\int_0^T\phi(s,s')D_sK_sD_{s'}\xi_{s'}dsds'\Big\}. \qquad (4.15)
\end{aligned}
$$

Using the Fubini theorem

$$
\int_0^T\int_0^t\int_0^t g(t,s,s')dsds'dt=\int_0^T\int_s^T\int_0^t g(t,s',s)ds'dtds \qquad (4.16)
$$

we obtain

$$
\begin{aligned}
&\frac{d}{d\varepsilon}J(K.+\varepsilon\xi.)\Big|_{\varepsilon=0} \\
=\;&\int_0^T\Big\{\alpha B_s\int_s^T Q_t\Theta_{\alpha,t}dt+(\alpha^2-\alpha)\int_s^T\int_0^t Q_t\Theta_{\alpha,t}\phi(s',s)C_{s'}D_sds'dt \\
&+(\alpha^2-\alpha)\int_s^T\int_0^t Q_t\Theta_{\alpha,t}\phi(s',s)D_{s'}K_{s'}D_sds'dt+2\tilde{R}(s) \\
&+\sum_{i,j=1}^{m}\int_s^T R_{ij}(t)K_i(t)^{\beta_i}K_j(t)^{\beta_j}\Theta_{\beta_i+\beta_j,t}\Big[\beta_{ij}B_s \\
&+(\beta_{ij}^2-\beta_{ij})\int_0^t\phi(s',s)C_{s'}D_sds' \\
&+(\beta_{ij}^2-\beta_{ij})\int_0^t\phi(s',s)D_{s'}K_{s'}D_sds'\Big]dt \\
&+\gamma G\Theta_{\gamma,T}B_s+(\gamma^2-\gamma)G\Theta_{\gamma,T}\int_0^T\phi(s',s)C_{s'}D_sds' \\
&+(\gamma^2-\gamma)G\Theta_{\gamma,T}\int_0^T\phi(s',s)D_{s'}K_{s'}D_sds'\Big\}\xi_sds=0. \qquad (4.17)
\end{aligned}
$$

Since ξ_s is arbitrary, we obtain the desired equation (4.7). □

The necessary condition (4.7), albeit complicated, is very general. Let us now discuss two special (yet interesting) cases where this condition can be greatly simplified.

Case 1. Suppose there is no running cost, i.e., $Q_t\equiv 0$ and $R_t\equiv 0$. However $G\neq 0$ and $\gamma\neq 0$ (otherwise the problem would become trivial). This kind of situations arise often in the financial portfolio selection models where only the terminal wealth is of concern (see, e.g., [23, 29]). In this case the condition (4.7) reduces to

$$(\gamma-1)\int_0^T \phi(s',s)D_{s'}K_{s'}ds'\; D_s + (\gamma-1)\int_0^T \phi(s',s)C_{s'}ds'\; D_s + B_s = 0\,, \quad \text{a.e. } s\in[0,T]. \qquad (4.18)$$

The above equation (4.18) (with $K.$ being the unknown) is reduced to the *Carleman type equation* in the following way. Rewrite (4.18) as

$$\left[(\gamma-1)\int_0^T \phi(s',s)K_{s'}D_{s'}ds' + (\gamma-1)\int_0^T \phi(s',s)C_{s'}ds'\right] D_s = -B_s\,.$$

Since the term inside the bracket is a real-valued function, a necessary condition that (4.18) has a solution is

$$\int_0^T \phi(s',s)K_{s'}D_{s'}ds' + \int_0^T \phi(s',s)C_{s'}ds' = \frac{\lambda_s}{1-\gamma}\,, \qquad (4.19)$$

where λ_s is a real-valued function satisfying

$$\lambda_s D_s = B_s\,.$$

Equation (4.19) is equivalent to (noting that $\phi(s,t)\equiv\phi(t,s)$)

$$\int_0^T \phi(s,t)K_tD_tdt = \frac{\lambda_s}{1-\gamma} - \int_0^T \phi(t,s)C_tdt\,. \qquad (4.20)$$

This equation, with $K.D.$ being the unknown, is of the following general Carleman type:

$$\int_0^T \phi(x,y)h(y)dy = g(x)\,,$$

where g is a given function, h is the unknown function to be determined by this equation and $\phi(x,y) = H(2H-1)|x-y|^{2H-2}$. The solution of this equation can be expressed explicitly as

$$h(x) = -a_H x^{\frac{1}{2}-H}\frac{d}{dx}\int_x^T \left[w^{2H-1}(w-x)^{\frac{1}{2}-H}\frac{d}{dw}\int_0^w z^{\frac{1}{2}-H}(w-z)^{\frac{1}{2}-H}g(z)dz\right]dw\,,$$

where

$$a_H := \frac{\Gamma(2-2H)}{2H\Gamma(\frac{1}{2}+H)\Gamma(\frac{3}{2}-H)^3};$$

see [11], [12], [13]. Applying this general formula to (4.20) we obtain $K.D.$. Note that in general $K.$ may not be unique if its dimension is strictly great than 1.

A particular case for equation (4.18) is when $\gamma = 1$ *or* $D_s = 0$, a.e. $s\in[0,T]$. In this case (4.18) implies that it is necessary (for there to be an extreme achieved by a Markovian control) that $B_s = 0$, a.e. $s\in[0,T]$; hence the system (3.1) becomes uncontrolled and the cost functional (4.1) is constant-valued.

Case 2. Suppose $\alpha = \beta_{ij} = \gamma \neq 0 \ \ \forall i, j$. Then (4.7) specializes to

$$\frac{2}{\alpha}\tilde{R}(s) + B_s \int_s^T \Theta_{\alpha,t} \left(Q_t + \sum_{i,j=1}^m R_{ij}(t) K_i(t)^{\beta_i} K_j(t)^{\beta_j} \right) dt$$

$$+(\alpha-1) D_s \int_s^T \int_0^t \Theta_{\alpha,t} \phi(s', s)(C_{s'} + D_{s'} K_{s'}) \left(Q_t + \sum_{i,j=1}^m R_{ij}(t) K_i(t)^{\beta_i} K_j(t)^{\beta_j} \right) ds' dt$$

$$+G\Theta_{\alpha,T} \left[B_s + (\alpha-1) D_s \int_0^T \phi(s', s)(C_{s'} + D_{s'} K_{s'}) ds' \right] = 0\,, \text{ a.e. } s \in [0, T]. \tag{4.21}$$

If $\alpha = \gamma = 2$ and $\beta_i = \beta_j = 1 \ \ \forall i, j$ (i.e., the usual LQ case), then the above equation can be further simplified to

$$K_s^* R_s \Theta_s + B_s \int_s^T \Theta_t (Q_t + K_t^* R_t K_t) dt$$

$$+D_s \int_s^T \int_0^t \Theta_t \phi(s', s)(C_{s'} + D_{s'} K_{s'})(Q_t + K_t^* R_t K_t) ds' dt$$

$$+G\Theta_T \left[B_s + D_s \int_0^T \phi(s', s)(C_{s'} + D_{s'} K_{s'}) ds' \right] = 0\,, \text{a.e. } s \in [0, T], \tag{4.22}$$

where $\Theta_t := \Theta_{2,t}$.

Therefore, in the LQ case where the state is scalar and $D_t \neq 0$, one needs to solve the functional differential equation (4.22) in order to obtain an optimal Markovian feedback control. Recall that in Section 3 we have derived a Riccati equation (3.5) for the LQ control when $D_t \equiv 0$. Let us now show that (3.5) can also be recovered from (4.22) under the assumptions of Theorem 3.1. To this end, first note that (4.22) reduces to the following when $D_t \equiv 0$:

$$K_s^* R_s \Theta_s + B_s \int_s^T \Theta_t (Q_t + K_t^* R_t K_t) dt + G B_s \Theta_T = 0, \text{ a.e. } s \in [0, T]. \tag{4.23}$$

Define

$$p_s := \Theta_s^{-1} \left[\int_s^T \Theta_t (Q_t + K_t^* R_t K_t) dt + G\Theta_T \right], \tag{4.24}$$

and

$$K_s := -R_s^{-1} B_s^* p_s. \tag{4.25}$$

It is clear that $K.$ defined above satisfies the necessary condition (4.23). Now we derive the equation that governs $p.$. Multiplying (4.24) by Θ_s and then take derivative in s, we obtain

$$\dot{p}_s \Theta_s + p_s \dot{\Theta}_s + \Theta_s (Q_s + K_s^* R_s K_s) = 0. \tag{4.26}$$

However, (4.6) gives (noting $D_t \equiv 0$)

$$\dot{\Theta}_s = \Theta_s \left[2(A_s + B_s K_s) + 2(C_s + D_s K_s) \int_0^s \phi(s, s')(C_{s'} + D_{s'} K_{s'}) ds' \right]. \tag{4.27}$$

Plugging (4.27) into (4.26), noting (4.25), and manipulating, we get that $p.$ satisfies the Riccati equation (3.5) (that $p_T = G$ is evident from (4.24)).

It should be noted that the above derivation only shows that $K.$ defined by (4.25), where $p.$ is the solution to the Riccati equation (3.5), satisfies the *necessary condition* of an optimum. Hence, it can by no means supersede Theorem 3.1.

Another interesting case for the necessary condition (4.22) is when $H = \frac{1}{2}$ (i.e., the Brownian motion) whereas $D_t \neq 0$. In this case $\phi(s,t) = \delta(s,t)$, the Dirac delta function, and (4.22) becomes

$$\begin{aligned} K_s^* R_s \Theta_s + [B_s + (C_s + K_s^* D_s^*) D_s] \int_s^T \Theta_t (Q_t + K_t^* R_t K_t) dt \\ + G\Theta_T [B_s + (C_s + K_s^* D_s^*) D_s] = 0 \,. \end{aligned} \tag{4.28}$$

Let

$$p_s := \Theta_s^{-1} \left[\int_s^T \Theta_t (Q_t + K_t^* R_t K_t) dt + G\Theta_T \right], \tag{4.29}$$

and

$$R_s K_s = -\left[B_s^* + D_s^*(C_s + D_s K_s)\right] p_s \tag{4.30}$$

or equivalently

$$K_s = -(R_s + p_s D_s^* D_s)^{-1} [B_s^* + D_s^* C_s] p_s \tag{4.31}$$

assuming that $R_s + p_s D_s^* D_s$ is invertible. On the other hand, in the present case Θ_s satisfies

$$\dot{\Theta}_s = \Theta_s \left[2(A_s + B_s K_s) + (C_s + D_s K_s)^2 \right]. \tag{4.32}$$

Going through the same argument as above we can show that $p.$ follows

$$\begin{cases} \dot{p}_s + (2A_s + C_s^2) p_s + Q_s - (B_s + C_s^* D_s)(R_s + p_s D_s^* D_s)^{-1} (B_s^* + D_s^* C_s) p_s^2 = 0 \,, \\ p_T = G, \\ R_s + p_s D_s^* D_s \succ 0. \end{cases} \tag{4.33}$$

This is the standard (stochastic) Riccati equation (see [4]).

5 Control Model 3: Vector State, Quadratic Cost, and General Control with Stratonovich Integral

In the two models previously studied, it is assumed that the state is scalar-valued and only the type of Markovian feedback controls is considered. In this section, we investigate a general model where the state is multi-dimensional and admissible controls are not necessarily Markovian. In addition, all the coefficients are allowed to be stochastic processes. The system is described by the following linear dynamics

$$\begin{cases} dx_t = (A_t x_t + B_t u_t) dt + C_t x_t \circ dW_t^H \\ x_0 \in \mathbb{R}^n \qquad \text{be given and deterministic,} \end{cases} \tag{5.1}$$

where $A_t, C_t\,, 0 \le t \le T$, are $n\times n$ matrix-valued $\mathcal{F}_t$-adapted stochastic processes, $B_t\,, 0 \le t \le T$, is an $n \times m$ matrix-valued $\mathcal{F}_t$-adapted process. An *admissible control* u_t, $0 \le t \le T$, is an $\mathbb{R}^m$-valued $\mathcal{F}_t$-adapted process such that (5.1) has a unique solution. Notice that, different from the previous two models, the Stratonovich type integral is involved in the dynamics.

Let Q_t and $R_t\,, 0 \le t \le T$, be given $n \times n$ and $m \times m$ matrix-valued $\mathcal{F}_t$-adapted stochastic processes respectively, and G be an $n\times n$ $\mathcal{F}_T$-measurable random matrix. Define the cost functional

$$J(x_0, u.) := \mathbb{E}\left[\int_0^T (x_t^* Q_t x_t + u_t^* R_t u_t)dt + x_T^* G x_T\right]. \tag{5.2}$$

The optimal stochastic control problem is to minimize the cost functional (5.2), subject to the dynamics (5.1), over the set of all admissible controls for each given x_0.

5.1 Optimal control

In this subsection we present the solution to the optimal control problem formulated above. Let W_t be the Brownian motion in the representation of the FBM W_t^H as given by Lemma 2.1. Introduced the following matrix-valued backward stochastic differential equation (BSDE):

$$\begin{cases} dP_t + (A_t^* P_t + P_t A_t + Q_t - P_t B_t R_t^{-1} B_t^* P_t)dt + (C_t^* P_t + P_t C_t)\circ dW_t^H - \Lambda_t dW_t = 0 \\ P_T = G. \end{cases} \tag{5.3}$$

This is a Riccati type of equation. A *pair* of $n \times n$ matrix-valued $\mathcal{F}_t$-adapted processes, $(P., \Lambda.)$, is called a solution of (5.3) if they satisfy

$$\begin{aligned} P_t = \ & G + \textstyle\int_t^T (A_t^* P_t + P_t A_t + Q_t - P_t B_t R_t^{-1} B_t^* P_t)dt \\ & + \textstyle\int_t^T (C_t^* P_t + P_t C_t)\circ dW_t^H - \int_t^T \Lambda_t dW_t, \quad \text{a.e. } t \in [0,T], \end{aligned}$$

with all the integrals above well defined. Notice that, different from a standard BSDE (see [29]), equation (5.3) involves simultaneously both the Brownian motion and the FBM.

Now, assume that the Riccati equation (5.3) admits a solution $(P., \Lambda.) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n\times n}) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n\times n})$ with $\Lambda.$ being a semimartingale of the form

$$\Lambda_t = \Lambda_0 + \int_0^t \Theta_s ds + \int_0^t \Xi_s dW_s\,, \tag{5.4}$$

where Θ_s and Ξ_s are continuous $\mathcal{F}_t$-adapted processes. Then

$$d\langle \Lambda\,, W\rangle_t = \Xi_t dt\,,$$

where the bracket $\langle \Lambda\,, W\rangle$ denotes the quadratic variational process corresponding to Λ_t and W_t. Hence we may rewrite the Riccati equation (5.3) as

$$\begin{cases} dP_t = -(A_t^* P_t + P_t A_t + Q_t - P_t B_t R_t^{-1} B_t^* P_t)dt \\ \qquad\quad -(C_t^* P_t + P_t C_t)\circ dW_t^H + \Lambda_t \circ dW_t - \frac{1}{2}\Xi_t dt \\ P_T = G. \end{cases} \tag{5.5}$$

Let $u.$ be any given admissible control and $x.$ be the corresponding state process. Applying the Itô formula for the Stratonovich differential ([9, Theorem 4.7]), we have

$$\begin{aligned} d(x_t^* P_t x_t) &= (A_t x_t + B_t u_t)^* P_t x_t dt + x_t^* C_t^* P_t x_t \circ dW_t^H \\ &\quad - x_t^*(A_t^* P_t + P_t A_t + Q_t - P_t B_t R_t^{-1} B_t^* P_t) x_t dt - \frac{1}{2} x_t^* \Xi_t x_t dt \\ &\quad - x_t^* (C_t^* P_t + P_t C_t) x_t \circ dW_t^H + x_t^* \Lambda_t x_t d^o W_t \\ &\quad + x_t^* P_t (A_t x_t + B_t u_t) dt + x_t^* P_t C_t x_t \circ dW_t^H \\ &= \Big[2u_t^* B_t^* P_t x_t - x_t^* (Q_t - P_t B_t R_t^{-1} B_t^* P_t) x_t \Big] dt \\ &\quad - \frac{1}{2} x_t^* \Xi_t x_t dt + x_t^* \Lambda_t x_t \circ dW_t \,. \end{aligned} \tag{5.6}$$

Integrating from 0 to T and taking expectation, we get

$$\begin{aligned} \mathbb{E}\ [x_T^* P_T x_T] = \ & x_0^* P_0 x_0 + \mathbb{E} \ \textstyle\int_0^T \Big[2u_t^* B_t^* P_t x_t - x_t^* (Q_t - P_t B_t R_t^{-1} B_t^* P_t) x_t \Big] dt \\ & - \tfrac{1}{2} \mathbb{E} \ \textstyle\int_0^T x_t^* \Xi_t x_t dt + \mathbb{E} \ \Big[\textstyle\int_0^T x_t^* \Lambda_t x_t \circ dW_t \Big] \,. \end{aligned} \tag{5.7}$$

In view of the relation between the Itô integral and the Stratonovich integral [9, Theorem 3.12], we have

$$\begin{aligned} x_t &= x_0 + \int_0^t (A_s x_s + B_s u_s) ds + \int_0^t C_s x_s \circ dW_s^H \\ &= x_0 + \int_0^t [A_s x_s + B_s u_s + D_s^H (C_s x_s)] ds + \int_0^t C_s x_s dW_s^H \,. \end{aligned}$$

Writing it in component form we have

$$x_t^i = x_0^i + \int_0^t g_s^i ds + \int_0^t h_s^i dW_s^H \,, \quad i = 1, 2, \cdots, n \,,$$

where g_s^i is the ith component of $A_s x_s + B_s u_s + D_s^H (C_s x_s)$ and h_s^i the ith component of $C_s x_s$. By the Itô formula ([9, Theorem 4.6]) we have

$$\begin{aligned} d(x_t^i x_t^j) &= x_t^i g_t^j dt + x_t^i h_t^j dW_t^H + x_t^j g_t^i dt + x_t^j h_t^i dW_t^H \\ &\quad + h_t^j D_t^H x_t^i dt + h_t^i D_t^H x_t^j dt \\ &= g_t^{ij} dt + h_t^{ij} dW_t^H \,, \end{aligned}$$

where

$$g_t^{ij} := x_t^i g_t^j + x_t^j g_t^i + h_t^j D_t^H x_t^i + h_t^i D_t^H x_t^j$$

and

$$h_t^{ij} := x_t^i h_t^j + x_t^j h_t^i \,.$$

We also write $d\Lambda_t$ in component form:

$$d\Lambda_t^{ij} = \Theta_t^{ij} dt + \Xi_t^{ij} dW_t \,, \quad 1 \le i, j \le n \,.$$

Then taking into account that $dW_t^H dW = 0$ (which can be proved by the same estimate as that in the proof of Theorem 2.2), we have

$$d[\Lambda_t^{ij} x_t^i x_t^j] = \Lambda_t^{ij} g_t^{ij} dt + \Lambda_t^{ij} h_t^{ij} dW_t^H + x_t^i x_t^j \Theta_t^{ij} dt + x_t^i x_t^j \Xi_t^{ij} dW_t\,.$$

Taking the sum over i and j and then integrating, we obtain

$$x_t^* \Lambda_t x_t = x_0^* \Lambda_0 x_0 + \int_0^t M_s ds + \int_0^t N_s dW_s^H + \int_0^t x_s^* \Xi_s x_s dW_s,$$

where

$$M_s := \sum_{i,j=1}^n (\Lambda_s^{ij} g_s^{ij} + x_s^i x_s^j \Theta_s^{ij}), \quad N_s := \Lambda_s^{ij} h_s^{ij}\,.$$

Setting

$$Y_t := x_0^* \Lambda_0 x_0 + \int_0^t M_s ds + \int_0^t N_s dW_s^H\,, \tag{5.8}$$

which is of the form (2.9). Assuming that the processes M_s and N_s satisfy the conditions (2.10), then it follows from Theorem 2.2 that

$$\mathbb{E} \int_0^T Y_s \circ dW_s = 0\,.$$

On the other hand the well-known relation between the Itô and Stratonovich integrals for Brownian motion yields (see [21])

$$\mathbb{E} \left[\int_0^T \left(\int_0^t x_s^* \Xi_s x_s dW_s \right) \circ dW_t \right] = \frac{1}{2} \mathbb{E} \int_0^T x_t^* \Xi_t x_t dt\,.$$

Therefore we obtain

$$\mathbb{E} \left[\int_0^T x_t^* \Lambda_t x_t \circ dW_t \right] = \frac{1}{2} \mathbb{E} \int_0^T x_t^* \Xi_t x_t dt\,.$$

It then follows from (5.7) that

$$\mathbb{E}\, [x_T^* P_T x_T] = \mathbb{E} \left[x_0^T P_0 x_0 \right] + \mathbb{E} \int_0^T \left[2u_t^* B_t^* P_t x_t - x_t^* (Q_t - P_t B_t R_t^{-1} B_t^* P_t) x_t \right] dt\,.$$

Hence

$$\begin{aligned} J(x_0, u.) &= x_0^* P_0 x_0 + \mathbb{E} \int_0^T (2u_t^* B_t^* P_t x_t + u_t^* R u_t + x_t^* P_t B_t R_t^{-1} B_t^* P_t x_t) dt \\ &= x_0^* P_0 x_0 + \mathbb{E} \int_0^T \left(u_t + R_t^{-1} B_t^* P_t x_t \right)^* R_t \left(u_t + R_t^{-1} B_t^* P_t x_t \right) dt\,. \end{aligned}$$

Assuming that $R_t \succ 0$, the minimum of the above functional is achieved when $u_t = -R_t^{-1} B_t^* P_t x_t$ with the minimum value being $x_0^* P_0 x_0$, provided that the control $u_t = -R_t^{-1} B_t^* P_t x_t$ induces an admissible control.

Summarizing, we have obtained the following result.

Theorem 5.1 *Assume that $R_t \succ 0$, a.e. $t \in [0,T]$, a.s., and that the Riccati equation (5.3) admits a solution $(P_\cdot, \Lambda_\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n\times n}) \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n\times n})$ with $\Lambda_\cdot$ being a semimartingale of the form (5.4). Moreover, assume that the process Y_t defined by (5.8) satisfies the conditions (2.10) and that (5.6) holds. Then, the stochastic control problem (5.1)–(5.2) is solvable and the optimal control $\hat{u}_\cdot$ must be of Markovian feedback type, given explicitly by*

$$\hat{u}_t = -R_t^{-1} B_t^* P_t x_t \,, \tag{5.9}$$

provided that the control $\hat{u}_\cdot$ (5.9) is admissible. Moreover the minimum value is achieved as $J(x_0, \hat{u}_\cdot) = x_0^* P_0 x_0$.

Remark 5.2 Even in the case when the system is driven by Brownian motion, to the best of our knowledge, equations involving the Stratonovich type integral (5.1) have not been considered in the literature of stochastic LQ control.

Remark 5.3 In the Brownian motion case, the Riccati equation is a deterministic differential equation when all the coefficients A_t, B_t, and C_t are deterministic (see [4]). However, In the FBM case, the Riccati equation (5.3) is still stochastic even when all the coefficients are deterministic.

5.2 Origin of Riccati equation

In the previous subsection we established the optimal solution to the control problem (5.1)–(5.2) by making use of the Riccati equation (5.3). The approach is a standard completion-of-square commonly employed for LQ control once the Riccati equation is in place. However, one may be curious how (5.3) was derived in the first place. In this subsection we present, in a rather formal way, how we get (5.3).

We first approximate the underlying FBM using an adapted convolution approach of Malliavin (see also [18]). Set $W_s^H = 0$ for $s \le 0$. Define

$$W_t^{H,\varepsilon} := \frac{1}{\varepsilon} \int_{t-\varepsilon}^t W_s^H ds \,. \tag{5.10}$$

The derivative of $W_t^{H,\varepsilon}$ is

$$\dot{W}_t^{H,\varepsilon} = \frac{1}{\varepsilon} \left(W_t^H - W_{t-\varepsilon}^H \right) \,. \tag{5.11}$$

Approximate the stochastic differential system (5.1) by

$$\dot{x}_t^\varepsilon = A_t x_t^\varepsilon + B_t u_t + C_t x_t^\varepsilon \dot{W}_t^{H,\varepsilon} = (A_t + C_t \dot{W}_t^{H,\varepsilon}) x_t^\varepsilon + B_t u_t \,. \tag{5.12}$$

The above can be regarded as a controlled linear system with the normal Brownian motion (but the diffusion coefficient is zero) and *random* coefficient in drift. Consider a LQ control problem with the dynamics (5.12) and cost functional (5.2). The corresponding Riccati equation, following [4], is

$$\begin{cases} dP_t^\varepsilon + \left[(A_t + C_t\dot{W}_t^{H,\varepsilon})^* P_t^\varepsilon + P_t^\varepsilon (A_t + C_t\dot{W}_t^{H,\varepsilon}) + Q_t - P_t^\varepsilon B_t R_t^{-1} B_t^* P_t^\varepsilon\right] dt - \Lambda_t^\varepsilon dW_t = 0, \\ P_T^\varepsilon = G. \end{cases} \tag{5.13}$$

Arranging this equation properly, we get

$$\begin{cases} dP_t^\varepsilon + (A_t^* P_t^\varepsilon + P_t^\varepsilon A_t + Q_t - P_t^\varepsilon B_t R_t^{-1} B_t^* P_t^\varepsilon)dt - \Lambda_t^\varepsilon dW_t + C_t^* P_t^\varepsilon \dot{W}_t^{H,\varepsilon} dt + P_t^\varepsilon C_t \dot{W}_t^{H,\varepsilon} dt = 0, \\ P_T^\varepsilon = G. \end{cases} \tag{5.14}$$

A (formal) limiting equation of (5.14) is exactly the Riccati equation (5.3) introduced earlier. It should be noted that it is hard to prove the convergence of the equation (5.14). Fortunately, once we have obtained the Riccati equation, albeit formally, we can then use the completion of square technique to solve the original control problem, as shown in the previous subsection.

It is also worth noting the difficulty in dealing with the following controlled system

$$\begin{cases} dx_t = (A_t x_t + B_t u_t)dt + (C_t x_t + D_t u_t) \circ dW_t^H \\ x_0 \in \mathbb{R}^n \qquad \text{be given and deterministic} \end{cases} \tag{5.15}$$

instead of (5.1). Going through the same approximation procedure as above, we get that equation (5.13) now becomes

$$\begin{aligned} dP^\varepsilon &+ \left[(A_t + C_t\dot{W}_t^{H,\varepsilon})^* P_t^\varepsilon + P_t^\varepsilon (A_t + C_t\dot{W}_t^{H,\varepsilon}) + Q_t\right] dt \\ &- P_t^\varepsilon (B_t + D_t W_t^{H,\varepsilon}) R_t^{-1} (B_t + D_t W_t^{H,\varepsilon})^* P_t^\varepsilon dt - \Lambda_t^\varepsilon dW_t = 0\,. \end{aligned}$$

Rearranging, we have

$$\begin{aligned} dP_t^\varepsilon &+ \left[A_t^* P_t^\varepsilon + P_t^\varepsilon A_t + Q_t - P_t^\varepsilon (B_t + D_t W_t^{H,\varepsilon}) R_t^{-1} (B_t + D_t W_t^{H,\varepsilon})^* P_t^\varepsilon\right] dt \\ &- \Lambda_t^\varepsilon dW_t + C_t^* P_t^\varepsilon \dot{W}_t^{H,\varepsilon} dt + P_t^\varepsilon C_t \dot{W}_t^{H,\varepsilon} dt \\ &- P_t^\varepsilon D_t R_t B_t^* P_t^\varepsilon \dot{W}_t^{H,\varepsilon} - P_t^\varepsilon B_t R_t D_t^* P_t^\varepsilon \dot{W}_t^{H,\varepsilon} - P_t^\varepsilon D_t R_t^{-1} D_t P_t^\varepsilon \left(\dot{W}_t^{H,\varepsilon}\right)^2 dt = 0\,. \end{aligned}$$

When $D_t \not\equiv 0$, the term $\left(\dot{W}_t^{H,\varepsilon}\right)^2 dt$ presents, whose limit does not exist. This explains the difficulty to study the case when $D_t \not\equiv 0$.

Let us conclude this section by remarking that the discussion in this section, to some extent, is rather formal, in the sense that Theorem 5.1 depends upon the existence of solution to the Riccati equation (5.3) along with some required (very technical) conditions of the solution. Nevertheless, the highlight of this section is indeed the introduction of the new equation (5.3), which is a highly unconventional BSDE involving stochastic integrals with respect to both the normal and fractional Brownian motions. Research on this type of equations, as we believe, will prove to be very interesting and challenging from both the BSDE and fractional analysis points of view.

6 Concluding Remarks

In this paper we have attempted to systematically tackle an important class of stochastic control problems with linear dynamic systems involving fractional Brownian motion. Three specific control models have been studied. It should be noted that the results in this paper are far from complete. Indeed the paper is intended to be more inspirational – in the sense that it will inspire more researches along the line – than exhaustive and conclusive. Many interesting and challenging problems remain open. The foremost one is the optimal control over the set of general controls – not necessarily that of the Markovian controls. The reason is that the FBM has the long range dependence; hence it is natural to expect that the optimal control would not only depend on the present time, but on the past as well. In fact, we conjecture that the optimal control is generally linear but non-Markovian (in a special case, this fact has been proved in a recent paper [22]). In particular, it is interesting to seek the optimal control in the class of the following controls

$$u_t = \int_0^t K(t,s)x_s ds\,.$$

We will study such problems in the forthcoming papers.

Acknowledgment: We thank the two anonymous referees for their careful reading of an earlier version of the paper and for their constructive comments that led to an improved version. In particular, we are grateful to one of the referees for bringing the paper [22] to the authors' attention.

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