A constrained nonlinear regular-singular stochastic control problem, with applications

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Abstract

This paper investigates a mixed regular–singular stochastic control problem where the drift of the dynamics is quadratic in the regular control variable. More importantly, the regular control variable is constrained. The value function of the problem is derived in closed form via solving the corresponding constrained Hamilton–Jacobi–Bellman (HJB) equation, and optimal controls are obtained explicitly. Applications and economic interpretations of the general results to two applied problems, from which the mathematical problem was originated, are discussed.

Key words. regular-singular stochastic control, value function, Hamilton–Jacobi–Bellman (HJB) equation, Skorohod problem, personnel management, re-insurance

1 Introduction

Consider the following controlled process $X(\cdot) := \{X(t) : 0 \leq t < +\infty\}$ with

$$dX(t) = [\mu U(t) - aU^2(t) - \delta]dt + \sigma U(t)dW(t) - dZ(t), \quad X(0-) = x. \tag{1}$$

Here μ, a, σ, δ are given constants, $W(\cdot)$ is a one-dimensional standard Brownian motion on a given filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, $Z(\cdot)$ is a nonnegative, non-decreasing $\mathcal{F}_t$-adapted process that is right continuous with left limits, and $U(\cdot)$ is an $\mathcal{F}_t$-adapted process

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constrained by

$$l \leq U(t) \leq u,$$

for some constants l and u. $\pi = (U(\cdot), Z(\cdot))$ is called an admissible control (pair). The objective is to find an optimal control $\pi^* = (U^*(\cdot), Z^*(\cdot))$ that achieves the supremum of

$$E \int_0^{\tau} e^{-rt} dZ(t), \tag{2}$$

where $\tau = \inf\{t > 0 : X(t) = 0\}$, over all admissible controls. In the terminology of stochastic control, $U(\cdot)$ is a regular control whereas $Z(\cdot)$ a singular control. Hence the problem is one with mixed regular–singular stochastic control.

Study on the model starts with the linear (in the control variable) dynamic case (i.e., $a = 0$), with practical motivations from controlling risks and dividend distributions, as well as from managing personnel (hiring/firing) policies for cooperations. Explicit solutions for the special linear case ($a = 0, \delta = 0$) (Radner and Shepp [9], Shepp and Shiryayev [10], and Jeanblanc-Picqué and Shiryayev [7]) have inspired further studies for the case of $a = 0, \delta \neq 0$ and its applications in the area of, for example, re-insurance (Asmussen and Taksar [1], Taksar and Zhou [11], and Choulli et. al. [3]). Later, Asmussen et. al. [2] and Choulli et. al. [4] investigated a certain non-linear diffusion process for this type of optimization problems in the context of the excess-of-loss re-insurance. Their parameterization method, however, relies on the monotonic structure between the drift and the diffusion terms. Most recently, Guo [5] introduced the nonlinear dynamics (1) motivated by a workforce control problem, and interpreted the factor a as the "internal competition factor", a friction coefficient when the company is over-sized. Her result, obtained under the assumptions that $l = 0$ and $u = +\infty$ (i.e., there is no upper bound on the regular control $U(\cdot)$), demonstrates that the monotonic structure is necessary neither for explicitly deriving the optimal value function nor for the concavity of the value solution.

The objective of this paper is to study the control problem (1)–(2) with bounded controls ($0 < l < u < +\infty$). Mathematically, the additional constraint would impose great difficulty in solving the problem. Indeed, as shown in the final results the solution structure in the bounded control case is very different from, in fact much more complex than, its unbounded counterpart (see also Remark 14). Besides mathematical curiosity, however, there are practical motivations for imposing a constraint on $U(\cdot)$. A control upper bound u normally reflects an *explicit* limit on the resource the decision-maker (or controller) can utilize, whereas a lower bound $l > 0$ covers the situations where there are institutional or statutory reasons (e.g., the company is public) that its business activities cannot be reduced to zero (unless the company faces bankruptcy). This is illustrated by the two specific examples which we are going to discuss in this paper. In the case of personnel management where $U(t)$ signifies the number of employees at time t, it is reasonable to assume that a company has an explicit headcount limitation. In the re-insurance setting, where $1-U(t)$ is interpreted as the re-insurance proportion at t, it is natural to restrict $U(t)$ to be in $[0,1]$.

Supplemented by the knowledge of the solution structure for the unbounded case, our analysis is led through by some crucial observations/intuitions. Compared with the unbounded case, however, the analysis for the bounded one is much more involved.

The rest of the paper is organized as follows. Section 2 presents the mathematical model and some preliminaries, and divides the problem into 6 cases according to the key parameter values. Section 3 is devoted to a detailed analysis and complete solution for the main case, whereas Section 4 is for other cases. In Section 5 the special case when $\delta=0$ is discussed. Applications of the general results to two examples, personnel management and re-insurance business, along with their economic insights, are given in Section 6. Finally, Section 7 concludes the paper.

2 Mathematical Model

We are given a filtered probability space $(\Omega,\mathcal{F},P;\mathcal{F}_t)$ and a one-dimensional standard Brownian motion $W(\cdot)$ (with $W(0)=0$) on it, adapted to the filtration $\mathcal{F}_t$. The system equation under consideration is

$$dX(t)=[\mu U(t)-aU^2(t)-\delta]dt+\sigma U(t)dW(t)-dZ(t),\quad X(0-)=x, \tag{3}$$

where $\mu,a,\sigma>0$ and $\delta\geq 0$ are given parameters, and $x\geq 0$ is the initial state. The control in this model is described by a pair of $\mathcal{F}_t$-adapted measurable processes $\pi=(U(\cdot),Z(\cdot))\equiv(U(t),Z(t);t\geq 0)$. A control $\pi=(U(\cdot),Z(\cdot))$ is admissible if $l\leq U(t)\leq u$, $\forall t\geq 0$, and $Z(t)\geq 0$ is non-decreasing, right continuous having left limits, where $0<l<u<+\infty$ are given scalars. We denote the set of all admissible controls by $\mathcal{A}$. Given an admissible control policy π, the time of bankruptcy is defined as

$$\tau=\tau^{\pi}:=\inf\{t\geq 0: X(t)=0\}. \tag{4}$$

We make the convention that

$$X(t)=0\ \ \forall t\geq\tau^{\pi}. \tag{5}$$

Our objective is to find $\pi\in\mathcal{A}$ so as to maximize the performance index

$$J(x,\pi)=E\int_0^{\tau}e^{-rt}dZ(t) \tag{6}$$

over $\mathcal{A}$, where $r > 0$ is an a priori given discount factor. Define the value function

$$V(x) = \sup_{\pi \in \mathcal{A}} J(x, \pi). \tag{7}$$

To solve this problem, our starting point is the dynamic programming Hamilton–Jacobi–Bellman (HJB) equation:

$$\begin{cases} \max\left\{1 - v'(x), \max_{l \le U \le u}[\frac{1}{2}\sigma^2 U^2 v''(x) + (\mu U - aU^2 - \delta)v'(x) - rv(x)]\right\} = 0, & \forall x > 0, \\ v(0) = 0. \end{cases} \tag{8}$$

The ultimate importance of the HJB equation lies in the following verification theorem:

Lemma 1 *If v is a twice continuously differential concave function of (8), then $V(x) \le v(x)$.*

Proof: The proof of [11, Theorem 5.3] readily applies to the present case. □

The idea of solving the original stochastic control problem is to first find a concave, smooth solution to the HJB equation (8), and then construct a control policy whose performance functional coincides with this solution. Then, the above verification theorem establishes the optimality of the constructed control policy.

Hence, all we need to do is to find a concave, smooth solution to the HJB equation (8). While we could have presented such a solution immediately without any explanation, we choose to unfold the process of tracking down the solution for the benefit of the readers.

Before starting our analysis, we present the following property of the value function V, defined by (7).

Proposition 2 *If $\mu U - aU^2 - \delta \le 0 \; \forall U \in [l, u]$, then $V(x) = x \; \forall x \ge 0$.*

Proof: Clearly $J(x, \pi^*) = x$ if $\pi^* = (U^*(\cdot), Z^*(\cdot))$ with $Z^*(t) \equiv x$ and $U^*(\cdot)$ being arbitrary. So it suffices to prove $J(x, \pi) \le x$ for any admissible control π. To this end, let $X(\cdot)$ be the state trajectory corresponding to any given admissible control $\pi = (U(\cdot), Z(\cdot))$. Since

$$\begin{aligned} X(t) = & \; x + \int_0^t [\mu U(s) - aU^2(s) - \delta] ds + \int_0^t \sigma U(s) dW(s) - \int_0^t dZ(s) \\ = & \; x - Z(t) + Z(0-) + \int_0^t [\mu U(s) - aU^2(s) - \delta] ds + \int_0^t \sigma U(s) dW(s), \end{aligned}$$

and $\mu U(s) - aU^2(s) - \delta \le 0$ due to the assumption, it follows that

$$M(t) := X(t) - x + Z(t) - Z(0-)$$

is an $\mathcal{F}_t$-supermartingale. Then,

$$\begin{aligned} J(x,\pi) = & \ E\int_0^\tau e^{-rt}dZ(t) = E\int_0^\tau e^{-rt}d[M(t) - X(t) + x + Z(0-)] \\ = & \ E\int_0^\tau e^{-rt}dM(t) - E\int_0^\tau e^{-rt}dX(t) \leq -E\int_0^\tau e^{-rt}dX(t) \\ = & \ x - rE\int_0^\tau e^{-rt}x(t)dt \\ \leq & \ x. \end{aligned}$$

This completes the proof. ∎

Remark 3 Although our focus throughout the paper concerns only the quadratic case, the forgoing proof is readily extended to the case where the quadratic form $\mu U - aU^2 - \delta$ is replaced by a general continuous function $f(\cdot)$ for which $f(U) \leq 0$ for any $U \in [l, u]$. On the other hand, the result of Proposition 2 is intuitive. When the outlook for growth is negative the best way is simply to get all the initial endowment and run.

Since the assumption of Proposition 2 holds true when $\mu^2 \leq 4a\delta$, we may exclude this trivial case from consideration by assuming the following throughout this paper:

$$\mu^2 > 4a\delta. \tag{9}$$

Considering $[l, u] \cap [\frac{2\delta}{\mu}, \frac{\mu}{2a}]$, we may categorize several cases to be investigated:

- Case 1: $\frac{2\delta}{\mu} < l < u < \frac{\mu}{2a}$;
- Case 2: $l \leq \frac{2\delta}{\mu} \leq u < \frac{\mu}{2a}$;
- Case 3: $\frac{2\delta}{\mu} < l \leq \frac{\mu}{2a} \leq u$;
- Case 4: $l < \frac{2\delta}{\mu} < \frac{\mu}{2a} < u$;
- Case 5: $l > \frac{\mu}{2a}$, and $f(l) > 0$ where $f(x) = \mu x - ax^2 - \delta > 0$;
- Case 6: $u < \frac{2\delta}{\mu}$, and $f(u) > 0$ where $f(x) = \mu x - ax^2 - \delta > 0$.

Among these cases, Case 1 is the most difficulty one that requires a comprehensive study. Therefore, we will first deal with Case 1 in details, and then present results for all the other cases, mentioning the necessary amendments.

3 Solution for Case 1: $\frac{2\delta}{\mu} < l < u < \frac{\mu}{2a}$

First of all, by its very definition the value function $V(x)$ must be non-decreasing in x. Our focus is therefore to find a non-decreasing, concave smooth function v which is the solution to the HJB equation (8). Remark that, although the concavity is not yet proved at this moment (it will be proved after the solution is found), our principle is to keep track of the concavity of the (possible) solution.

Let v be a candidate for the solution to (8) that we are searching for, and $x^* := \inf\{x \geq 0 : v'(x) = 1\}$. In view of the concavity of v as well as the form of (8), $x^* \in [0, +\infty]$ with $v'(x) > 1$ for $x < x^*$ and $v'(x) = 1$ for $x \geq x^*$.

First our attention is on $x < x^*$. The HJB equation (8) reduces to

$$\begin{cases} \max_{l \leq U \leq u}[\frac{1}{2}\sigma^2 U^2 v''(x) + (\mu U - aU^2 - \delta)v'(x) - rv(x)] = 0, & 0 < x < x^*, \\ v(0) = 0. \end{cases} \tag{10}$$

Our first objective is to find

$$U^*(x) := \text{argmax}_{l \leq U \leq u}[\frac{1}{2}\sigma^2 U^2 v''(x) + (\mu U - aU^2 - \delta)v'(x) - rv(x)], \quad 0 \leq x < x^*. \tag{11}$$

It follows from the continuity of $v''(x)$ that $U^*(x)$ is continuous in x. To get the expression for $U^*(x)$ define

$$\theta(x) := \text{argmax}_{-\infty < U < +\infty}[\frac{1}{2}\sigma^2 U^2 v''(x) + (\mu U - aU^2 - \delta)v'(x) - rv(x)], \quad 0 \leq x < x^*. \tag{12}$$

Then clearly

$$U^*(x) = \begin{cases} l, & \theta(x) \leq l, \\ \theta(x), & l < \theta(x) < u, \\ u, & \theta(x) \geq u. \end{cases} \tag{13}$$

Now, applying the zero-derivative condition to (12) we obtain

$$[\mu - 2a\theta(x)]v'(x) + \sigma^2\theta(x)v''(x) = 0, \tag{14}$$

which gives

$$\theta(x) = \frac{\mu v'(x)}{2av'(x) - \sigma^2 v''(x)}, \quad 0 \leq x < x^*. \tag{15}$$

If it turns out $l < \theta(x) < u$ then substitute the above to (10). After some manipulation we get (recalling that $v'(x) > 1$ for $0 \leq x < x^*$)

$$\theta(x) = \frac{2rv(x)}{\mu v'(x)} + \frac{2\delta}{\mu}. \tag{16}$$

Now, taking $x \to 0+$ we have by (16) that $\theta(x) \to \frac{2\delta}{\mu} < l$, whereas taking $x \to x^*-$ we have by (15) that $\theta(x) \to \frac{\mu}{2a} > u$. This along with (13) shows that there exists $0 \le x_1 < x_2 \le x^*$ so that

$$U^*(x) = \begin{cases} l, & 0 \le x \le x_1, \\ \theta(x), & x_1 < x < x_2, \\ u, & x_2 \le x \le x^*. \end{cases} \tag{17}$$

Keeping in mind that the desired value function $v'(x)$ is strictly positive near the neighborhood of 0, concave, and will have linear growth when $x \to \infty$.

Next, we are going to determine the values of x_1, x_2 and x^*. It will be carried out by solving the HJB equation (8) in several intervals, and then "paste" all the solution segments nicely by the principle of smooth fit.

Step 1: $x \in [0, x_1]$.

When $0 \le x \le x_1$, we have $U^*(x) = l$. Therefore, (10) reads

$$\frac{1}{2}\sigma^2 l^2 v''(x) + (\mu l - al^2 - \delta)v'(x) - rv(x) = 0, \quad v(0) = 0.$$

The general form of the solution is

$$v(x) = A(e^{\alpha_1 x} - e^{\beta_1 x}), \quad 0 \le x \le x_1, \tag{18}$$

where

$$\alpha_1 = \frac{-(\mu l - al^2 - \delta) + \sqrt{(\mu l - al^2 - \delta)^2 + 2r\sigma^2 l^2}}{\sigma^2 l^2}, \tag{19}$$

$$\beta_1 = \frac{-(\mu l - al^2 - \delta) - \sqrt{(\mu l - al^2 - \delta)^2 + 2r\sigma^2 l^2}}{\sigma^2 l^2}. \tag{20}$$

Here A is a constant to be determined later.

Step 2: $x \in (x_1, x_2)$

When $x \in (x_1, x_2)$, we have $U^*(x) = \theta(x)$. Using (16) we have

$$-rv(x) + (\frac{1}{2}\mu\theta(x) - \delta)v'(x) = 0. \tag{21}$$

Taking derivative on the above equation and then combining with (14), we arrive at

$$[\mu\theta'(x) - 2r]\theta(x)\sigma^2 = [\mu - 2a\theta(x)][\mu\theta(x) - 2\delta]. \tag{22}$$

Here, the boundary condition $\theta(x_1) = l$ will determine $\theta(x)$, once x_1 is solved. This, in turn, will define x_2 by setting $\theta(x_2) = u$. At last, $v(x)$ can be obtained on (x_1, x_2) by solving (21) together with the boundary condition at x_2 (which will be available in Step 3 below).

Step 3: $x \in [x_2, x^*)$

When $x_2 \le x < x^*$ we have $U^*(x) = u$. Therefore, (10) specializes to

$$\frac{1}{2}\sigma^2 u^2 v''(x) + (\mu u - au^2 - \delta)v'(x) - rv(x) = 0,$$

for which we have

$$v(x) = Be^{\alpha_2 x} + Ce^{\beta_2 x}, \quad x_2 \le x < x^*, \tag{23}$$

with

$$\alpha_2 = \frac{-(\mu u - au^2 - \delta) + \sqrt{(\mu u - au^2 - \delta)^2 + 2r\sigma^2 u^2}}{\sigma^2 u^2}, \tag{24}$$

$$\beta_2 = \frac{-(\mu u - au^2 - \delta) - \sqrt{(\mu u - au^2 - \delta)^2 + 2r\sigma^2 u^2}}{\sigma^2 u^2}. \tag{25}$$

Here B, C are unknown constants to be determined.

Step 4: $x \in [x^*, +\infty)$.

In this case, since $v'(x) \equiv 1$, we have

$$v(x) = x + v(x^*) - x^* \tag{26}$$

with x^* to be determined.

Step 5: Determine x_1, x_2, x^*, A, B, C, and $\theta(x)$. To this end, we apply the principle of smooth fit to $v(x)$ at $x = x_1, x_2, x^*$, respectively.

First off, at $x = x_1$, we have

$$v(x_1+) = v(x_1-), \quad v'(x_1+) = v'(x_1-). \tag{27}$$

Applying these conditions to equations (18) and (21) (and noting $\theta(x_1+) = l$), we obtain

$$A(\alpha_1 e^{\alpha_1 x_1} - \beta_1 e^{\beta_1 x_1}) = \frac{rA(e^{\alpha_1 x_1} - e^{\beta_1 x_1})}{\frac{1}{2}\mu l - \delta}. \tag{28}$$

Lemma 4 *$x_1 > 0$ for x_1 determined by (28).*

Proof: We first verify that

$$\frac{r - \beta_1(\frac{1}{2}\mu l - \delta)}{r - \alpha_1(\frac{1}{2}\mu l - \delta)} > 1. \tag{29}$$

Indeed, since $\beta_1 < 0$ and $\frac{1}{2}\mu l - \delta > 0$ we have $r - \beta_1(\frac{1}{2}\mu l - \delta) > 0$. Using the expressions of α_1 and β_1 as given by (19) and (20) respectively, one has

$$(\alpha_1 - \beta_1)(\frac{1}{2}\mu l - \delta) > 0.$$

This proves (29). Hence from (28) we conclude

$$x_1 = \frac{1}{\alpha_1 - \beta_1} \log \left[\frac{r - \beta_1(\frac{1}{2}\mu l - \delta)}{r - \alpha_1(\frac{1}{2}\mu l - \delta)} \right] > 0. \tag{30}$$

□

Next, we determine $U^*(x) = \theta(x)$ on (x_1, x_2).

Lemma 5 *The unique strictly increasing solution $\theta(x)$ to the following equation*

$$\begin{cases} [\mu\theta'(x) - 2r]\theta(x)\sigma^2 = [\mu - 2a\theta(x)][\mu\theta(x) - 2\delta], \\ \theta(x_1) = l, \end{cases} \tag{31}$$

can be determined by the following

$$[G - \theta(x)]^{\frac{G}{G-H}} [\theta(x) - H]^{\frac{-H}{G-H}} = K \exp\left[-\frac{2a}{\sigma^2}(x - x_1) \right], \tag{32}$$

where x_1 is given by (30),

$$G = \frac{2r\sigma^2 + \mu^2 + 4a\delta + \sqrt{(2r\sigma^2 + \mu^2 + 4a\delta)^2 - 16a\mu^2\delta}}{4a\mu},$$
$$H = \frac{2r\sigma^2 + \mu^2 + 4a\delta - \sqrt{(2r\sigma^2 + \mu^2 + 4a\delta)^2 - 16a\mu^2\delta}}{4a\mu},$$

and

$$K = (G - l)^{\frac{G}{G-H}} (l - H)^{\frac{-H}{G-H}}.$$

Proof: First, we show $H < \frac{2\delta}{\mu} < l < u < \frac{\mu}{2a} < G$. Indeed,

$$\begin{aligned} H &= \frac{2r\sigma^2 + \mu^2 + 4a\delta - \sqrt{(2r\sigma^2 + \mu^2 + 4a\delta)^2 - 16a\mu^2\delta}}{4a\mu} \\ &< \frac{2r\sigma^2 + \mu^2 + 4a\delta - \sqrt{(2r\sigma^2 + \mu^2 - 4a\delta)^2}}{4a\mu} \\ &= \frac{2\delta}{\mu}. \end{aligned}$$

A similar proof yields $\frac{\mu}{2a} < G$. Since $l < \theta(x) < u$ whenever $x \in (x_1, x_2)$, we have

$$H < \theta(x) < G, \quad x \in (x_1, x_2). \tag{33}$$

Now, after some manipulations the differential equation (31) can be rewritten as

$$\frac{\theta(x)d\theta(x)}{[G-\theta(x)][\theta(x)-H]} = \frac{2a}{\sigma^2}dx.$$

Taking integration from x_1 to x and taking (33) into account we obtain (32).

To show that $\theta(x)$ is a strictly increasing function of x, let $\xi(z) := (G-z)^{\frac{G}{G-H}}(z-H)^{-\frac{H}{G-H}}$, $H < z < G$. Since $\xi'(z) = -z(G-z)^{\frac{G}{G-H}-1}(z-H)^{-\frac{H}{G-H}-1} < 0$, $\xi(z)$ is strictly decreasing on $z \in (H, G)$. However, $\xi(\theta(x)) = K \exp\left[-\frac{2a}{\sigma^2}(x-x_1)\right]$, hence $\theta(x)$ is strictly increasing in x.

□

Now define

$$x_2 := \inf\{x > x_1 | \theta(x) = u\}. \tag{34}$$

One can express x_2 explicitly by setting $x = x_2$ in (32) and then solving for x_2:

$$x_2 = x_1 + \frac{\sigma^2}{2a}\left[\frac{G}{G-H}\log\left(\frac{G-l}{G-u}\right) - \frac{H}{G-H}\log\left(\frac{l-H}{u-H}\right)\right]. \tag{35}$$

By the very definition (34) and the strict monotonicity of $\theta(x)$ it is clear that $x_2 > x_1$ since $l \neq u$.

Having found x_1 and x_2, we are now in the position to find A, B, C and x^*. First, smooth fit at $x = x_2$, together with (21), yields

$$-rv(x_2) + (\frac{1}{2}\mu u - \delta)v'(x_2) = 0 \tag{36}$$

since $\theta(x_2) = u$. Using (23), we get

$$-r(Be^{\alpha_2 x_2} + Ce^{\beta_2 x_2}) + \left(\frac{1}{2}\mu u - \delta\right)(B\alpha_2 e^{\alpha_2 x_2} + C\beta_2 e^{\beta_2 x_2}) = 0,$$

or equivalently,

$$B\left[-r + (\frac{1}{2}\mu u - \delta)\alpha_2\right]e^{\alpha_2 x_2} + C\left[-r + (\frac{1}{2}\mu u - \delta)\beta_2\right]e^{\beta_2 x_2} = 0. \tag{37}$$

Remark 6 It follows from (14) that the smooth fit condition at $x = x_1$ as given in (27) implies $v''(x_1+) = v''(x_1-)$, namely, v thus obtained in fact is C^2 at x_1. Similarly, v is C^2 at x_2.

On the other hand, smoothness at $x = x^*$ dictates

$$v'(x^*-) = 1, v''(x^*-) = 0. \tag{38}$$

As a consequence, we see

$$B\alpha_2^2 e^{\alpha_2 x^*} + C\beta_2^2 e^{\beta_2 x^*} = 0, \tag{39}$$

$$B\alpha_2 e^{\alpha_2 x^*} + C\beta_2 e^{\beta_2 x^*} = 1. \tag{40}$$

¿From (37) and (39) it follows

$$\frac{-C}{B} = \frac{\alpha_2^2}{\beta_2^2} e^{(\alpha_2-\beta_2)x^*} = \frac{\alpha_2(\frac{1}{2}\mu u - \delta) - r}{\beta_2(\frac{1}{2}\mu u - \delta) - r} e^{(\alpha_2-\beta_2)x_2}. \tag{41}$$

Hence,

$$\begin{aligned} x^* = &\ \tfrac{1}{\alpha_2-\beta_2} \log\left[\tfrac{\alpha_2\beta_2^2(\frac{1}{2}\mu u-\delta)-r\beta_2^2}{\alpha_2^2\beta_2(\frac{1}{2}\mu u-\delta)-r\alpha_2^2} e^{(\alpha_2-\beta_2)x_2}\right] \\ = &\ x_2 + \tfrac{1}{\alpha_2-\beta_2} \log\left[\tfrac{\alpha_2\beta_2^2(\frac{1}{2}\mu u-\delta)-r\beta_2^2}{\alpha_2^2\beta_2(\frac{1}{2}\mu u-\delta)-r\alpha_2^2}\right]. \end{aligned} \tag{42}$$

Before we proceed, we must verify the following

Lemma 7 *$x^* > x_2$ for x^* given by (42).*

Proof: In view of (41) it suffices to show that

$$\frac{\alpha_2(\frac{1}{2}\mu u - \delta) - r}{\beta_2(\frac{1}{2}\mu u - \delta) - r} > \frac{\alpha_2^2}{\beta_2^2},$$

or equivalently (noting that $\beta_2(\frac{1}{2}\mu u - \delta) - r < 0$)

$$\beta_2^2\left[\alpha_2\left(\frac{1}{2}\mu u - \delta\right) - r\right] < \alpha_2^2\left[\beta_2\left(\frac{1}{2}\mu u - \delta\right) - r\right].$$

Indeed,

$$\begin{aligned} &\alpha_2^2\left[\beta_2\left(\frac{1}{2}\mu u - \delta\right) - r\right] - \beta_2^2\left[\alpha_2\left(\frac{1}{2}\mu u - \delta\right) - r\right] \\ = &\ (\alpha_2-\beta_2)\left[\left(\frac{1}{2}\mu u - \delta\right)\alpha_2\beta_2 - r(\alpha_2+\beta_2)\right] \\ = &\ (\alpha_2-\beta_2)\left[\left(\frac{1}{2}\mu u - \delta\right)\frac{-2r}{\sigma^2 u^2} + r\frac{2(\mu u - au^2 - \delta)}{\sigma^2 u^2}\right] \\ = &\ \frac{2r(\alpha_2-\beta_2)}{\sigma^2 u^2}\left(\frac{1}{2}\mu u - au^2\right) > 0 \end{aligned}$$

because of $u < \frac{\mu}{2a}$. This proves the desired claim. ∎

Once x^* is determined, we solve (39) and (40) to obtain

$$B = -\frac{\beta_2}{\alpha_2(\alpha_2-\beta_2)e^{\alpha_2 x^*}}, \tag{43}$$

$$C = \frac{\alpha_2}{\beta_2(\alpha_2-\beta_2)e^{\beta_2 x^*}}. \tag{44}$$

Lemma 8 *There is a unique non-decreasing function $v(x)$ that solves the following equation on $[x_1, x_2)$:*

$$\begin{cases} -rv(x) + (\frac{1}{2}\mu\theta(x) - \delta)v'(x) = 0, & x \in [x_1, x_2), \\ v(x_2) = Be^{\alpha_2 x_2} + Ce^{\beta_2 x_2}. \end{cases} \tag{45}$$

Proof: First note that $\frac{1}{2}\mu\theta(x) - \delta \geq \frac{1}{2}\mu l - \delta > 0$ since $\theta(x)$ is non-decreasing on $[x_1, x_2)$. Thus (45) has a unique solution

$$\begin{aligned} v(x) = & \ v(x_2)\exp\left(-\int_x^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta}dy\right) \\ = & \ (Be^{\alpha_2 x_2} + Ce^{\beta_2 x_2})\exp\left(-\int_x^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta}dy\right), \quad x \in [x_1, x_2). \end{aligned} \tag{46}$$

Clearly it is a non-decreasing function. ◻

Finally, we can now determine A. The continuity of v at x_1 indicates that

$$A(e^{\alpha_1 x_1} - e^{\beta_1 x_1}) = v(x_1-) = v(x_1+) = (Be^{\alpha_2 x_2} + Ce^{\beta_2 x_2})\exp\left(-\int_{x_1}^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y) - \delta}dy\right).$$

Thus,

$$A = \frac{Be^{\alpha_2 x_2} + Ce^{\beta_2 x_2}}{e^{\alpha_1 x_1} - e^{\beta_1 x_1}}\exp\left(-\int_{x_1}^{x_2} \frac{1}{\frac{1}{2}\mu\theta(y) - \delta}dy\right). \tag{47}$$

To summarize, we have now obtained a solution to (8) as follows:

$$v(x) = \begin{cases} A(e^{\alpha_1 x} - e^{\beta_1 x}), & 0 \leq x < x_1, \\ (Be^{\alpha_2 x_2} + Ce^{\beta_2 x_2})\exp\left(-\int_x^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta}dy\right), & x_1 \leq x < x_2, \\ Be^{\alpha_2 x} + Ce^{\beta_2 x}, & x_2 \leq x < x^*, \\ x - x^* + Be^{\alpha_2 x^*} + Ce^{\beta_2 x^*}, & x \geq x^*, \end{cases} \tag{48}$$

where, respectively, α_1 and β_1 are given by (19) and (20), α_2 and β_2 by (24) and (25), $\theta(x)$ determined by (32), x_1, x_2 and x^* by (30), (35) and (42), and A, B and C by (47), (43) and (44).

Lemma 9 *$v(x)$ as given by (48) is a twice continuously differentiable concave function with $v(0) = 0$.*

Proof: First, that $v(x)$ is C^1 comes from the construction, via the smooth fit principle. Moreover, from the explicit expression (48) it is evident that $v(x)$ is C^2 on each of the intervals $(0, x_1), (x_1, x_2), (x_2, x^*)$ and (x^*, ∞). Now, the C^2 property of $v(x)$ at x_1 and x_2 follows from Remark 6, and that at x^* is seen directly from (48). Secondly, that $v(0) = 0$ is obvious. Concavity on each of the intervals other than $[x_1, x_2)$ is evident by the explicit expressions of $v(x)$. On $[x_1, x_2)$, recall that v satisfies (14). Thus $v''(x) \leq 0$ on $[x_1, x_2)$ since $v'(x) \geq 0$ and $\mu - 2a\theta(x) \geq \mu - 2au > 0$. Finally, concavity on the whole positive half line follows from the continuity of its derivatives at the turning points x_1, x_2 and x^*. □

Now we construct the optimal control policies based on the solutions to the HJB equations. There are two components, $U(\cdot)$ and $Z(\cdot)$, in a control. From the foregoing analysis, the first component $U(\cdot)$ is obtained in a state feedback form:

$$U^*(x) = \begin{cases} l, & 0 \leq x \leq x_1, \\ \theta(x), & x_1 < x < x_2, \\ u, & x_2 \leq x \leq x^*, \end{cases} \tag{49}$$

where the critical reserve levels x_1 and x_2, together with the function $\theta(x)$, are specified in (48). To determine the other component $Z(\cdot)$, we need to involve the so-called Skorohod problem for the one dimensional diffusion. Let $(X^*(t), Z^*(t); t \geq 0)$ be a solution to the following Skorohod problem on $t \geq 0$:

$$\begin{cases} dX^*(t) = [\mu U^*(X^*(t)) - aU^{*2}(X^*(t)) - \delta]dt + \sigma U^*(X^*(t))dW(t) - dZ^*(t), \quad X(0-) = x, \\ X^*(t) \leq x^*, \\ \int_0^\infty 1_{\{X^*(t) < x^*\}} dZ^*(t) = 0, \end{cases} \tag{50}$$

where $U^*(x)$ is given by (49) and x^* by (42). Equation (50) essentially determines the diffusion process $X^*(\cdot)$ reflected at x^* via a so-called *regulatory process* $Z^*(\cdot)$ (see, e.g., [6]). Existence and uniqueness of a solution pair $(X^*(t), Z^*(t); t \geq 0)$ to such a Skorohod problem follows from Theorem 3.1 in [8].

Theorem 10 *Let v be the concave, twice continuously differentiable solution of the HJB equation (8) given by (48), and $(X^*(t), Z^*(t); t \geq 0)$ be a solution to the Skorohod problem (50). Then $\pi^* = (U^*(X^*(t)), Z^*(t); t \geq 0)$ is an optimal control for the problem (3)–(6), and v equals the value function V.*

Proof: In view of Lemma 1 the proof of this result is the same as that of [11, Theorem 5.3]. □

Remark 11 By now, it is clear that the threshold $\frac{\mu}{2a}$ does not come arbitrarily. In fact, it is the level for which the optimal value function were about to change its concavity, had it not been the regulatory process at x^* (see equation (14)). This also suggests that, the very feature of our mixed regular–singular control problem, which in general more difficult to solve than its regular control counterpart, helps us in overcoming the difficulty of the nonlinearity of the underlying model and in deriving the optimal control policy. Thanks to the regulatory process $Z(x)$ at x^*, we are able to keep the concavity of the function and yield the optimality.

4 Other Cases

All the other cases can be regarded as degenerate cases of the main case, Case 1, that we have solved. Since the computation is similar and in most cases substantially simpler, we simply present the results, mentioning the necessary changes.

4.1 Case 2: $l \leq \frac{2\delta}{\mu} \leq u < \frac{\mu}{2a}$

In this case, by virtue of (16), we have

$$l \leq \frac{2\delta}{\mu} = \theta(0) \leq u. \tag{51}$$

Hence (17) should be modified to

$$U^*(x) = \begin{cases} \theta(x), & 0 \leq x < x_2, \\ u, & x_2 \leq x \leq x^*. \end{cases} \tag{52}$$

In other words, x_1 appearing in Case 1 degenerates to 0. As a consequence, $\theta(x)$ for $0 \leq x < x_2$ can be obtained by modifying the initial condition of (31) by $\theta(0) = \frac{2\delta}{\mu}$. Therefore, $\theta(x)$ can be determined by the following

$$[G - \theta(x)]^{\frac{G}{G-H}}[\theta(x) - H]^{\frac{-H}{G-H}} = K_1 \exp(-\frac{2a}{\sigma^2}x), \;\; 0 \leq x < x_2, \tag{53}$$

where

$$K_1 = (G - \frac{2\delta}{\mu})^{\frac{G}{G-H}}(\frac{2\delta}{\mu} - H)^{\frac{-H}{G-H}}.$$

Accordingly, x_2 must be revised to

$$x_2 = \frac{\sigma^2}{2a}\left[\frac{G}{G-H}\log\left(\frac{G - \frac{2\delta}{\mu}}{G-u}\right) - \frac{H}{G-H}\log\left(\frac{\frac{2\delta}{\mu} - H}{u-H}\right)\right]. \tag{54}$$

Now, we can present a solution to (8)

$$v(x) = \begin{cases} (Be^{\alpha_2 x_2} + Ce^{\beta_2 x_2})\exp\left(-\int_x^{x_2} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta}dy\right), & 0 \leq x < x_2, \\ Be^{\beta_2 x} + Ce^{\alpha_2 x}, & x_2 \leq x < x^*, \\ x - x^* + Be^{\beta_2 x^*} + Ce^{\alpha_2 x^*}, & x \geq x^*, \end{cases} \tag{55}$$

where, respectively, α_2 and β_2 by (24) and (25), $\theta(x)$ by (53), x_2 and x^* by (54) and (42), and B and C by (43) and (44).

Finally, in this case the optimal regular control is given by (52) and the optimal singular control $Z^*(\cdot)$ is the corresponding regulatory process at x^*. Notice that in this case $U^*(x) \in [\frac{2\delta}{\mu}, u]$.

4.2 Case 3: $\frac{2\delta}{\mu} < l \le \frac{\mu}{2a} \le u$

In this case, it follows from (15) that

$$l \le \frac{\mu}{2a} = \theta(x^*) \le u. \tag{56}$$

Hence

$$U^*(x) = \begin{cases} l, & 0 \le x < x_1, \\ \theta(x) & 0 \le x \le x^*. \end{cases} \tag{57}$$

Thus, x_2 in Case 2 degenerates to x^* in the present case. The point x_1 is determined by exactly the same formula, (30). Moreover, x^* can be obtained by setting $\theta(x^*) = \frac{\mu}{2a}$ where $\theta(x)$ is given by (32). Hence

$$x^* = x_1 + \frac{\sigma^2}{2a}\left[\frac{G}{G-H}\log\left(\frac{G-l}{G-\frac{\mu}{2a}}\right) - \frac{H}{G-H}\log\left(\frac{l-H}{\frac{\mu}{2a}-H}\right)\right]. \tag{58}$$

On the other hand,

$$v(x) = x - x^* + A_1, \quad x \ge x^*$$

for some constant A_1. Noting that $U^*(x^*-) = \theta(x^*-) = \frac{\mu}{2a}$ and substituting it to (10), we get

$$A_1 = \frac{\mu^2 - 4a\delta}{4ar}. \tag{59}$$

To proceed, we solve equation (45) with the terminal condition changed to $v(x^*) = A_1$ and obtain

$$v(x) = A_1 \exp\left(-\int_x^{x^*} \frac{r}{\frac{1}{2}\mu\theta(y) - \delta}dy\right), \quad x \in [x_1, x^*). \tag{60}$$

Thus, (47) should be modified to

$$A = \frac{A_1}{e^{\alpha_1 x_1} - e^{\beta_1 x_1}} \exp\left(-\int_{x_1}^{x^*} \frac{1}{\frac{1}{2}\mu\theta(y) - \delta}dy\right). \tag{61}$$

To summarize, we have the following solution to (8) in this case:

$$v(x) = \begin{cases} A(e^{\alpha_1 x} - e^{\beta_1 x}), & 0 \le x < x_1, \\ A_1 \exp\left(-\int_x^{x^*} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta}dy\right), & x_1 \le x < x^*, \\ x - x^* + A_1, & x \ge x^*, \end{cases} \tag{62}$$

where α_1, β_1, $\theta(x)$, x_1, and x^* are as in (19), (20), (32), (30), and (58) respectively.

The optimal regular control is (57) and the optimal singular control $Z^*(\cdot)$ is the corresponding regulatory process at x^*. Notice that in this case $U^*(x) \in [l, \frac{\mu}{2a}]$.

4.3 Case 4: $l < \frac{2\delta}{\mu} < \frac{\mu}{2a} < u$

In this case,

$$U^*(x) = \theta(x), \quad 0 \le x \le x^*.$$

Therefore, similar arguments as in Cases 2 and 3 lead to

$$v(x) = \begin{cases} A_1 \exp\left(-\int_x^{x^*} \frac{r}{\frac{1}{2}\mu\theta(y)-\delta} dy\right), & 0 \le x < x^*, \\ x - x^* + A_1, & x \ge x^*, \end{cases} \tag{63}$$

where A_1 is given by (59), $\theta(x)$ by (53), and

$$x^* = \frac{\sigma^2}{2a}\left[\frac{G}{G-H}\log\left(\frac{G-\frac{2\delta}{\mu}}{G-\frac{\mu}{2a}}\right) - \frac{H}{G-H}\log\left(\frac{\frac{2\delta}{\mu}-H}{\frac{\mu}{2a}-H}\right)\right]. \tag{64}$$

The optimal regular control is $U^*(x) = \theta(x)\ \ \forall 0 \le x \le x^*$ and the optimal singular control $Z^*(\cdot)$ is the corresponding regulatory process at x^*. Notice that in this case $U^*(x) \in [\frac{2\delta}{\mu}, \frac{\mu}{2a}]$.

4.4 Case 5: $l > \frac{\mu}{2a}$ and $f(l) = \mu l - al^2 - \delta > 0$

First it is easy to see that Case 5 defined earlier is equivalent to $l > \frac{\mu}{2a}$ and $f(l) > 0$. In this case, since $\theta(x^*) = \frac{\mu}{2a} < l$, we have

$$U^*(x) = l, \quad 0 \le x \le x^*.$$

Thus

$$v(x) = A(e^{\alpha_1 x} - e^{\beta_1 x}), \quad 0 \le x \le x^*, \tag{65}$$

where α_1 and β_1 are defined by (19) and (20) respectively. Smooth fit at x^* yields

$$\begin{cases} A(\alpha_1 e^{\alpha_1 x^*} - \beta_1 e^{\beta_1 x^*}) = 1, \\ A(\alpha_1^2 e^{\alpha_1 x^*} - \beta_1^2 e^{\beta_1 x^*}) = 0. \end{cases}$$

Solving the above equations leads to

$$\begin{aligned} x^* &= \frac{\log(\beta_1^2) - \log(\alpha_1^2)}{\alpha_1 - \beta_1}, \\ A &= \frac{1}{\alpha_1 e^{\alpha_1 x^*} - \beta_1 e^{\beta_1 x^*}}. \end{aligned} \tag{66}$$

Hence, a solution to (8) is

$$v(x) = \begin{cases} A(e^{\alpha_1 x} - e^{\beta_1 x}), & 0 \le x < x^*, \\ x - x^* + A(e^{\alpha_1 x^*} - e^{\beta_1 x^*}), & x \ge x^*, \end{cases} \tag{67}$$

where x^* and A are defined in (66).

The optimal regular control is $U^*(x) = l\ \ \forall 0 \le x \le x^*$ and the optimal singular control $Z^*(\cdot)$ is the corresponding regulatory process at x^*.

4.5 Case 6: $u < \frac{2\delta}{\mu}$ and $f(u) = \mu u - au^2 - \delta > 0$

In this case, we have

$$U^*(x) = u, \quad 0 \le x \le x^*.$$

By a completely parallel analysis to that for Case 5, we obtain

$$v(x) = \begin{cases} A(e^{\alpha_2 x} - e^{\beta_2 x}), & 0 \le x < x^*, \\ x - x^* + A(e^{\alpha_2 x^*} - e^{\beta_2 x^*}), & x \ge x^*, \end{cases} \tag{68}$$

where

$$\begin{aligned} x^* &= \frac{\log(\beta_2^2) - \log(\alpha_2^2)}{\alpha_2 - \beta_2}, \\ A &= \frac{1}{\alpha_2 e^{\alpha_2 x^*} - \beta_2 e^{\beta_2 x^*}}, \end{aligned} \tag{69}$$

with α_2 and β_2 defined in (24) and (25).

The optimal regular control is $U^*(x) = u \ \ \forall 0 \le x \le x^*$ and the optimal singular control $Z^*(\cdot)$ is the corresponding regulatory process at x^*.

Before we conclude this section, we have the following remark.

Remark 12 We see that in all the cases (including Case 1 dealt with in the previous section), the optimal regular control $U^*(\cdot)$ has the minimum value $l' = \max\{l, \frac{2\delta}{\mu}\}$ and the maximum value $u' = \min\{u, \frac{\mu}{2a}\}$. Since $U^*(\cdot)$ can be understood to be the one controlling the risk, l' and u' are the minimum and maximum risks to be taken, respectively. Also, we can conclude from the solutions obtained that the optimal risk range is such that it should be as close to the interval $[\frac{2\delta}{\mu}, \frac{\mu}{2a}]$ as possible.

5 A Special Case: $\delta = 0$

The special case when $\delta = 0$ is an interesting case that warrants additional attention. Practically it corresponds to the no-debt situation (see Section 6 below). Theoretically, the value function becomes completely explicit.

For simplicity, we only present the result for the main case $0 < l < u < \frac{\mu}{2a}$. In this case, equation (31) specializes to

$$\begin{cases} \theta'(x) + \frac{2a}{\sigma^2}\theta(x) - \frac{\mu^2 + 2r\sigma^2}{\mu\sigma^2} = 0, \\ \theta(x_1) = l, \end{cases} \tag{70}$$

which has the solution

$$\theta(x) = Ke^{-\frac{2a}{\sigma^2}x} + \frac{\mu^2 + 2r\sigma^2}{2a\mu}, \tag{71}$$

where $K = -e^{\frac{2a}{\sigma^2}x_1}\left(\frac{\mu^2+2r\sigma^2}{2a\mu} - l\right) < \frac{2a}{\mu}$. Setting $\theta(x_2) = u$, we obtain

$$x_2 = x_1 + \frac{\sigma^2}{2a}\log\left(\frac{\frac{\mu^2+2r\sigma^2}{2a\mu} - l}{\frac{\mu^2+2r\sigma^2}{2a\mu} - u}\right).$$

With the expression of $\theta(x)$, the resulting solution to the HJB equation becomes explicit as well. Indeed, with the aid of (21), we have

$$\int_{x_1}^{x}\frac{v'(x)}{v(x)}dx = \frac{2r}{\mu}\int_{x_1}^{x}\frac{1}{\theta(x)}dx.$$

After simple calculation, we get

$$v(x) = v(x_1)\left(\frac{e^{\frac{2a}{\sigma^2}(x-x_1)} - 1 + \frac{2a\mu l}{\mu^2+2r\sigma^2}}{\frac{2a\mu l}{\mu^2+2r\sigma^2}}\right)^{\frac{2r\sigma^2}{\mu^2+2r\sigma^2}}.$$

Going through a similar smooth-fit procedure, we get the following result.

Theorem 13 *If $\delta = 0$ and $0 < l < u \le \frac{\mu}{2a}$, then*

1) The optimal regular control $U^*(x)$ is given as a feedback control

$$U^*(x) = \begin{cases} l, & 0 \le x < x_1, \\ \frac{\mu^2+2r\sigma^2}{2a\mu} - (\frac{\mu^2+2r\sigma^2}{2a\mu} - l)e^{\frac{2a}{\sigma^2}(x_1-x)}, & x_1 \le x < x_2, \\ u, & x_2 \le x \le x^*, \end{cases}$$

and the optimal singular control $Z^*(\cdot)$ is the regulatory process at x^*, where

$$\begin{aligned} x_1 &= \tfrac{1}{\alpha_1-\beta_1}\log\left(\tfrac{r-\frac{1}{2}\mu l\beta_1}{r-\frac{1}{2}\mu l\alpha_1}\right), \\ x_2 &= x_1 + \tfrac{2a}{\sigma^2}\log\left(\tfrac{\frac{\mu^2+2r\sigma^2}{2a\mu}-l}{\frac{\mu^2+2r\sigma^2}{2a\mu}-u}\right), \\ x^* &= x_2 + \tfrac{1}{\alpha_2-\beta_2}\log\left(\tfrac{\beta_2^2 r-\frac{1}{2}\mu u\alpha_2}{\alpha_2^2 r-\frac{1}{2}\mu u\beta_2}\right). \end{aligned} \tag{72}$$

2) The value function is

$$V(x) = \begin{cases} A(e^{\alpha_1 x} - e^{\beta_1 x}), & 0 \le x < x_1, \\ A(e^{\alpha_1 x} - e^{\beta_1 x})\left(\frac{e^{\frac{2a}{\sigma^2}(x-x_1)}-1+\frac{2a\mu l}{\mu^2+2r\sigma^2}}{\frac{2a\mu l}{\mu^2+2r\sigma^2}}\right)^{\frac{2r\sigma^2}{\mu^2+2r\sigma^2}}, & x_1 \le x < x_2, \\ \frac{\alpha_2}{\beta_2(\alpha_2-\beta_2)}e^{\beta_2(x-x^*)} - \frac{\beta_2}{\alpha_2(\alpha_2-\beta_2)}e^{\alpha_2(x-x^*)}, & x_2 \le x < x^*, \\ x - x^* + \frac{\alpha_2+\beta_2}{\alpha_2\beta_2}, & x \ge x^*, \end{cases}$$

where $\alpha_1, \beta_1, \alpha_2$ and β_2 are as defined in Section 3, and A is such that

$$A(e^{\alpha_1 x_2} - e^{\beta_1 x_2})\left(\frac{e^{\frac{2a}{\sigma^2}(x_2-x_1)} - 1 + \frac{2a\mu l}{\mu^2+2r\sigma^2}}{\frac{2a\mu l}{\mu^2+2r\sigma^2}}\right)^{\frac{2r\sigma^2}{\mu^2+2r\sigma^2}} = \frac{\alpha_2}{\beta_2(\alpha_2-\beta_2)}e^{\beta_2(x_2-x^*)} - \frac{\beta_2}{\alpha_2(\alpha_2-\beta_2)}e^{\alpha_2(x_2-x^*)}$$

Remark 14 By now we can tell some fundamental difference between the results in this paper and those in [5]. First of all, in [5] (where the control constraint is absent) the optimal control and the value function are dictated by one single critical state level x^*, whereas in our case they are determined by x^* as well as two *additional* points, x_1 and x_2. (It is also interesting to note that x_1 and x_2 are both no greater than x^*.) Moreover, even the point x^* in this paper is qualitatively different from the one in [5]. To see this, consider the simple case where $\delta = 0$ and then compare Theorem 13 with [5, Theorem 3]. One easily sees that our $x^* \to 0$ as $u \to 0$, meaning that x^* shifts to the left with the additional upper control constraint. On the other hand, consider Case 5. It follows from (66) that $x^* \to \infty$ when $l \to \infty$, suggesting that x^* moves to the right with the additional lower control constraint.

6 Applications and Economic Interpretations

Besides its obvious mathematical interest, the study of the constrained nonlinear control problem (1)–(2) not only fits the needs of applications, but also provides some interesting implications and insights for practice. In this section we discuss two applications.

6.1 Hiring and firing

The problem (1)–(2) can model a dynamic optimization problem involving personnel decision and dividend distribution policy [10, 5]. Specifically, $X(t)$ is the cash reserve of the firm under consideration and $U(t)$ the number of hires both at time t, and $Z(t)$ is the total dividend distributed up to time t. Moreover, μ is the expected net profit per head with risk factor σ, and δ is the debt rate. Finally, a is the so-called internal competition factor, introduced in [5], to reflect the counter-productivity phenomenon of over-hiring.

It was shown [5] that, when $l = 0$, $u = +\infty$, the maximum size of the company is $\frac{\mu}{2a}$ and the minimum size is $\frac{2\delta}{\mu}$ under an optimal personnel policy. This is consistent with our Case 4: $l < \frac{2\delta}{\mu} < \frac{\mu}{2a} < u$. Namely, when the affordable workforce level is sufficiently high, the higher the debt, the larger the minimum number of employees that the firm must maintain. On the other hand, "too many people may be counter-productive": the less efficient a company (viz. the greater the friction a), the smaller the maximum number the company should keep.

However, since in this paper $U(t)$ is constrained, the optimal policy is largely different from that in the unconstrained case. To start with, if the debt rate δ and/or the personnel friction a are too large (compared with the potential profit) so that $\mu^2 \le 4a\delta$, then Proposition 2 suggests that the best policy would be to "take the money and run"; that is, to declare bankruptcy immediately distributing the whole reserve as dividend. If this is not the case, then the optimal hiring/firing policy is determined by two critical reserve levels: x_1 and x_2.

The values of x_1 and x_2 are in turn dictated by the allowable range of the workforce level (l and u), the debt–profit ratio ($\frac{\delta}{\mu}$) and the profit–friction ratio ($\frac{\mu}{a}$). If the firm has very little debt as well as very little friction compared to the potential profit so that $\frac{2\delta}{\mu} < l < u < \frac{\mu}{2a}$ (Case 1), then both the critical reserve levels, x_1 and x_2, are positive and finite. In this case, the firm will hire the minimum number of employees when the reserve is below x_1, then gradually increase the hiring when the reserve is between x_1 and x_2, and finally take in the maximum number of employees when the reserve reaches or goes beyond level x_2. Clearly, the above policy matches the intuition and common practice.

Personnel policies for other cases can be discussed similarly. We only mention that when $U(t)$ is restricted beyond the desirable interval $[\frac{2\delta}{\mu}, \frac{\mu}{2a}]$ (Cases 5 and 6), then the optimal hiring strategy is to stay as close as possible to this interval. This is again very reasonable and intuitive.

On the other hand, the optimal dividend policy is always of a threshold type with the threshold being equal to x^*. In other words, the reserve should be kept below x^* while distributing any excess as dividends. ¿From our explicit solutions presented earlier we also see that the maximum workforce level is always reached *before* dividend distributions ever take place.

6.2 Preferred re-insurance

In the area of re-insurance, Taksar and Zhou [11] viewed the linear diffusion model (i.e., $a = 0$) as a continuous-time approximation of the well-known Lundeberge–Cramer compound Poisson model for insurance claims. There, $1 - U(t)$ is interpreted as the proportion of the insurance premium to be diverted to the third party (the re-insurer) at t, and other parameters have the same interpretations as those in the forgoing workforce model. There is a natural constraint on $U(t)$, namely, $0 \leq U(t) \leq 1$. Because the risk and profit are hypothesized as linearly dependent, hence the name of the "proportional re-insurance".

Proportional re-insurance is a simple and nice model that captures, to a certain extent, the essence of the complex re-insurance practice. Nevertheless, it is an obvious oversimplification of reality. For example, although risk and return should be highly positively correlated, over-risking is not necessarily the recipe for high return. This leads to questions as to how an optimal strategy would change when the risk and return are not linearly (or, monotonically) dependent on each other. Moreover, while a risk-averse re-insurer may have its preferred risk level and impose additional service charge on firms seeking services beyond the target level, other re-insurer may demand additional charges for those seeking services with risk level lower than its preferred level as an aggressive move to gain market shares. This leads to the consideration of the following equation:

$$dX(t) = [\tilde{\mu}U(t) - a(U(t) - p)^2 - \tilde{\delta}]dt + \sigma U(t)dW(t) - dZ(t), \quad X(0-) = x. \tag{73}$$

Here p is the preferred reinsurance level imposed by the re-insurer and a is the additional rate of charge for the deviation from the preferred level (the quadratic term ensures that larger deviation is penalized more heavily). Note that in this model $U(t)$ is constrained to be $l \leq U(t) \leq u$. Hence, when $u > 1$ the insurance company can take an extra insurance business from other companies (that is act as a reinsurer for other cedents).

We see that (73) is a special case of (1) with $\mu = \tilde{\mu} + 2ap, \delta = \tilde{\delta} + ap^2$. Recasting the main mathematical results obtained in the previous sections, we obtain some new and intriguing insights. First, the general stay-in-business condition $\mu^2 > 4a\delta$ translates into $\tilde{\mu}^2 + 4ap\tilde{\mu} > 4a\tilde{\delta}$. This shows that whether an insurance company should initially run the business at all depends not only on the potential profit $\tilde{\mu}$, but also on the choice of the preferred level p (which, as discussed earlier, is sometime set by its counter-party – the reinsurer). An appropriate choice of p may provide incentive for both parties in the re-insurance business. Second, with an optimal reinsurance policy (which is essentially an optimal risk control policy) the maximal level of the optimal control $U^*(t)$ is $\min\{u, \frac{\tilde{\mu}+2ap}{2a}\} = \min\{u, p + \frac{\tilde{\mu}}{2a}\}$. Hence, if the upper bound $u \leq p + \frac{\tilde{\mu}}{2a}$, then one should stick to this upper bound constraint. However, whenever feasible (i.e., $u > p + \frac{\tilde{\mu}}{2a}$), one should be aggressive instead of risk-averse. In fact, one could exceed p by as much as $\frac{\tilde{\mu}}{2a}$. ¿From an economic point of view, since a surcharge a is imposed by the re-insurer for deviation from the preferred level, it is better to take more risk and (hopefully) get more return, as long as the business is lucrative. This in turn shows that although p may be chosen by a reinsurer, the risk control can be flexibly adjusted by the insurance company according to its individual business situation. This may serve as another incentive for the adoption of the scheme of "re-insurance with risk preference".

Finally, economic interpretation of the explicit optimal policies obtained for the reinsurance model can be elaborated in a way similar to the workforce level. Details are left to the interested readers.

7 Concluding Remarks

In this paper we have studied the optimization problem with a controlled diffusion process and have obtained explicit solutions via solving a Hamilton–Jacobi–Bellman equation. A natural question is what the optimal solution will be if one changes the form of $\mu x - ax^2 - \delta$ into other quadratic forms such as $\mu x - ax^2 + \delta$? This question turns out to be a more complex one than expected. We feel that it is relevant to the fact that when $U(t)$ is unbounded, the value function is singular with respect to δ at 0 (see [5]). Moreover, taking this particular form of $\mu x - ax^2 + \delta$ implies $V(0) > 0$. This means that at $x = 0$ with no endowment one may have non-zero profit and this in some sense is an arbitrage opportunity in the case of reinsurance, hence practically unrealistic and undesirable. However, complete understanding of this particular case as a mathematical problem as well as of its implications in practice remains an interesting open problem.

Also, a choice of the preferred level in the reinsurance model may be settled through negotiations between the cedent and the reinsurer, which is itself an interesting research topic from a game theoretic viewpoint.

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References

[1] S. Asmussen and M. Taksar (1997): Controlled diffusion models for optimal dividend pay-out. *Insurance: Mathematics and Economics* 20: 1–15.

[2] S. Asmussen, B. Hojgaard, and M. Taksar (2000): Optimal risk control and dividend distribution policies: Example of excess of loss reinsurance. *Finance and Stochastics* 4: 299–324.

[3] T. Choulli, M. Taksar and X. Y. Zhou (2003): A diffusion model for optimal dividend distribution for a company with constraints on risk control. *SIAM Journal on Control and Optimization* 41: 1946–1979.

[4] T. Choulli, M. Taksar and X. Y. Zhou (2001): Excess-of-loss reinsurance for a company with debt liability and constraints on risk reduction. *Quantitative Finance* 1:573–596.

[5] X. Guo (2002): A stochastic control problem for companies with internal competitions and debt. *Journal of Applied Probability* 39(1): 55-69.

[6] M. J. Harrison (1985): *Brownian Motion and Stochastic Flow System.* John Wiley and Sons.

[7] M. Jeanblanc-Picqué and A. N. Shiryaev (1995): Optimization of the flow of dividends. *Russian Math. Surveys* 50:2 257–277.

[8] P.-L. Lions and A.S. Sznitman (1984): Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.* 37: 511–537.

[9] R. Radner and L. Shepp (1996): Risk vs. profit potential: A model for corporate strategy. *Journal of Econ. Dynam. Control* 20: 1373–1393.

[10] L. Shepp and A. N. Shiryayev (1996): Hiring and firing optimally in a corporation. *Journal of Econ. Dynam. Control* 20: 1523–1529.

[11] M. Taksar and X. Y. Zhou (1998): Optimal risk and dividend control for a company with debt liability. *Insurance: Mathematics and Economics* 22: 105–122.