Indefinite Stochastic Riccati Equations*

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Abstract

This paper is concerned with stochastic Riccati equations (SREs), which are a class of matrix-valued, nonlinear backward stochastic differential equations (BSDEs). The SREs under consideration are in general indefinite, in the sense that certain parameter matrices are indefinite. This kind of equations arises from the stochastic linear quadratic (LQ) optimal control problem with random coefficients and indefinite state and control weighting costs, the latter having profound implications in both theory and applications. While the solvability of the SREs is the key to solving the indefinite stochastic LQ control, it remains in general an extremely difficult, open problem. This paper attempts to solve the problem of existence and uniqueness of solutions to the indefinite SREs for a number of special, yet important, cases.

Key words– Stochastic Riccati equation, backward stochastic differential equation, stochastic linear–quadratic optimal control.

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1 Introduction

In this paper, we study the following matrix-valued equation, called a stochastic Riccati equation (SRE) on a finite time horizon $[0, T]$ (the time variable t is suppressed):

$$\begin{cases} dP = -\Big\{P\,A + A'\,P + \sum_{j=1}^{k}(\Lambda_j\,C_j + C_j'\,\Lambda_j + C_j'\,P\,C_j) + Q \\ \qquad - \Big[P\,B + \sum_{j=1}^{k}(C_j'P + \Lambda_j)\,D_j\Big]\,K^{-1}\,\Big[B'P + \sum_{j=1}^{k} D_j'\,(PC_j + \Lambda_j)\Big]\Big\}\,dt \\ \qquad + \sum_{j=1}^{k}\Lambda_j\,dW^j, \quad t \in [0,\,T], \\ P(T) = H, \\ K = R + \sum_{j=1}^{k} D_j'\,P\,D_j \;>\; 0. \end{cases} \tag{1}$$

Note that the last (matrix) positive definiteness constraint is *part* of the equation that must be satisfied by any solution. Thus, strictly speaking, this is an equation with mixed equality/inequality constraints. The first two constraints constitute what is known as a nonlinear backward stochastic differential equation (BSDE). The unknown of the SRE is the matrix-valued stochastic process $(P(t), \Lambda_1(t), \cdots, \Lambda_k(t))$ adapted to the filtration $\mathcal{F}_t$ that is generated by the underlying Brownian motion $W(t) = (W^1(t), \cdots, W^k(t))$. The coefficients of this equation, $A(t), B(t), C_j(t), D_j(t), Q(t)$ and $R(t)$, are matrix-valued $\mathcal{F}_t$-adapted stochastic processes, and H is an $\mathcal{F}_T$-measurable random matrix.

The SRE (1) relates intimately to the following optimal linear–quadratic (LQ) control problem with *random* coefficients:
Minimize the quadratic cost functional

$$J(x_0, u(\cdot)) = E\Big\{\int_0^T \Big(x(t)'Q(t)x(t) + u(t)'R(t)u(t)\Big)\,dt + x(T)'Hx(T)\Big\}, \tag{2}$$

subject to the stochastic linear system dynamics

$$\begin{cases} dx(t) = [A(t)\,x(t) + B(t)\,u(t)\,]\,dt \\ \qquad + \sum_{j=1}^{k}[C_j(t)\,x(t) + D_j(t)\,u(t)]\,dW^j(t), \quad t \in [0,\ T] \\ x(0) = x_0. \end{cases} \tag{3}$$

In general, if the SRE (1) admits a solution, then the LQ problem (2)–(3) is solvable, and an optimal control can be represented explicitly in terms of the solution to (1) (see Section 3 for details). Bismut [5] was the first to study the stochastic LQ problem with random coefficients and the associated SRE. Using a fixed-point argument, he proved the existence and uniqueness of solutions to the SRE under the assumptions, (among others), that the random coefficients are independent of the Brownian motion and that R is uniformly positive definite and Q and

H are positive semidefinite. The definiteness assumption is particularly crucial in proving the result, though the assumption *per se* seems rather natural because it had been taken for granted in the stochastic LQ literature, [3, 4], which was presumably inherited from its deterministic counterpart [2]. Note, however, the SRE investigated by Bismut is not really a BSDE in the sense of Pardoux and Peng [17] since the coefficients are independent of, rather than adapted to, the driving Brownian motion. Later, Peng [18] studied the SRE (1) as a nonlinear BSDE, but again under the definiteness condition, i.e., $R > 0, Q \geq 0, H \geq 0$, in addition to a strong assumption that $D = 0$.

Recently, it was found by Chen, Li and Zhou [6] that a stochastic LQ problem where R is possibly *indefinite* may still be solvable as long as the corresponding SRE (1) is solvable. This finding has triggered an extensive research on the so-called *indefinite* LQ control problem, [9, 1, 7, 8, 19]. The problem not only stands out on its own as an interesting theoretical problem, but also has promising applications in practical areas, especially in finance. In the mean–variance portfolio selection [21, 15], or hedging [13], problems, for example, the matrix R is inherently zero, which is a special indefinite case. Moreover, a pollution control model where R is *negative* definite is formulated in [6].

The indefinite stochastic LQ problem leads to an *indefinite* SRE, adding new twists and greater difficulty to the study of the backward SRE, an already hard problem in terms of the existence of its solutions. To make it more precise, in the definite case, i.e., $R > 0, Q \geq 0, H \geq 0$, the third inequality constraint normally becomes redundant which can therefore be eliminated from consideration. In the indefinite case, though, this constraint becomes a hard one that must be taken care of. In [6], a special case when all the coefficients are deterministic (so the equation reduces to an ordinary differential equation) and $C = 0$ is solved. However, even that case is non-trivial due to the indefiniteness of R. Most of the subsequent research has centered around solving the indefinite SRE, analytically or computationally, with deterministic coefficients [9, 1, 13, 19]. For the case with random coefficients a very special indefinite SRE was solved in [15] where P is scalar-valued, $C_j = Q = 0$, R is a zero matrix, and $H = 1$. This special SRE arises from a mean–variance portfolio selection problem for a market with random parameters. The existence of a solution to this special SRE is proved in [15] using an *ad hoc* approach. On the other hand, *local* solvability (i.e., existence of solution in a small neighborhood of the terminal time) of the general SRE (1) is established in [7, 8]. Note, however, that the local solvability is not very useful in view of the LQ control application because the time horizon in a control problem is typically given *a priori*.

The *global* existence of the general, indefinite SRE (1) on the entire time interval $[0, T]$ remains an extremely challenging, if not at all insurmountable, open problem due to the following reasons. First of all, it is a highly nonlinear BSDE, especially in view of the matrix inverse term $(R + D'PD)^{-1}$, for which the normal Lipschitz/linear growth conditions for solvability [17, 16, 20] are not valid. Secondly, the indefiniteness of R makes possible the singularity of the term $R + D'PD$ when one tries to use the typical approximation scheme to construct a solution. Thirdly, the final constraint in (1) must be satisfied by any solution. This is a feature not typically dealt with in the BSDE literature. Finally, (1) is a *matrix* equation. Hence certain terms do not commute which adds substantial difficulty to the analysis.

This paper represents the *first* systematic attempt to tackle the indefinite SRE (1), where Q, R, and H are all allowed to be indefinite. While the results of this paper are still far away from being a complete solution to the solvability of (1) in its greatest generality, we are able to identify some special cases where a global solution is available.

It should be mentioned that, after finishing this work, we have received the preprints [11, 12] by Kohlmann and Tang. In these Peng's result for definite SRE [18] is improved to the case where R is possibly singular, i.e., $R \geq 0, Q \geq 0, H \geq 0$, albeit with various other assumptions. The main approach of [11, 12] is an approximation scheme, which does not apply to the indefinite case due to the possible singularity mentioned above.

The remainder of this paper is organized as follows: In Section 2, we give preliminaries including two known results needed in the subsequent study. In Section 3 we recall the origin of the SRE, namely, the stochastic linear-quadratic optimal control problem, via which we also address the uniqueness of solutions to SRE. Sections 4 and 5 are devoted to the study of one-dimensional SREs and higher-dimensional SREs, respectively. Finally, Section 6 concludes the paper.

2 Preliminaries

We assume throughout that $(\Omega,\, \mathcal{F},\, \{\mathcal{F}_t\}_{t\geq 0},\, P)$ is a given complete, filtered probability space and that $W(\cdot)$ is a k-dimensional standard Brownian motion on this space with $W(0) = 0$. In addition, we assume that $\mathcal{F}_t$ is the augmentation of $\sigma\{W(s)\,|\,0 \leq s \leq t\}$ by all the P-null sets of $\mathcal{F}$.

Throughout this paper, we denote by $\mathbb{R}^{m\times n}$ the set of $m \times n$ real matrices, and by $\mathbb{S}^n$ the set of symmetric $n \times n$ real matrices. If $M = (m_{ij}) \in \mathbb{R}^{m\times n}$, we denote its norm by $||M|| = \sqrt{\sum_{i,j} m_{ij}^2}$. If $M \in \mathbb{S}^n$ is positive (positive semi-) definite, we write $M > (\geq)\ 0$. Suppose $\eta : \Omega \to \mathbb{R}^n$ is an $\mathcal{F}_T$-random variable. We write $\eta \in L^2_{\mathcal{F}_T}(\Omega;\, \mathbb{R}^n)$ if η is square integrable (i.e. $E|\eta|^2 < \infty$), and $\eta \in L^\infty_{\mathcal{F}_T}(\Omega;\, \mathbb{R}^n)$ if η is uniformly bounded. Consider now the case when $f : [0,\, T] \times \Omega \to \mathbb{R}^n$ is an $\{\mathcal{F}_t\}_{t\geq 0}$ adapted process. If $f(\cdot)$ is square integrable (i.e. $E\int_0^T |f(t)|^2\, dt < \infty$) we shall write $f(\cdot) \in L^2_{\mathcal{F}}(0,\, T;\, \mathbb{R}^n)$; if $f(\cdot)$ is uniformly bounded (i.e. ess $\sup_{(t,\omega)\in[0,\,T]\times\Omega} |f(t)| < \infty$) then $f(\cdot) \in L^\infty_{\mathcal{F}}(0,\, T;\, \mathbb{R}^n)$. If $f(\cdot)$ has (P-a.s.) continuous sample paths and $E\, \sup_{t\in[0,\,T]} |f(t)|^2 < \infty$ we write $f(\cdot) \in L^2_{\mathcal{F}}(\Omega;\, C(0,\, T;\, \mathbb{R}^n))$. These definitions generalize in the obvious way to the case when $f(\cdot)$ is $\mathbb{R}^{n\times m}$- or $\mathbb{S}^n$-valued. Finally, we say that $N \in L^2_{\mathcal{F}}(0,\,T;\,\mathbb{S}^n)$ is positive (positive semi-) definite, which is sometimes denoted simply by $N > (\geq)\ 0$, if $N(t,\omega) > (\geq)\ 0$ for a.e. $t \in [0,\,T]$ and P-a.s. ω, and say that N is uniformly positive definite if $N \geq \delta I$ for a.e. $t \in [0,\,T]$ and P-a.s. ω with some given $\delta > 0$.

Definition 2.1 A stochastic process $(P, \Lambda_1, \cdots, \Lambda_k) \in L^2_{\mathcal{F}}(\Omega;\, C(0,\,T;\,\mathbb{S}^n)) \times \big(L^2_{\mathcal{F}}(0,\,T;\,\mathbb{S}^n)\big)^k$ is called a solution to the SRE (1) if it satisfies the first equation of (1) in the Itô sense as well as the second (the terminal condition) and third (the positive definiteness) constraints of (1). A solution $(P, \Lambda_1, \cdots, \Lambda_k)$ of (1) is called bounded if $P \in L^\infty_{\mathcal{F}}(0,\,T;\,\mathbb{S}^n)$.

Throughout this paper, we impose the following assumptions on the parameters of the SRE (1):

Assumption:

(A1)

$$\left\{\begin{array}{rcl} A,\, C_j & \in & L^\infty_{\mathcal{F}}(0,\,T;\mathbb{R}^{n\times n}),\ \ j = 1,2,\cdots,k, \\ B,\, D_j & \in & L^\infty_{\mathcal{F}}(0,\,T;\mathbb{R}^{n\times m}),\ \ j = 1,2,\cdots,k, \\ Q & \in & L^\infty_{\mathcal{F}}(0,\,T;\mathbb{S}^n), \\ R & \in & L^\infty_{\mathcal{F}}(0,\,T;\mathbb{S}^m), \\ H & \in & L^\infty_{\mathcal{F}_T}(\Omega;\,\mathbb{S}^n). \end{array}\right.$$

In particular, we do not assume that $Q \geq 0$, $R > 0$ or $H \geq 0$. In addition, A, B, C_j, D_j, Q, R, H are random. Later, we shall impose other specific assumptions for various special cases considered in this paper.

Next, we collect two known results needed in our subsequent study of indefinite SREs. The first result, due to Kobylanski [10], concerns the existence of solution and comparison theorem for one-dimensional backward stochastic differential equations with quadratic growth.

Let $\alpha_0, \beta_0, b \in \mathbb{R}$, and $c : [0,+\infty) \to [0,+\infty)$ be a continuous increasing function. We say that the coefficient F satisfies condition (H1) with α_0, β_0, b, c, if F is continuous, and for all $(t,y,z,\omega) \in [0,+\infty) \times \mathbb{R} \times \mathbb{R}^k \times \Omega$,

$$F(t,y,z,\omega) = a_0(t,y,z,\omega)y + F_0(t,y,z,\omega),$$

where $a_0(\cdot,y,z,\cdot)$ and $F_0(\cdot,y,z,\cdot)$ are $\{\mathcal{F}_t\}_{t\geq 0}$ adapted processes for fixed $(y,z) \in \mathbb{R}\times\mathbb{R}^k$, and

$$\beta_0 \leq a_0(t,y,z,\omega) \leq \alpha_0,\ \ P-a.s.\omega,$$

$$|F_0(t,y,z,\omega)| \leq b + c(|y|)||z||^2,\ \ P-a.s.\omega.$$

Lemma 2.1 *Let (F,ξ) be a set of parameters of the following BSDE:*

$$Y(t) = \xi + \int_t^T F(s,Y(s),Z(s))ds - \int_t^T Z(s)dW(s), \quad 0 \leq t \leq T. \tag{4}$$

Suppose that the coefficient F satisfies (H1) with α_0, β_0, b, c, and $\xi \in L^\infty_{\mathcal{F}_T}(\Omega)$. Then,

(i) *The BSDE (4) has at least one solution* $(Y, Z) \in [L^\infty_{\mathcal{F}}(0,T;\mathbb{R}) \cap L^2_{\mathcal{F}}(\Omega;\, C(0,\, T;\, \mathbb{R}))] \times L^2_{\mathcal{F}}(0,T;\mathbb{R}^k)$.

(ii) *There exists a maximal solution* (Y^*, Z^*) *of (4) in the following sense: For any set of parameters* (G, η) *where* G *satisfies (H1) with* α_0, β_0, b, c*, if*

$$F \geq G \text{ and } \xi \geq \eta,$$

then for any solution (Y_G, Z_G) *of the BSDE (4) with the parameters* (G, η)*, one must have*

$$Y^* \geq Y_G.$$

The second result, due to Peng [18, Theorem 5.1], is about the unique solvability of the SRE (1) in the definite case (i.e., $Q \geq 0, H \geq 0, R > 0$) where $D = 0$ is additionally assumed.

Lemma 2.2 *The following SRE with* $Q \geq 0, H \geq 0, R > 0$*:*

$$\left\{\begin{array}{rcl} dP &=& -\Big\{PA + A'P + \sum_{j=1}^k(\Lambda_j C_j + C_j'\Lambda_j + C_j' P C_j) + Q - PBR^{-1}B'P\Big\}\,dt \\ && + \sum_{j=1}^k \Lambda_j\, dW^j, \quad t \in [0,\, T], \\ P(T) &=& H, \end{array}\right.$$

admits a unique bounded positive semi-definite solution (P, Λ).

3 Stochastic LQ control and uniqueness of solutions to SRE

In this section we recall the connection between the SRE (1) and the stochastic LQ control problem. A general result of the uniqueness of solutions to SRE will also be addressed via LQ control.

We first recall the formulation of the stochastic LQ control [6]. Let $(\Omega,\, \mathcal{F},\, \{\mathcal{F}_t\}_{t\geq 0},\, P)$ be a complete filtered probability space on which a k-dimensional standard Brownian motion $W(\cdot)$ is defined such that $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration of $W(t)$ augmented by all the P-null sets of $\mathcal{F}$. For any given $(s, \xi) \in [0, T) \times L^2_{\mathcal{F}_s}(\Omega; \mathbb{R}^n)$, consider the following linear SDE on $[s, T]$:

$$\left\{\begin{array}{rcl} dx(t) &=& [A(t)\,x(t) + B(t)\,u(t)\,]\,dt \\ && + \sum_{j=1}^k [C_j(t)\,x(t) + D_j(t)\,u(t)]\,dW^j(t), \quad t \in [s,\, T] \\ x(s) &=& \xi, \end{array}\right. \tag{5}$$

where A, B, C_j and D_j are the same parameters as appearing in the SRE (1). The class of *admissible controls* is the set $\mathcal{U} = L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$. If $u(\cdot) \in \mathcal{U}$ and $x(\cdot)$ is the associated solution of (5), then we refer to $(x(\cdot), u(\cdot))$ as an *admissible pair.*

Suppose that the cost functional is given by

$$J(s, \xi; u(\cdot)) = E\Big\{ \int_0^T \Big(x(t)'Q(t)x(t) + u(t)'R(t)u(t) \Big)\, dt + x(T)'Hx(T) \Big| \mathcal{F}_s \Big\}. \tag{6}$$

Again, $Q, R,$ and H are the same as given in (1). To summarize, the *stochastic LQ control problem* associated with (5)-(6) is as follows:

$$\begin{cases} \min J(s, \xi; u(\cdot)), \\ \text{subject to: } (x(\cdot),\, u(\cdot)) \text{ admissible for (5).} \end{cases} \tag{7}$$

The problem (7) is said to be *solvable* if for any $(s, \xi) \in [0, T) \times L^2_{\mathcal{F}_s}(\Omega; \mathbb{R}^n)$ there exists a control $u^*(\cdot) \in \mathcal{U}$ such that

$$-\infty < J(s, \xi; u^*(\cdot)) \le J(s, \xi; u(\cdot)), \quad P - a.s.\omega, \quad \forall\, u(\cdot) \in \mathcal{U}.$$

In this case, the control $u^*(\cdot)$ is referred to as an *optimal control.*

Suppose $(P, \Lambda_1, \cdots, \Lambda_k)$ is a solution to (1). We introduce the following (forward) SDE (the argument t is suppressed):

$$\begin{cases} dx &= \Big[A - BK^{-1}(B'P + \sum_{j=1}^k D_j'(P\,C_j + \Lambda_j))\Big] x\, dt \\ & \quad + \sum_{j=1}^k \Big[C_j - D_jK^{-1}(B'P + \sum_{j=1}^k D_j'(P\,C_j + \Lambda_j))\Big] x\, dW^j, \quad t \in [s, T], \\ x(s) &= \xi. \end{cases} \tag{8}$$

Lemma 3.1 *Assume that* **(A1)** *holds. Suppose that the BSDE (1) has a solution* $(P, \Lambda) \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^{n\times n}) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n\times n})^k$*, where* $\Lambda = (\Lambda_1, \cdots, \Lambda_k)$*, and the SDE (8) has a solution* $x(\cdot) \in L^2_{\mathcal{F}}(\Omega;\, C(0, T; \mathbb{R}^n))$*. Then the stochastic LQ problem (7) is solvable. A (unique) optimal feedback control is*

$$u^*(t) = -K(t)^{-1} \Big[\Big(B(t)'P(t) + \sum_{j=1}^k D_j(t)'(P(t)\, C_j(t) + \Lambda_j(t)) \Big)\, x(t) \Big] \tag{9}$$

where $K(t) = R(t) + \sum_{j=1}^k D_j(t)'\, P(t)\, D_j(t)$*, and the associated optimal cost is:*

$$\inf_{u(\cdot)\in\mathcal{U}} J(s, \xi; u(\cdot)) \;=\; \xi'\, P(s)\, \xi, \;\; P - a.s.\omega. \tag{10}$$

Proof: This result has been proved in [6, Theorem 3.1] along with [6, Remark 3.1]. The only minor difference is that in [6] the initial state ξ is deterministic, but the argument there works for the case when ξ is $\mathcal{F}_s$-measurable, since ξ is almost surely deterministic under the the probability measure $P(\cdot|\mathcal{F}_s)$. ■

Theorem 3.1 *Assume that* **(A1)** *holds. If* (1) *has two solutions* (P^i, Λ^i), $i = 1, 2$, *such that the corresponding SDE (8) has solutions* $x_i(\cdot) \in L^2_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R}^n))$, $i = 1, 2$. *Then* $(P^1(s), \Lambda^1(s)) = (P^2(s), \Lambda^2(s))$, *P-a.s.*$\omega$, *a.e.*$s \in [0, T]$.

Proof: By Lemma 3.1, $\xi' P_1(s)\xi = \xi' P_2(s)\xi$, $\forall (s, \xi) \in [0, T) \times L^2_{\mathcal{F}_s}(\Omega; \mathbb{R}^n)$. Therefore $P^1(s) = P^2(s)$, P-a.s.ω, $\forall s \in [0, T]$. Now applying Itô's formula to $|P^1(s) - P^2(s)|^2$ and using their respective equations, we get

$$0 \equiv d|P^1(s) - P^2(s)|^2 = 2(P^1(s) - P^2(s))d(P^1(s) - P^2(s)) + |\Lambda^1(s) - \Lambda^2(s)|^2 ds = |\Lambda^1(s) - \Lambda^2(s)|^2 ds.$$

Hence $\Lambda^1(s) = \Lambda^2(s)$, P-a.s.$\omega$, a.e.$s \in [0, T]$. ■

Remark 3.1 Theorem 3.1 essentially gives the uniqueness to the SRE (1) among such solutions that (8) admits solutions. As stipulated in [6, Remark 3.1] the solvability of (8) depends on some moment estimates of its coefficients and may be available in some special cases.

4 Existence: The case when $n = 1$

In this section, we consider the case when $n = 1$ (therefore the unknown P of the SRE (1) is a scalar). However, it is still allowed that $m > 1, k > 1$. Note that this situation is typically encountered in many financial problems, where the state variable is the wealth which is scalar-valued; see, e.g., [21, 13, 15].

When $n = 1$, the equation (1) reduces to

$$\begin{cases} dP = -\Big\{(2A + \sum_{j=1}^k C_j^2)P + 2\sum_{j=1}^k C_j\Lambda_j + Q \\ \qquad - \Big[(B + \sum_{j=1}^k C_j D_j)P + \sum_{j=1}^k D_j\Lambda_j\Big] K^{-1} \Big[(B' + \sum_{j=1}^k C_j D_j')P + \sum_{j=1}^k D_j'\Lambda_j\Big]\Big\}dt \\ \qquad + \sum_{j=1}^k \Lambda_j dW^j, \quad t \in [0, T], \\ P(T) = H, \\ K = R + P\sum_{j=1}^k D_j' D_j > 0. \end{cases} \tag{11}$$

Let us set:

$\alpha = 2A + \sum_{j=1}^k C_j^2$, $\beta \equiv (\beta_1, \beta_2, \cdots, \beta_k)' = (2C_1, 2C_2, \cdots, 2C_k)'$, $\Gamma = B + \sum_{j=1}^k C_j D_j$, $D = (D_1', D_2', \cdots, D_k')'$, $\Lambda = (\Lambda_1, \Lambda_2, \cdots, \Lambda_k)$.

Then, the above equation can be rewritten as

$$\begin{cases} dP = -\Big\{\alpha P + \Lambda\beta + Q - (\Gamma P + \Lambda D)(R + PD'D)^{-1}(\Gamma P + \Lambda D)'\Big\}dt \\ \qquad + \Lambda dW, t \in [0, T], \\ P(T) = H, \\ R + PD'D > 0. \end{cases} \tag{12}$$

4.1 Standard Case

The standard case (i.e. $H \geq 0, R > 0, Q \geq 0$) can be treated as an application of Lemma 2.1. In fact, we are going to study this case for a generalized version of the equation (12), which in turn will be useful for the case when R is possibly indefinite (see Section 4.3 below). To be specific, we add a parameter Δ, which is an m-dimensional-row-vector-valued, essentially bounded $\mathcal{F}_t$-adapted process, in (12):

$$\left\{\begin{array}{rcl} dP & = & -\Big\{\alpha P + \Lambda\beta + Q \\ & & -(\Gamma P + \Lambda D + \Delta)(R + PD'D)^{-1}(\Gamma P + \Lambda D + \Delta)'\Big\}dt \\ & & +\Lambda dW, t \in [0,T], \\ P(T) & = & H, \\ R + PD'D & > & 0. \end{array}\right. \tag{13}$$

Theorem 4.1 *Assume that $H \geq 0, R > 0,$ $D'D > 0,$ $Q - \Delta R^{-1}\Delta' \geq 0,$ $R^{-1} \in L^\infty_{\mathcal{F}}(0,\, T; \mathbb{R}^{m\times m})$ and $(D'D)^{-1} \in L^\infty_{\mathcal{F}}(0,\, T; \mathbb{R}^{m\times m})$. Then equation (13) admits a bounded nonnegative solution. In particular, the Riccati equation (12) admits a bounded nonnegative solution if $H \geq 0, R > 0,$ $D'D > 0,$ $Q \geq 0,$ $R^{-1} \in L^\infty_{\mathcal{F}}(0,\, T; \mathbb{R}^{m\times m})$ and $(D'D)^{-1} \in L^\infty_{\mathcal{F}}(0,\, T; \mathbb{R}^{m\times m})$.*

Proof: Let us consider the following equation:

$$\left\{\begin{array}{rcl} dP & = & -F(t, P, \Lambda)dt + \Lambda dW, \quad t \in [0,T], \\ P(T) & = & H, \end{array}\right. \tag{14}$$

where

$$F(t, P, \Lambda) = \alpha(t)P + \Lambda\beta(t) + Q(t) - (\Gamma(t)P^+ + \Lambda D(t) + \Delta(t))(R + P^+D'D)^{-1}(\Gamma(t)P^+ + \Lambda D(t) + \Delta(t))'$$

In the above, $P^+ = \max(P, 0)$. Noting the following inequalities (by virtue of the assumptions that $R > 0$ and $D'D > 0$):

$$||(R + P^+D'D)^{-1}|| \leq ||R^{-1}||, \;\; ||(R + P^+D'D)^{-1}|| \leq \frac{||(D'D)^{-1}||}{|P^+|} \;\; (\text{for } P \neq 0),$$

we can easily check that the equation (14) satisfies the assumption of Lemma 2.1. Hence it admits a bounded maximal solution (P, Λ). Moreover, we have (t is suppressed)

$$\begin{array}{rcl} F(t, P, \Lambda) & = & \alpha P + \Lambda\beta + Q - 2\Delta(R + P^+D'D)^{-1}(\Gamma P^+ + \Lambda D)' \\ & & +\Delta[R^{-1} - (R + P^+D'D)^{-1}]\Delta' - (\Gamma P^+ + \Lambda D)(R + P^+D'D)^{-1}(\Gamma P^+ + \Lambda D)' \\ & & +Q - \Delta R^{-1}\Delta'. \end{array}$$

On the other hand, the following BSDE

$$\begin{cases} dP &= -\Big\{\alpha P + \Lambda\beta - 2\Delta(R + P^+D'D)^{-1}(\Gamma P^+ + \Lambda D)' \\ &\quad +\Delta[R^{-1} - (R + P^+D'D)^{-1}]\Delta' - (\Gamma P^+ + \Lambda D)(R + P^+D'D\ )^{-1}(\Gamma P^+ + \Lambda D)'\Big\}dt \\ &\quad +\Lambda dW, t \in [0,T], \\ P(T) &= 0 \end{cases}$$

satisfies the assumption of Lemma 2.1 and admits a bounded solution $(0, 0)$. Applying Lemma 2.1-(ii) to (14), we deduce that $P \geq 0$. Hence, P is also a bounded nonnegative solution of (13). The assertion regarding the original equation (12) is straightforward. ■

4.2 Case When $R = 0$

The case when $R = 0$ is studied in [15] under the additional assumptions that $C = 0$ (i.e., $\beta = 0$) and $Q = 0$, which arises naturally in a mean–variance portfolio selection problem. Now we discuss this case without those additional assumptions.

When $R = 0$, the equation (12) specializes to

$$\begin{cases} dP &= -\Big\{\alpha P + \Lambda\beta + Q - \frac{1}{P}(\Gamma P + \Lambda D)(D'D)^{-1}(\Gamma P + \Lambda D)'\Big\}dt + \Lambda dW, t \in [0,T], \\ P(T) &= H, \\ P &> 0. \end{cases} \tag{15}$$

The key idea is that if (15) has a bounded solution, then, assuming that $H > 0, H^{-1} \in L^\infty(\Omega, \mathcal{F}_T, P; \mathbb{R})$ and $D'D > 0, (D'D)^{-1} \in L^\infty_\mathcal{F}(0, T; \mathbb{R}^{m\times m})$, the process $(Y, Z) = (\frac{1}{P}, -\frac{\Lambda}{P^2})$ should satisfy the following equation (by virtue of the Itô formula):

$$\begin{cases} dY &= -\Big\{[\Gamma(D'D)^{-1}\Gamma' - \alpha]Y + Z[\beta - 2D(D'D)^{-1}\Gamma] - QY^2 - \frac{Z[D(D'D)^{-1}D' - I]Z'}{Y}\Big\}dt \\ &\quad +ZdW, t \in [0,T], \\ Y(T) &= \frac{1}{H}. \end{cases} \tag{16}$$

Theorem 4.2 *Suppose that $H > 0$, $H^{-1} \in L^\infty(\Omega, \mathcal{F}_T, P; \mathbb{R})$, $Q \geq 0$, $m = k$, and that $D'D > 0, (D'D)^{-1} \in L^\infty_\mathcal{F}(0, T; \mathbb{R}^{m\times m})$. Then (15) admits a unique bounded, uniformly positive solution.*

Proof: Since $m = k$ and D is invertible, the last term of the drift coefficient of the equation (16) vanishes. Thus (16) is a type of SRE (1) for which Lemma 2.2 applies. So it has a unique bounded nonnegative solution. Rewrite this equation as

$$\begin{cases} dY &= -\Big\{[\Gamma(D'D)^{-1}\Gamma' - \alpha - QY]Y + Z[\beta - 2D(D'D)^{-1}\Gamma]\Big\}dt + ZdW, \quad t \in [0,T], \\ Y(T) &= \frac{1}{H}. \end{cases} \tag{17}$$

Put

$$\tilde{\alpha} = \Gamma(D'D)^{-1}\Gamma' - \alpha - QY, \; \tilde{\beta} = \beta - 2D(D'D)^{-1}, \; \tilde{W}_t = W_t - \int_0^t \tilde{\beta}(s)ds.$$

Then from the well-known Girsanov theorem, $\tilde{W}$ is a Brownian motion under a certain probability measure $\tilde{P}$. The above equation now becomes:

$$\begin{cases} dY &= -\tilde{\alpha}Ydt + Zd\tilde{W}, \quad t \in [0,T], \\ Y(T) &= \frac{1}{H}. \end{cases}$$

Hence,

$$Y(t) = \tilde{E}(\frac{1}{H}e^{\int_t^T \tilde{\alpha}_s ds}|\mathcal{F}_t) \geq \frac{1}{||H||_\infty}e^{-||\tilde{\alpha}||_\infty(T-t)} \geq \frac{1}{||H||_\infty}e^{-||\tilde{\alpha}||_\infty T} > 0.$$

Thus we can set

$$(P,\Lambda) = (\frac{1}{Y}, -\frac{Z}{Y^2}).$$

Applying Itô's formula, we deduce easily that (P,Λ) is a bounded solution of (15) with P uniformly positive. Finally, the uniqueness follows from that of (17). ∎

Remark 4.1 Under the assumption that $D'D$ is invertible, it is necessary that $m \leq k$ ($m = \text{rank}(D'D) \leq \text{rank } D \leq k$). When $m < k$, it means that, in the context of the stochastic control that leads to the SRE (1), the number of the independent controllable directions is less than that of the independent random sources. In the special case of a stochastic market, $m < k$ implies that the number of the stocks available for selection is less than that of the independent random sources that constitute the market or, in other words, the market is incomplete. Hence, assuming $m = k$ (and $D'D$ is invertible) really stipulates that we are in the realm of the complete market in the context of the finance problem. On the other hand, when $m < k$, then (16) suggests that one should handle the last term of its drift coefficient (in particular, the matrix-valued process $I - D(D'D)^{-1}D'$ which is only nonnegative with possible zero eigenvalues).

Remark 4.2 In [11], similar results to Theorems 4.1 and 4.2 are derived using approximation technique which are quite involved. Here we provide completely different yet much simpler proofs. It should be emphasized, however, that our ultimate objective is to prove the existence for some indefinite SREs, for which Theorems 4.1 and 4.2 will be utilized. See next subsection for details.

4.3 Case when R is indefinite

The case with an indefinite R is more complicated. Unfortunately we are able to only treat the case when $m = k = 1$. In this case, the equation (12) further simplifies to

$$\begin{cases} dP = -\Big\{\alpha P + \Lambda\beta + Q - \frac{(\Gamma P+\Lambda D)^2}{R+D^2P}\Big\}dt + \Lambda dW, \quad t \in [0,T], \\ P(T) = H, \\ R + D^2 P > 0, \end{cases} \tag{18}$$

where

$$\alpha = 2A + C^2, \ \beta = 2C, \ \Gamma = B + CD.$$

In this subsection we give two sets of sufficient conditions that guarantee the solvability of the Riccati equation (18).

Theorem 4.3 *Assume that* $\Gamma = 0, D \neq 0, Q - \frac{\alpha R}{D^2} \geq 0, \frac{R}{D^2} + H > 0,$ *and* $(\frac{R}{D^2} + H)^{-1} \in L^\infty(\Omega, \mathcal{F}_T, P; \mathbb{R})$. *Moreover, assume that* $\frac{R}{D^2}$ *is constant. Then the Riccati equation (14) admits a unique bounded solution.*

Proof: Let us consider the following BSDE:

$$\begin{cases} dY = -\Big\{-\alpha Y + \beta Z - (Q - \frac{\alpha R}{D^2})Y^2\Big\}dt + ZdW, \quad t \in [0,T], \\ Y(T) = \frac{D^2}{R+HD^2}. \end{cases}$$

As in the proof of Theorem 4.2, this equation has a unique bounded solution by virtue of the assumptions. Moreover, we can prove in the same manner that there exists a constant $\delta > 0$ such that $Y \geq \delta$. Now, we set

$$(P, \Lambda) = (\frac{1}{Y} - \frac{R}{D^2}, -\frac{Z}{Y^2}).$$

Applying Itô's formula and noting that $\frac{R}{D^2}$ is constant, we deduce easily that (P, Λ) is a solution of (14), which is bounded. The uniqueness comes from the inverse procedure. ∎

Remark 4.3 From the above result we can see that the SRE may admit a solution even when *both* Q and R are negative (in the context of the stochastic LQ control, this amounts to saying that an LQ control may be solvable even when both the running state and control costs are negative.). To see this, take an example where all the assumptions of Theorem 4.3 are satisfied. Moreover, let $\alpha = 2A + C^2 > 0$. Then we see that the critical condition $Q - \frac{\alpha R}{D^2} \geq 0$ may be satisfied even when both Q and R are negative (but then H must be positive and large enough).

In the above result it is assumed that $\Gamma \equiv B + CD = 0$ which is rather strict. The next theorem replaces this condition by others.

Theorem 4.4 *Assume that there exists a constant $\epsilon > 0$ such that*

$$|D| > 0, \quad H + \frac{R}{D^2} \geq \epsilon, \quad Q + \alpha(\epsilon - \frac{R}{D^2}) - \frac{\Gamma^2(\epsilon - \frac{R}{D^2})^2}{D^2\epsilon} \geq 0. \tag{19}$$

Moreover, assume that $\frac{R}{D^2}$ is constant, and $D^{-1} \in L^{\infty}_{\mathcal{F}}(0, T; \mathbb{R})$. Then there exists a bounded solution for the Riccati equation (18).

Proof: Consider the following Riccati equation:

$$\begin{cases} dY &= -\Big\{\alpha Y + \beta Z + Q + \alpha(\epsilon - \frac{R}{D^2}) - \frac{[\Gamma Y + \Lambda D + \Gamma(\epsilon - \frac{R}{D^2})]^2}{D^2\epsilon + D^2 Y}\Big\}dt \\ & \quad + ZdW, t \in [0, T], \\ Y(T) &= H + \frac{R}{D^2} - \epsilon. \end{cases}$$

This equation is of the same type as (13) with $\Delta = \Gamma(\epsilon - \frac{R}{D^2})$. The assumption of Theorem 4.1 is satisfied due to (19). Thus by Theorem 4.1 the above equation admits a bounded nonnegative solution $(Y, Z) \in L^{\infty}_{\mathcal{F}}(0, T; \mathbb{R}) \times L^{2}_{\mathcal{F}}(0, T; \mathbb{R})$.

We set

$$(P, \Lambda) = (Y + \epsilon - \frac{R}{D^2}, Z).$$

It is straightforward to verify that (P, Λ) is a bounded solution of the Riccati equation (14). Moreover, since $\frac{R}{D^2}$ is a constant, the solution $P = Y + \epsilon - \frac{R}{D^2}$ is bounded. ∎

Remark 4.4 Condition (19) gives the overall requirement for the coefficients of the SRE in order for it to admit a solution. The positiveness/nonnegativeness of *individual* coefficients is no longer required.

Example 4.1 Let us take an example to illustrate Theorem 4.4. Take $H \equiv 1$, $D \equiv 1$, and R being negative with $1 + R \geq \epsilon$. In this case, the conditions (19) specialize to

$$Q + (\epsilon - R)(\alpha - \Gamma^2 + \frac{R\Gamma^2}{\epsilon}) \geq 0. \tag{20}$$

Now if $\alpha \equiv 2A + C^2 \leq \Gamma^2 \equiv (B + C)^2$, then Q has to be positive in order for the above inequality holds. However, if $\alpha > \Gamma^2$, then Q can also be negative while still satisfying (20). Indeed, as long as $\alpha > \frac{\Gamma^2}{\epsilon}$ one can show that $(\epsilon - R)(\alpha - \Gamma^2 + \frac{R\Gamma^2}{\epsilon}) > 0$. Hence there is some room for Q to be negative.

5 Existence: The case when $n > 1$

In this section, we will investigate the case when the unknown P is a matrix. For technical reasons, we assume that $k = 1$ and $m = n$. Note that even in the standard case (i.e. $Q \geq 0$, $H \geq 0$, $R > 0$), the solvability of the Riccati equation remains generally unsolved. In this section we shall study two cases where $R = 0$ and R is indefinite respectively.

5.1 Case when $R = 0$

In this subsection, we treat the case when $R = 0$. We need to assume in addition that $C = 0$.

First we have the following technical lemma.

Lemma 5.1 *We have the following assertions.*

(i) *If X is a square matrix, then $X + X' + 2||X||I \geq 0$.*

(ii) *If X is a positive definite matrix, then $X^{-1} - ||X||^{-1}I \geq 0$.*

Proof: (i) For any column vector x of appropriate size, we have by the Cauchy-Schwartz inequality:

$$-(x'Xx + x'X'x) \leq |x'Xx| + |x'X'x| \leq 2||x||^2||X||,$$

or equivalently

$$x'(X + X' + 2||X||I)x \geq 0, \quad \forall x.$$

This proves the claim.

(ii) For any column vector x of appropriate size, again by the Cauchy-Schwartz inequality:

$$(X^{-\frac{1}{2}}x)'X(X^{-\frac{1}{2}}x) \leq ||X||||X^{-\frac{1}{2}}x||^2 = ||X||(x'X^{-1}x),$$

or equivalently

$$x'(X^{-1} - ||X||^{-1}I)x \geq 0, \quad \forall x.$$

This proves the claim. ∎

Theorem 5.1 *Suppose that $Q \geq 0$, $H > 0$, $H^{-1} \in L^{\infty}_{\mathcal{F}_T}(\Omega; \mathbb{S}^n)$, D is nonsingular and $D^{-1} \in L^{\infty}_{\mathcal{F}}(0, T; \mathbb{R}^{n\times n})$. Then the Riccati equation (21) admits a unique bounded, uniformly positive definite solution.*

Proof: Let us first assume that $D \equiv I$. Then the Riccati equation becomes:

$$\begin{cases} dP = -\Big\{PA + A'P + Q - (PB + \Lambda)P^{-1}(B'P + \Lambda)\Big\}dt + \Lambda dW, t \in [0, T], \\ P(T) = H, \\ P > 0. \end{cases} \tag{21}$$

Consider the following BSDE:

$$\begin{cases} dY &= -\Big\{-AY - YA' - BZ - ZB' + BYB' - YQY\Big\}dt + ZdW, \quad t \in [0,T], \\ Y(T) &= H^{-1}. \end{cases}$$

This equation admits a unique bounded positive semidefinite solution according to Lemma 2.2. Rewrite the equation as

$$\begin{cases} dY &= -\Big\{Y\tilde{A} + \tilde{A}'Y + \tilde{C}'Y\tilde{C} + Z\tilde{C} + \tilde{C}'Z\Big\}dt + ZdW, t \in [0,T], \\ Y(T) &= H^{-1}, \end{cases} \tag{22}$$

where $\tilde{A} = -A' - \frac{1}{2}QY$ and $\tilde{C} = -B'$.

We need to prove that $Y > 0$. To this end, introduce

$$\bar{Y} = ||H||_\infty^{-1} e^{-2(T-t)||\tilde{A}||_\infty} I, \quad \bar{Z} = 0,$$

where

$$||\tilde{A}||_\infty = \text{esssup}_{(t,\omega)} ||\tilde{A}(t,\omega)||, \quad ||H||_\infty = \text{esssup}_\omega ||H(\omega)||.$$

Then $(\bar{Y}, \bar{Z})$ is a bounded solution of the following BSDE:

$$\begin{cases} d\bar{Y} &= -\Big\{-2||\tilde{A}||_\infty \bar{Y} + \bar{Z}\tilde{C} + \tilde{C}'\bar{Z}\Big\}dt + \bar{Z}dW, \quad t \in [0,T], \\ \bar{Y}(T) &= ||H||_\infty^{-1} I. \end{cases}$$

Now we put

$$\hat{Y} = Y - \bar{Y}, \hat{Z} = Z - \bar{Z}.$$

Then $(\hat{Y}, \hat{Z})$ satisfies the following BSDE:

$$\begin{cases} d\hat{Y} &= -\Big\{\hat{Y}\tilde{A} + \tilde{A}'\hat{Y} + \hat{Z}\tilde{C} + \tilde{C}'\hat{Z} + \tilde{C}'\hat{Y}\tilde{C} \\ &\quad +\bar{Y}(\tilde{A} + \tilde{A}' + 2||\tilde{A}||_\infty I)\Big\}dt + \hat{Z}dW, \quad t \in [0,T], \\ \hat{Y}(T) &= H^{-1} - ||H||_\infty^{-1} I. \end{cases}$$

By Lemma 5.1, we have

$$\tilde{A}(t,\omega) + \tilde{A}'(t,\omega) + 2||\tilde{A}||_\infty I \geq 0, \quad \forall (t,\omega),$$

and

$$H(\omega)^{-1} - ||H||_\infty^{-1} I \geq 0, \quad \forall \omega.$$

It follows then from Lemma 2.2 that

$$\hat{Y} \geq 0,$$

namely,

$$Y \geq \bar{Y} > 0.$$

Now set

$$(P, \Lambda) = (Y^{-1}, -Y^{-1}ZY^{-1}).$$

It is easy to check, via the Itô formula, that (P, Λ) is a bounded, uniformly positive definite solution of (21). Again, the uniqueness follows from the inverse procedure.

Now, for a general nonsingular D, the Riccati equation can be rewritten as:

$$\begin{cases} dP &= -\Big\{PA + A'P + Q - (PBD^{-1} + \Lambda)P^{-1}((D^{-1})'B'P + \Lambda)\Big\}dt + \Lambda dW, t \in [0,T], \\ P(T) &= H, \\ P &> 0. \end{cases}$$

This is in the same form of (21). The proof is completed. ■

5.2 Case when $B = 0, C = 0$

In this subsection we study another special case when $B = 0, C = 0$. We also need to assume that R is a constant matrix. However, R is allowed to be *indefinite*.

Theorem 5.2 *Assume that* $Q - RA - A'R \geq 0$*,* $R + H > 0$*,* $(R + H)^{-1} \in L^{\infty}(\Omega, \mathcal{F}_T, P; \mathbb{S}^n)$*,* D *is nonsingular and* $D^{-1} \in L^{\infty}_{\mathcal{F}}(0, T; \mathbb{R}^{n\times n})$*. Then the Riccati equation (23) admits a unique bounded solution.*

Proof: First let us take $D = I$. Then the Riccati equation under consideration is:

$$\begin{cases} dP &= -\Big\{PA + A'P + Q - \Lambda(R + P)^{-1}\Lambda\Big\}dt + \Lambda dW, \quad t \in [0,T], \\ P(T) &= H, \\ R + P &> 0. \end{cases} \tag{23}$$

Consider the following BSDE:

$$\begin{cases} dY &= -\Big\{ - AY - YA' - Y(Q - RA - A'R)Y\Big\}dt + ZdW, \quad t \in [0,T], \\ Y(T) &= (R + H)^{-1}. \end{cases} \tag{24}$$

By Lemma 2.2, this equation admits a unique bounded solution $(Y, Z) \in L^\infty_{\mathcal{F}}(0, T; \mathbb{S}^n) \times L^2_{\mathcal{F}}(0, T; \mathbb{S}^n)$. As in the proof of the preceding theorem, there exists a constant $c > 0$, such that $Y \geq cI$. Hence we may set $(P, \Lambda) = (Y^{-1} - R, -Y^{-1}ZY^{-1})$. It is easy to verify via the Itô formula that (P, Λ) is a bounded solution of (23). On the other hand, the uniqueness comes from the inverse procedure.

Now for a general nonsingular D, the Riccati equation can be rewritten as

$$\begin{cases} dP = -\Big\{PA + A'P + Q - \Lambda((D^{-1})'RD^{-1} + P)^{-1}\Lambda\Big\}dt + \Lambda dW, \quad t \in [0, T], \\ P(T) = H, \\ (D^{-1})'RD^{-1} + P > 0. \end{cases}$$

This is in the same form of (23), which completes the proof. ■

Remark 5.1 In [12] the unique solvability of a *definite* SRE is proved under the assumptions (among others) that $B = C = 0$ and R is uniformly positive definite.

6 Concluding remarks

In this paper we have investigated the stochastic Riccati equation (1) with random coefficients where the matrix-valued stochastic processes Q, R and H are possibly indefinite. The existence of its solution is a prerequisite for solving the corresponding indefinite stochastic linear quadratic control problem, which in turn has important applications in many applied areas especially in finance. Here we have identified several special cases where the existence is proved. The general global existence remains an extremely challenging open problem.

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