Constrained Stochastic LQ Control with Random Coefficients, and Application to Portfolio Selection*

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Abstract

This paper is devoted to the study of a stochastic linear–quadratic (LQ) optimal control problem where the control variable is constrained in a cone, and all the coefficients of the problem are random processes. Employing Tanaka's formula, optimal control and optimal cost are explicitly obtained via solutions to two extended stochastic Riccati equations (ESREs). The ESREs, introduced for the first time in this paper, are highly nonlinear backward stochastic differential equations (BSDEs), whose solvability is proved based on a truncation function technique and Kobylanski's results. The general results obtained are then applied to a mean–variance portfolio selection problem for a financial market with random appreciation and volatility rates, and with short-selling prohibited. Feasibility of the problem is characterized, and efficient portfolios and efficient frontier are presented in closed forms.

Key words. Stochastic LQ control, extended stochastic Riccati equation, backward stochastic differential equation, mean–variance portfolio selection, efficient portfolio, efficient frontier.

Abbreviated title. Constrained Stochastic LQ Control and Application

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1 Introduction

Linear–quadratic (LQ) optimal control is a problem where the system dynamics are linear in state and control variables and the cost functional is quadratic in the two variables. It is a classical yet fundamental problem in control theory, pioneered by Kalman [11] (for deterministic control). Extension to stochastic LQ control was first carried out by Wonham [25]. Bismut [3] performed a detailed analysis for stochastic LQ control with random coefficients. With the joint effort of many researchers in the last 40 years, there has been an enormously rich theory on LQ control, deterministic and stochastic alike. Recently, starting with Chen, Li and Zhou [6], there has been emerging interest in the so-called *indefinite* stochastic LQ control, where, quite contrary to the conventional belief, the cost weighting matrices are allowed to be indefinite; see [7, 8, 1, 26]. This new theory turns out to be useful in solving the continuous-time version of Markowitz's Nobel-winning mean–variance portfolio selection model; see [28, 14, 16, 18, 13, 17]. For systematic accounts of the deterministic and stochastic LQ theory, refer to [2] and [27] respectively.

One of the elegant features of the LQ theory is that it is able to give in explicit forms the optimal state feedback control and the optimal cost value through the celebrated Riccati equation; hence the LQ control problem is completely solved. What essentially enables this closed-form solution, besides the special linear–quadratic structure, is that the control is *not* constrained. Specifically, since the control is unconstrained, the feedback control constructed via the Riccati equation is *automatically* admissible. If, on the other hand, there are pointwise control constraints, then the whole LQ approach would collapse.

One should acknowledge that LQ control with control constraints is a well-posed problem which is important in both theory and applications. For example, in many real applications the control variable is required to take only non-negative values. The mean–variance portfolio selection problem with no-shorting constraint, which is to be tackled in this paper, is exactly one of such problems. There were some attempts in attacking the deterministic LQ problems with positive controls; see for example [23, 5, 9]. In these works, however, only some implicit necessary and sufficient conditions for optimality were derived and some numerical schemes suggested, and the special LQ structure was not fully taken advantage of and no explicit result was obtained. On the other hand, to our best knowledge research on pointwisely constrained stochastic LQ control has been completely absent in the literature.

The main purpose of this paper is to tackle a stochastic LQ control problem where the control variable is constrained in a cone (which, certainly, includes the non-negative orthant in a Euclidean space as a special case), and all the coefficient matrices of the model are random. Moreover, the problem is allowed to be "singular" in the sense that the control weighting matrix in the cost functional is possibly singular. However, we are able to treat only the case when the state variable is scalar-valued, although it is sufficient to cover many meaningful practical applications, in particular in financial area where the one-dimensional wealth process is typically taken as the state. We aim to obtain *explicit* solutions comparable to the classical unconstrained-control counterpart. To this end, we introduce two equations termed extended stochastic Riccati equations (ESREs). These two equations are highly nonlinear backward stochastic differential equations (BSDEs), the solvability of which is interesting in its own right. Based on a truncation function technique and a delicate result of Kobylanski [12], we are able to prove the existence of solutions to the introduced ESREs. Then, applying Tanaka's formula and going through some detailed analysis, we obtain explicit optimal feedback control as well as the optimal cost value in terms of the solutions to the two ESREs.

The other purpose of the paper is to solve the continuous-time mean–variance portfolio selection model with short selling prohibition and random market parameters. Indeed, this is the very problem that motivated us to tackle the general constrained stochastic LQ problem. Notice that a mean–portfolio selection model with no-shorting was solved in [16] using Hamilton–Jacobi–Bellman equation and viscosity solution theory; however among other additional assumptions all the market parameters are assumed to be deterministic in [16]. The partial differential equation approach there does not extend to the current case with random parameters. To overcome this difficulty, we reformulate the problem so that it falls exactly into the general constrained LQ problem that has been solved. Hence, by applying the general results obtained we are able to solve the portfolio selection problem, once again, explicitly and completely.

There are large number of papers devoted to applying stochastic control theory for mean–variance efficient/hedging portfolio selection models; see, to name a few recent ones, [14, 21, 13, 18, 17, 22, 4]. While some of these works are on more general asset price models (such as semimartingale ones) and/or markets (including incomplete markets), none of them deals with directly constrained portfolios. As a result, the value functions there remain quadratic, which is no longer the case, as will be demonstrated in this paper, with the conic constrained portfolios.

The rest of the paper is organized as follows. In Section 2 we formulate the constrained stochastic LQ model, and in Section 3 we present some mathematical preliminaries including Tanaka's formula and introduction of the two ESREs. Section 4 is devoted to the solvability of the ESREs in two common cases. Section 5 gives the solution to the LQ problem. Application of the general results to a mean–variance portfolio selection problem is presented in Section 6. Finally, Section 7 concludes the paper by some remarks.

2 Problem Formulation

We assume throughout that $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_t\}_{t\ge 0},\ P)$ is a given complete, filtered probability space and that $W(\cdot)$ is a k-dimensional standard Brownian motion on this space with $W(0)=0$. In addition, we assume that $\mathcal{F}_t$ is the augmentation of $\sigma\{W(s)\,|\,0\le s\le t\}$ by all the P-null sets of $\mathcal{F}$. When no confusion would occur, we omit to write P-a.s. for a statement that holds almost surely with respect to P.

Throughout this paper, we denote by $\mathbb{R}^m$ the set of m-dimensional *column* vectors, by $\mathbb{R}^m_+$ the set of m-dimensional column vectors whose components are nonnegative, by $\mathbb{R}^{m\times n}$ the set of $m\times n$ real matrices, and by $\mathbb{S}^n$ the set of symmetric $n\times n$ real matrices. Therefore, $\mathbb{R}^m\equiv\mathbb{R}^{m\times 1}$. If $M=(m_{ij})\in\mathbb{R}^{m\times n}$, we denote its transpose by M', and its norm by $|M|=\sqrt{\sum_{i,j}m_{ij}^2}$. If $M\in\mathbb{S}^n$ is positive (positive semi-) definite, we write $M>(\ge)\ 0$. Suppose $\eta:\Omega\to\mathbb{R}^n$ is a $\mathcal{G}$-measurable random variable. We write $\eta\in L^2_{\mathcal{G}}(\Omega;\ \mathbb{R}^n)$ if η is square integrable (i.e. $E|\eta|^2<\infty$), and $\eta\in L^\infty_{\mathcal{G}}(\Omega;\ \mathbb{R}^n)$ if η is uniformly bounded. Consider now the case when $f:[0,\ T]\times\Omega\to\mathbb{R}^n$ is an $\{\mathcal{F}_t\}_{t\ge 0}$-adapted process. If $f(\cdot)$ is square integrable (i.e. $E\int_0^T|f(t)|^2\,dt<\infty$) we shall write $f(\cdot)\in L^2_{\mathcal{F}}(0,\ T;\ \mathbb{R}^n)$; if $f(\cdot)$ is uniformly bounded (i.e. ess $\sup_{(t,\omega)\in[0,\ T]\times\Omega}|f(t)|<\infty$) then $f(\cdot)\in L^\infty_{\mathcal{F}}(0,\ T;\ \mathbb{R}^n)$. If $f(\cdot)$ has (P-a.s.) continuous sample paths and $E\sup_{t\in[0,\ T]}|f(t)|^2<\infty$ we write $f(\cdot)\in L^2_{\mathcal{F}}(\Omega;\ C(0,\ T;\ \mathbb{R}^n))$. These definitions generalize in the obvious way to the case when $f(\cdot)$ is $\mathbb{R}^{n\times m}$- or $\mathbb{S}^n$-valued. In addition, we say that $N\in L^2_{\mathcal{F}}(0,\ T;\ \mathbb{S}^n)$ is positive (positive semi-) definite, which is sometimes denoted simply by $N>(\ge)\ 0$, if $N(t,\omega)>(\ge)\ 0$ for a.e. $t\in[0,\ T]$ and P-a.s., and say that N is uniformly positive definite if $N\ge cI_n$ for a.e. $t\in[0,\ T]$ and P-a.s. with some given deterministic constant $c>0$, where I_n is the n-dimensional identity matrix.

Finally, for any real number we define $x^+:=\max\{x,0\}$ and $x^-:=\max\{-x,0\}$.

Consider the following linear SDE:

$$\begin{cases} dx(t) &= [A(t)\,x(t)+B(t)\,u(t)]\,dt+[x(t)C(t)'+u(t)'D(t)']\,dW(t),\quad t\in[0,\ T],\\ x(0) &= x_0,\end{cases}\tag{1}$$

where A, B, C and D are $\{\mathcal{F}_t\}_{t\ge 0}$-adapted processes (possibly matrix-valued), and $x_0\in\mathbb{R}$ is a non-random scalar. Precise assumptions on this data will be specified below. Let $\Gamma\subset\mathbb{R}^m$ be a given closed cone; i.e., Γ is closed, and if $u\in\Gamma$ then $\alpha u\in\Gamma\ \forall\alpha\ge 0$. Typical examples of such a cone are $\Gamma=\mathbb{R}^m_+$, $\Gamma=\{u\in\mathbb{R}^m\mid Mu\le 0\}$ and $\Gamma=\{u\in\mathbb{R}^m\mid Mu=0\}$ where $M\in\mathbb{R}^{n\times m}$. The class of *admissible controls* is the set

$$\mathcal{U}:=\Big\{u(\cdot)\in L^2_{\mathcal{F}}(0,\ T;\ \mathbb{R}^m)\ \Big|\ u(t)\in\Gamma,\ P-\text{a.s., a.e. } t\in[0,T],\ \text{and (1) has a unique solution under } u(\cdot)\Big\}.$$

If $u(\cdot)\in\mathcal{U}$ and $x(\cdot)$ is the associated solution of (1), then we refer to $(x(\cdot),\ u(\cdot))$ as an *admissible pair.*

Suppose that the cost functional is given by

$$J(x_0, u(\cdot)) := E\Big\{ \int_0^T \Big[Q(t)x(t)^2 + u(t)'R(t)u(t) \Big] \, dt + Gx(T)^2 \Big\}. \tag{2}$$

Throughout this paper, we shall assume the following:

Assumption (A1):

$$\begin{cases} A,\, Q & \in & L^\infty_{\mathcal{F}}(0,\, T; \mathbb{R}), \\ B & \in & L^\infty_{\mathcal{F}}(0,\, T; \mathbb{R}^{1\times m}), \\ C & \in & L^\infty_{\mathcal{F}}(0,\, T; \mathbb{R}^k), \\ D & \in & L^\infty_{\mathcal{F}}(0,\, T; \mathbb{R}^{k\times m}), \\ R & \in & L^\infty_{\mathcal{F}}(0,\, T; \mathbb{S}^m), \\ G & \in & L^\infty_{\mathcal{F}_T}(\Omega; \, \mathbb{R}). \end{cases}$$

Note that all the parameters involved may be random. Also, by standard SDE theory, equation (1) admits a unique solution $x(\cdot) \in L^2_{\mathcal{F}}(\Omega; \, C(0, \, T; \, \mathbb{R}))$ for any $u(\cdot) \in L^2_{\mathcal{F}}(0, \, T; \, \mathbb{R}^m)$ under Assumption (A1). The *stochastic LQ problem* associated with (1)–(2) is as follows:

$$\begin{cases} \text{Minimize} & J(x_0, u(\cdot)), \\ \text{subject to} & (x(\cdot),\, u(\cdot)) \text{ admissible for (1).} \end{cases} \tag{3}$$

The problem (3) is said to be *finite* (w.r.t. x_0) if there exists some finite constant $K \in \mathbb{R}$ such that

$$J(x_0, u(\cdot)) \geq K, \quad \forall\, u(\cdot) \in \mathcal{U},$$

and *solvable* (w.r.t. x_0) if there exists a control $u^*(\cdot) \in \mathcal{U}$ such that

$$J(x_0, u^*(\cdot)) \leq J(x_0, u(\cdot)), \quad \forall\, u(\cdot) \in \mathcal{U}.$$

In this case, the control $u^*(\cdot)$ is referred to as the *optimal control* (w.r.t. x_0). We say that (3) is *uniquely solvable* if it is solvable and the optimal control is unique. Note that a finite LQ problem is not necessarily solvable.

3 Preliminaries

In this section we present some mathematical preliminaries required in the sequel, including in particular Tanaka's formula which plays a critical technical role in the subsequent analysis.

Lemma 3.1 (**Tanaka's formula**) *Let $X(t)$ be a continuous semimartingale. Then*

$$\begin{aligned} dX^+(t) &= 1_{(X(t)>0)} dX(t) + \tfrac{1}{2} dL(t), \\ dX^-(t) &= -1_{(X(t)\leq 0)} dX(t) + \tfrac{1}{2} dL(t), \end{aligned} \tag{4}$$

where $L(\cdot)$ is an increasing continuous process, called the local time of $X(\cdot)$ at 0, satisfying

$$\int_0^t |X(s)|dL(s) = 0, \quad P - a.s. \tag{5}$$

Proof: See, for example, [24, Chapter VI, Theorem 1.2 and Proposition 1.3]. ■

Next, define the following mappings:

$$\begin{aligned} H_+(t,\omega,v,P,\Lambda) := \;& v'[R(t,\omega)+PD(t,\omega)'D(t,\omega)]v \\ & +2v'[B(t,\omega)'P+D(t,\omega)'PC(t,\omega)+D(t,\omega)'\Lambda], \\ H_-(t,\omega,v,P,\Lambda) := \;& v'[R(t,\omega)+PD(t,\omega)'D(t,\omega)]v \\ & -2v'[B(t,\omega)'P+D(t,\omega)'PC(t,\omega)+D(t,\omega)'\Lambda], \\ & (t,\omega,v,P,\Lambda) \in [0,T]\times\Omega\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^k, \end{aligned} \tag{6}$$

and

$$\begin{aligned} &H_+^*(t,\omega,P,\Lambda) := \inf_{v\in\Gamma} H_+(t,\omega,v,P,\Lambda), \\ &H_-^*(t,\omega,P,\Lambda) := \inf_{v\in\Gamma} H_-(t,\omega,v,P,\Lambda), \quad (t,\omega,P,\Lambda)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^k. \end{aligned} \tag{7}$$

Remark 3.1 $H_+^*(t,\omega,P,\Lambda)$ and $H_-^*(t,\omega,P,\Lambda)$ have finite values if $R(t,\omega)+PD(t,\omega)'D(t,\omega) > 0$. Indeed, in this case, there exists $C_1(t,\omega,P,\Lambda)>0$, $C_2(t,\omega,P,\Lambda)>0$ such that

$$H_+(t,\omega,v,P,\Lambda) \geq C_1|v|^2 - C_2|v| = C_1|v|(|v| - \frac{C_2}{C_1}).$$

If $|v| > \frac{C_2}{C_1}$, then $H_+ > 0$. Recall that $0\in\Gamma$, hence $\inf_{v\in\Gamma} H_+(t,\omega,v,P,\Lambda) \leq 0$. These facts imply

$$H_+^*(t,\omega,P,\Lambda) = \inf_{v\in\Gamma, |v|\leq\frac{C_2}{C_1}} H_+(t,\omega,v,P,\Lambda) \geq \min_{|v|\leq\frac{C_2}{C_1}} H_+(t,\omega,v,P,\Lambda) > -\infty.$$

Hence, $\inf_{v\in\Gamma} H_+(t,\omega,v,P,\Lambda)$ is finite. The same is true for H_-.

Now we introduce the following two nonlinear backward stochastic differential equations – BSDEs (the arguments t and ω are suppressed):

$$\begin{cases} dP_+ = -\Big\{(2A+C'C)P_+ + 2C'\Lambda_+ + Q + H_+^*(P_+,\Lambda_+)\Big\}dt + \Lambda_+' dW, \quad t\in[0,\,T], \\ P_+(T) = G, \\ R + P_+D'D > 0, \end{cases} \tag{8}$$

$$\begin{cases} dP_- = -\Big\{(2A+C'C)P_- + 2C'\Lambda_- + Q + H_-^*(P_-,\Lambda_-)\Big\}dt + \Lambda_-' dW, \quad t\in[0,\,T], \\ P_-(T) = G, \\ R + P_-D'D > 0. \end{cases} \tag{9}$$

The equations (8) and (9) are referred to as the *extended stochastic Riccati equations* (ESREs). The following gives a precise definition of their solutions.

Definition 3.1 A stochastic process $(P_+, \Lambda_+) \in L^2_{\mathcal{F}}(\Omega;\, C(0,\, T;\, \mathbb{R})) \times L^2_{\mathcal{F}}(0,\, T;\, \mathbb{R}^k)$ is called a *solution* to the ESRE (8) if it satisfies the first equation of (8) in the Itô sense as well as the second (the terminal condition) and third (the positive definiteness) constraints of (8). A solution (P_+, Λ_+) of (8) is called *bounded* if $P_+ \in L^\infty_{\mathcal{F}}(0,\, T;\, \mathbb{R})$, called *positive* (respectively *nonnegative*) is $P_+(t) > 0$ (respectively $P_+(t) \geq 0$) for all $t \in [0,\, T]$ and P-a.s., and called *uniformly positive* if $P_+(t) \geq c$ for all $t \in [0,\, T]$ and P-a.s. with some deterministic constant $c > 0$. Similar terms can be defined for the other ESRE (9).

4 Existence of solution to the ESREs

As the existence of solution to the ESREs (8) and (9) is essential to solving the underlying stochastic LQ problem, we devote this section to this issue. Note that the existence problem is interesting in its own right from BSDE point of view, for both (8) and (9) are nonlinear BSDEs that do not satisfy the standard assumptions for existence.

We will deal with the following two cases.

Standard case. $Q \geq 0$, $R > 0$ with $R^{-1} \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^{m\times m})$, and $G \geq 0$.

Singular case. $Q \geq 0, R \geq 0, G > 0$ with $G^{-1} \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R})$, and $D'D > 0$ with $(D'D)^{-1} \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^{m\times m})$.

4.1 Standard Case

In this subsection we solve the standard case.

Theorem 4.1 *For the standard case, there exists a bounded, nonnegative solution* (P_+, Λ_+) *(respectively* (P_-, Λ_-) *) to the ESRE (8) (respectively (9)).*

Proof: Let us first consider the following BSDE:

$$\begin{cases} dP_1 = -\Big\{(2A + C'C)P_1 + 2C'\Lambda_1 + Q\Big\}dt + \Lambda_1' dW, \quad t \in [0,\, T], \\ P_1(T) = G. \end{cases} \tag{10}$$

This is a linear BSDE with bounded coefficients (by virtue of Assumption (A1)), and with $Q \geq 0$ and $G \geq 0$. Hence there exists a unique nonnegative, bounded solution (P_1, Λ_1). Denote by $c_1 > 0$ an upper bound of P_1. Now, consider the following BSDE:

$$\begin{cases} dP = -F_1(t, P, \Lambda)dt + \Lambda' dW, \quad t \in [0,\, T], \\ P(T) = G, \end{cases} \tag{11}$$

where the function F_1 is defined by

$$\begin{aligned} F_1(t, \omega, P, \Lambda) := \quad & [2A(t,\omega) + C(t,\omega)'C(t,\omega)]P + 2C(t,\omega)'\Lambda + Q(t,\omega) + H_+^*(t,\omega,P^+,\Lambda)g_1(P^+), \\ & (t, \omega, P, \Lambda) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^k \end{aligned}$$

(recall that P^+ denotes the positive part of P), whereas $g_1 : \mathbb{R}^+ \to [0,1]$ is a smooth truncation function satisfying $g_1(x) = 1$ for $x \in [0, c_1]$, and $g_1(x) = 0$ for $x \in [2c_1, +\infty)$.

The function F_1 is continuous in (P, Λ). To see this, for a given $n \in \mathbb{N}$ (the set of positive integers), if $|P| \le n, |\Lambda| \le n$, then by the assumption on R there exist two constants $C_1 > 0$ and $C_2(n) > 0$, such that

$$H_+(t, v, P^+, \Lambda) \ge C_1|v|^2 - C_2(n)|v|.$$

Thus, for $|P| \le n, |\Lambda| \le n$, the argument in Remark 3.1 results in

$$H_+^*(t, P^+, \Lambda) = \min_{v \in \Gamma, |v| \le \frac{C_2(n)}{C_1}} H_+(t, v, P^+, \Lambda).$$

This implies that F_1 is continuous in (P, Λ).

On the other hand, there exists a $C_2 > 0$ such that

$$H_+(t, v, P^+, \Lambda) \ge C_1|v|^2 - C_2(P^+ + |\Lambda|)|v|,$$

which yields

$$0 \ge H_+^*(t, P^+, \Lambda) \ge -\frac{C_2^2(P^+ + |\Lambda|)^2}{4C_1}.$$

Hence, F_1 satisfies the hypothesis (H1) of Kobylanski [12] noting the role of the truncation function g_1. According to [12, Theorem 2.3], there is a bounded, maximal solution (see [12, p. 565] for its definition) (P_+, Λ_+) to the BSDE (11). Now, as $H_+^*(t, P, \Lambda) \le 0$ and (P_1, Λ_1) is the only, hence maximal, bounded solution to (10), we get $P_+ \le P_1 \le c_1$. Furthermore, that $G \ge 0$, $Q \ge 0$ and $H_+^*(t, P^+, \Lambda) \ge -\frac{C_2^2(P^+ + |\Lambda|)^2}{4C_1}$ implies $P_+ \ge 0$, using the facts that (P_+, Λ_+) is the bounded, maximal solution to (11) and that (0,0) is an obvious solution to (11) with $G = 0$, $Q = 0$, and $H_+^*(t, P^+, \Lambda)g_1(P^+)$ replaced by $-\frac{C_2^2(P^+ + |\Lambda|)^2}{4C_1}g_1(P^+)$. This proves that (P_+, Λ_+) is a bounded nonnegative solution of the ESRE (8). The same argument concludes also the existence of solution to the ESRE (9).

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4.2 Singular Case

The singular case is the one that will be used in the financial application in the second half of the paper.

Theorem 4.2 *For the singular case, there exists a bounded, uniformly positive solution* (P_+, Λ_+) *(respectively* (P_-, Λ_-)*) to the ESRE (8) (respectively (9)).*

Proof Let us first consider the following BSDE (the argument t is suppressed)

$$\begin{cases} dP_2 = -[(2A + C'C)P_2 + 2C'\Lambda_2 + H^*(P_2, \Lambda_2)]dt + \Lambda_2' dW, \quad t \in [0,\, T], \\ P_2(T) = G, \\ P_2 > 0, \end{cases} \tag{12}$$

where

$$H^*(t, P, \Lambda) \quad := \quad -[PB + (C'P + \Lambda')D]P^{-1}(D'D)^{-1}[B'P + D'(PC + \Lambda)].$$

This is a BSDE studied independently in [13] and [10], using different approaches. The existence of its solution was proved in [13] with a rather involved proof, and was proved in [10] with a short proof, nonetheless for the case $k = m$. For completeness, we will prove the existence of solution to this BSDE using a simpler method in the lemma following the end of the proof of this theorem. By this lemma, there exists a unique bounded, uniformly positive solution (P_2, Λ_2). In particular, there exists a constant $c_2 > 0$ such that $P_2(t) \geq c_2, \ \ \forall t \in [0, T]$, P-a.s..

Now, let us consider the following BSDE:

$$\begin{cases} dP = -F_2(t, P, \Lambda)dt + \Lambda' dW, \quad t \in [0,\, T], \\ P(T) = G, \end{cases} \tag{13}$$

where

$$\begin{aligned} F_2(t, \omega, P, \Lambda) := \quad & [2A(t,\omega) + C(t,\omega)'C(t,\omega)]P + 2C(t,\omega)'\Lambda + Q(t,\omega) + H_+^*(t,\omega,P,\Lambda)g_2(P^+), \\ & (t, \omega, P, \Lambda) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^k \end{aligned}$$

with $g_2 : \mathbb{R}^+ \to [0,1]$ being another smooth truncation function satisfying $g_2(x) = 0$, for $x \in [0, \frac{1}{2}c_2]$, and $g_2(x) = 1$ for $x \in [c_2, +\infty)$. As with the proof for the standard case we can show that the function F_2 is continuous in (P, Λ) and satisfies the assumption of [12]. Thus, there exists a bounded, maximal solution of the BSDE (13), denoted as (P_+, Λ_+).

Notice the following inequality

$$\begin{aligned} H_+^*(t, P, \Lambda) \geq \ & \inf\nolimits_{v \in \mathbb{R}^m} H_+(t, v, P, \Lambda) \\ \geq \ & \inf\nolimits_{v \in \mathbb{R}^m} \{v'PD(t)'D(t)v + 2v'[B(t)'P + D(t)'PC(t) + D(t)'\Lambda]\} \\ = \ & H^*(t, P, \Lambda). \end{aligned}$$

Hence, noting $Q \geq 0$, the maximal solution argument gives

$$P_+(t) \geq P_2(t) \geq c_2, \ \ \forall t \in [0, T].$$

This implies that (P_+, Λ_+) is actually a bounded, uniformly positive solution of the ESRE (8). The same argument also leads to the existence of solution to the ESRE (9).

■

Lemma 4.1 *Equation (12) has a bounded, uniformly positive solution* (P_2, Λ_2).

Proof: Set

$$\alpha := 2A + C'C - (B + C'D)(D'D)^{-1}(B' + D'C),\ \beta := 2C - 2D(D'D)^{-1}(B' + D'C).$$

Then, equation (12) can be rewritten as

$$\begin{cases} dP_2 = -[\alpha P_2 + \beta'\Lambda_2 - \frac{1}{P_2}\Lambda_2' D(D'D)^{-1}D'\Lambda_2]dt + \Lambda_2' dW, \quad t \in [0, T], \\ P_2(T) = G, \\ P_2 > 0. \end{cases} \tag{14}$$

Let $c_3 > 0$ and $c_4 > 0$ be two constants satisfying $G \geq c_3$ and $|\alpha| \leq c_4$, and set

$$c_2 := c_3 e^{-c_4 T}.$$

Now, consider the following BSDE:

$$\begin{cases} dP = -[\alpha P^+ + \beta'\Lambda - \frac{1}{P}\Lambda' D(D'D)^{-1}D'\Lambda g_2(P)]dt + \Lambda' dW, \quad t \in [0, T], \\ P(T) = G, \end{cases} \tag{15}$$

where g_2 is the truncation function defined in the proof of Theorem 4.2 corresponding to the constant c_2. According again to [12, Theorem 2.3], there exists a bounded, maximal solution to this BSDE denoted as (P_2, Λ_2).

Finally, the following BSDE:

$$\begin{cases} dP = -[-c_4 P^+ + \beta'\Lambda - \frac{1}{P}\Lambda' D(D'D)^{-1}D'\Lambda g_2(P)]dt + \Lambda' dW, \quad t \in [0, T], \\ P(T) = c_3, \end{cases} \tag{16}$$

has an obvious solution $(c_3 e^{-c_4(T-t)}, 0)$. As (P_2, Λ_2) is a maximal solution to (15), we deduce that $P_2(t) \geq c_3 e^{-c_4(T-t)} \geq c_3 e^{-c_4 T} = c_2$. This implies that (P_2, Λ_2) is also a bounded, uniformly positive solution to (14), hence to (12). ■

Remark 4.1 When there is no control constraint, i.e., when $\Gamma = \mathbb{R}^m$, then

$$H_+^*(t, P, \Lambda) = H_-^*(t, P, \Lambda) = -[PB + (C'P + \Lambda')D](R + PD'D)^{-1}[B'P + D'(PC + \Lambda)],$$

provided $R + PD'D > 0$. Hence both the ESREs (8) and (9) reduce to the normal stochastic Riccati equation

$$\begin{cases} dP = -\Big\{(2A + C'C)P + 2C'\Lambda + Q \\ \qquad -[PB + (C'P + \Lambda')D](R + PD'D)^{-1}[B'P + D'(PC + \Lambda)]\Big\}dt + \Lambda' dW, \quad t \in [0, T], \\ P(T) = G, \\ R + PD'D > 0. \end{cases} \tag{17}$$

The above equation has been studied in great details in [13, 10]. Moreover, in [10] the case when R is possibly indefinite was investigated.

Remark 4.2 The uniqueness of solutions to (8) and (9) will be proved in the next section, interestingly, as a direct consequence of the solution to the LQ problem. It should be noted that in [12], the uniqueness of solutions is proved under the additional assumption that the generator (i.e., the drift coefficient) is differentiable which is not satisfied here.

5 Solution to the LQ problem

In this section we give explicit solution to the LQ problem (1)–(2) in terms of the solutions to the two ESREs, for both the standard and singular cases defined in the previous section.

First, when $R + PD'D > 0$, define

$$\begin{aligned} &\xi_+(t, \omega, P, \Lambda) := \mathrm{argmin}_{v\in\Gamma} H_+(t, \omega, v, P, \Lambda), \\ &\xi_-(t, \omega, P, \Lambda) := \mathrm{argmin}_{v\in\Gamma} H_-(t, \omega, v, P, \Lambda), \quad (t, \omega, P, \Lambda) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^k. \end{aligned} \tag{18}$$

Note that the minimizers above are achievable due to a similar argument in Remark 3.1 and the assumption that Γ is closed.

Theorem 5.1 *In both the standard and singular cases, let* $(P_+, \Lambda_+) \in L^2_{\mathcal{F}}(\Omega;\, C(0, T; \mathbb{R})) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^k)$ *and* $(P_-, \Lambda_-) \in L^2_{\mathcal{F}}(\Omega;\, C(0, T; \mathbb{R})) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^k)$ *be bounded, nonnegative (in the standard case) or bounded, uniformly positive (in the singular case) solutions to the ESREs (8) and (9), respectively. Then the following state feedback control*

$$u^*(t) = \xi_+(t, P_+(t), \Lambda_+(t))x^+(t) + \xi_-(t, P_-(t), \Lambda_-(t))x^-(t) \tag{19}$$

is optimal for the problem (1)–(2). Moreover, in this case the optimal cost is:

$$J^*(x_0) := \inf_{u(\cdot)\in\mathcal{U}} J(x_0, u(\cdot)) = P_+(0)(x_0^+)^2 + P_-(0)(x_0^-)^2. \tag{20}$$

Proof: First note that Theorems 4.1 and 4.2 ensure that (8) and (9) admit bounded, nonnegative (in the standard case) or bounded, uniformly positive (in the singular case) solutions $(P_+, \Lambda_+) \in L^2_{\mathcal{F}}(\Omega;\, C(0, T; \mathbb{R})) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^k)$ and $(P_-, \Lambda_-) \in L^2_{\mathcal{F}}(\Omega;\, C(0, T; \mathbb{R})) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^k)$ respectively. Let $x(\cdot)$ be the solution of (1) under an arbitrary given admissible control $u(\cdot)$. By Tanaka's formula (Lemma 3.1), we obtain:

$$dx^+(t) = 1_{(x(t)>0)}[A(t)x(t)+B(t)u(t)]dt+1_{(x(t)>0)}[x(t)C(t)'+u(t)'D(t)']dW(t)+\frac{1}{2}dL(t), \tag{21}$$

where $L(\cdot)$ is the local time of $x(\cdot)$ at 0 as specified in Lemma 3.1. Applying Ito's formula to the above, we get

$$
\begin{aligned}
& dx^+(t)^2 \\
= \;& 2x^+(t)\Big\{1_{(x(t)>0)}[A(t)x(t)+B(t)u(t)]dt + 1_{(x(t)>0)}[x(t)C(t)'+u(t)'D(t)']dW(t) \\
& +\tfrac{1}{2}dL(t)\Big\} + 1_{(x(t)>0)}[x(t)C(t)'+u(t)'D(t)'][C(t)x(t)+D(t)u(t)]dt \\
= \;& \Big\{2A(t)x^+(t)^2 + 2u(t)'B(t)'x^+(t) + 1_{(x(t)>0)}[x(t)C(t)'+u(t)'D(t)'][C(t)x(t)+D(t)u(t)]\Big\}dt \\
& +2x^+(t)[x(t)C(t)'+u(t)'D(t)']dW(t),
\end{aligned} \tag{22}
$$

where we have used the fact that $x^+(t)dL(t)=0$ by virtue of (5). Using Ito's formula again to (8) and (22), and writing

$$
\Theta_+(t) := -\left\{[2A(t)+C(t)'C(t)]P_+(t) + 2C(t)'\Lambda_+(t) + Q(t) + H_+^*(t,P_+(t),\Lambda_+(t))\right\}, \tag{23}
$$

we have (after some reorganization)

$$
\begin{aligned}
& d[P_+(t)x^+(t)^2] \\
= \;& \Big\{u(t)'[1_{(x(t)>0)}P_+(t)D(t)'D(t)]u(t) + 2u(t)'[P_+(t)B(t)' + P_+(t)D(t)'C(t) + D(t)'\Lambda_+(t)]x^+(t) \\
& +[\Theta_+(t) + (2A(t)+C(t)'C(t))P_+(t) + 2C(t)'\Lambda_+(t)]x^+(t)^2\Big\} \\
& +\Big\{2P_+(t)x^+(t)[x(t)C(t)'+u(t)'D(t)'] + x^+(t)^2\Lambda_+(t)'\Big\}dW(t).
\end{aligned} \tag{24}
$$

Similarly, we can derive

$$
\begin{aligned}
& d[P_-(t)x^-(t)^2] \\
= \;& \Big\{u(t)'[1_{(x(t)\le 0)}P_-(t)D(t)'D(t)]u(t) - 2u(t)'[P_-(t)B(t)' + P_-(t)D(t)'C(t) + D(t)'\Lambda_-(t)]x^-(t) \\
& +[\Theta_-(t) + (2A(t)+C(t)'C(t))P_-(t) + 2C(t)'\Lambda_-(t)]x^-(t)^2\Big\} \\
& +\Big\{-2P_-(t)x^-(t)[x(t)C(t)'+u(t)'D(t)'] + x^-(t)^2\Lambda_-(t)'\Big\}dW(t),
\end{aligned} \tag{25}
$$

where

$$
\Theta_-(t) := -\left\{[2A(t)+C(t)'C(t)]P_-(t) + 2C(t)'\Lambda_-(t) + Q(t) + H_-^*(t,P_-(t),\Lambda_-(t))\right\}. \tag{26}
$$

Next, we define, for $n \ge 1$, the following stopping time τ_n:

$$
\begin{aligned}
\tau_n := \;& \inf\Big\{t\ge 0 \Big| \; \textstyle\int_0^t \{|2P_+(s)x^+(s)[x(s)C(s)'+u(s)'D(s)'] + x^+(s)^2\Lambda_+(s)'|^2\}ds \\
& + \textstyle\int_0^t \{|-2P_-(s)x^-(s)[x(s)C(s)'+u(s)'D(s)'] + x^-(s)^2\Lambda_-(s)'|^2\}ds \ge n\Big\} \wedge T,
\end{aligned} \tag{27}
$$

where $\inf\emptyset := T$. Obviously, $\tau_n,\ n\ge 1$, is an increasing sequence of stopping times converging to T almost surely.

Summing (24) and (25), taking integration from 0 to τ_n and then taking expectation, we have (t is suppressed)

$$
E\Big\{P_+(\tau_n)x^+(\tau_n)^2 + P_-(\tau_n)x^-(\tau_n)^2\Big\} + E\Big\{\int_0^{\tau_n}[Q(t)x(t)^2 + u(t)'R(t)u(t)]dt\Big\}
$$

$$
\begin{aligned}
&= P_+(0)(x_0^+)^2 + P_-(0)(x_0^-)^2 \\
&\quad + E\int_0^{\tau_n} \Big\{ u'(R + 1_{(x(t)>0)}P_+D'D + 1_{(x(t)\le 0)}P_-D'D)u \\
&\quad + 2u'(P_+B' + P_+D'C + D'\Lambda_+)x^+ - 2u'(P_-B' + P_-D'C + D'\Lambda_-)x^- \\
&\quad + [\Theta_+ + (2A + C'C)P_+ + 2C'\Lambda_+ + Q](x^+)^2 \\
&\quad + [\Theta_- + (2A + C'C)P_- + 2C'\Lambda_- + Q](x^-)^2 \Big\} dt \\
&= P_+(0)(x_0^+)^2 + P_-(0)(x_0^-)^2 \\
&\quad + E\int_0^{\tau_n} \Big\{ u'(R + 1_{(x(t)>0)}P_+D'D + 1_{(x(t)\le 0)}P_-D'D)u \\
&\quad + 2u'(P_+B' + P_+D'C + D'\Lambda_+)x^+ - 2u'(P_-B' + P_-D'C + D'\Lambda_-)x^- \\
&\quad - H_+^*(P_+, \Lambda_+)(x^+)^2 - H_-^*(P_-, \Lambda_-)(x^-)^2 \Big\} dt. \qquad (28)
\end{aligned}
$$

Let us now send $n \to \infty$. Then by noting that $x(\cdot) \in L^2_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R}))$ we get, from the dominated convergence theorem, that,

$$
J(x_0, u(\cdot)) = P_+(0)(x_0^+)^2 + P_-(0)(x_0^-)^2 + E\int_0^T \varphi(x(t), u(t))dt, \qquad (29)
$$

where $\varphi(x(t), u(t))$ denotes the integrand on the right hand side of (28).

Now, we are to show that $\varphi(x(t), u(t)) \ge 0$ for any $t \in [0, T]$. Indeed, if $x(t) > 0$ for some t, then set $u(t) = x(t)v(t)$. Notice $u(t) \in \Gamma$ if and only if $v(t) \in \Gamma$ since Γ is a cone. Then (again t is suppressed)

$$
\begin{aligned}
\varphi(x, u) &= u'(R + P_+D'D)u + 2u'(P_+B' + P_+D'C + D'\Lambda_+)x - H_+^*(P_+, \Lambda_+)x^2 \\
&= [v'(R + P_+D'D)v + 2v'(P_+B' + P_+D'C + D'\Lambda_+)]x^2 - H_+^*(P_+, \Lambda_+)x^2 \qquad (30) \\
&\ge H_+^*(P_+, \Lambda_+)x^2 - H_+^*(P_+, \Lambda_+)x^2 = 0.
\end{aligned}
$$

Moreover, the inequality becomes equality when $u^*(t) = x(t)v^*(t) = x^+(t)\xi_+(t, P_+(t), \Lambda_+(t)) \in \Gamma$. Next, if $x(t) < 0$ for some t, then put $u(t) = -x(t)v(t)$. In this case,

$$
\begin{aligned}
\varphi(x, u) &= u'(R + P_-D'D)u + 2u'(P_-B' + P_-D'C + D'\Lambda_-)x - H_-^*(P_-, \Lambda_-)x^2 \\
&= [v'(R + P_+D'D)v - 2v'(P_-B' + P_-D'C + D'\Lambda_-)]x^2 - H_-^*(P_-, \Lambda_-)x^2 \qquad (31) \\
&\ge H_-^*(P_-, \Lambda_-)x^2 - H_-^*(P_-, \Lambda_-)x^2 = 0,
\end{aligned}
$$

where the equality holds at $u^*(t) = -x(t)v^*(t) = x^-(t)\xi_-(t, P_-(t), \Lambda_-(t)) \in \Gamma$. Finally, when $x(t) = 0$, then $\varphi(x, u) = u'(R + P_-D'D)u \ge 0$ with the equality holds at $u^*(t) = 0$.

The above analysis together with (29) shows that

$$
J(x_0, u(\cdot)) \ge P_+(0)(x_0^+)^2 + P_-(0)(x_0^-)^2, \;\; \forall u(\cdot) \in \mathcal{U}, \qquad (32)
$$

whereas the equality is achieved when $u^*(\cdot)$ is defined by (19).

If we can prove that the control $u^*(\cdot)$ defined by (19) is in $L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$, then the proof of the theorem will be finished. To this end, first note that as shown before there exist two constants $C_1 > 0$ and $C_2 > 0$, such that

$$H_+(t,v,P,\Lambda) \geq C_1|v|^2 - C_2(|P|+|\Lambda|)v \geq C_1|v|[|v| - \frac{C_2}{C_1}(|P|+|\Lambda|)]. \tag{33}$$

Hence $H_+(t,v,P,\Lambda) > 0$ if $|v| > \frac{C_2}{C_1}(|P|+|\Lambda|)$. From the definition of $\xi_+(t,P,\Lambda)$, it follows that

$$|\xi_+(t,P,\Lambda)| \leq \frac{C_2}{C_1}(|P|+|\Lambda|). \tag{34}$$

The same is true for ξ_-.

Now, under the feedback control (19), the system dynamics (1) read

$$\begin{cases} dx(t) = & [A(t)x(t) + B(t)\xi_+(t,P_+(t),\Lambda_+(t))x^+(t) + B(t)\xi_-(t,P_-(t),\Lambda_-(t))x^-(t)]dt \\ & +[x(t)C(t)' + x^+(t)\xi_+(t,P_+(t),\Lambda_+(t))'D(t)' \\ & \quad +x^-(t)\xi_-(t,P_-(t),\Lambda_-(t))'D(t)']dW(t), \quad t \in [0,\,T], \\ x(0) = & x_0. \end{cases} \tag{35}$$

This equation has a unique continuous $\{\mathcal{F}_t\}_{t\geq 0}$-adapted solution; see the lemma following the end of the proof of this theorem. We denote this solution by $x^*(\cdot)$, and hence $u^*(t) = \xi_+(t,P_+(t),\Lambda_+(t))x^{*+}(t) + \xi_-(t,P_-(t),\Lambda_-(t))x^{*-}(t)$. The continuity of $x^*(\cdot)$ along with (34) leads to

$$\int_0^T (|x^*(t)|^2 + |u^*(t)|^2)dt < +\infty, \;\; P-a.s. \tag{36}$$

Denote by τ_n^*, $n \geq 1$, the sequence of stopping times defined by (27) where the state–control pair is taken as $(x^*(\cdot),u^*(\cdot))$. It follows from (36) that $\tau_n^* \to T$ as $n \to +\infty$, P-a.s.. On the other hand, (28) yields

$$\begin{aligned} & E\Big\{P_+(\tau_n^*)x^{*+}(\tau_n^*)^2 + P_-(\tau_n^*)x^{*-}(\tau_n^*)^2\Big\} + E\int_0^{\tau_n^*}[Q(t)x^*(t)^2 + u^*(t)'R(t)u^*(t)]dt \\ = \;& P_+(0)(x_0^+)^2 + P_-(0)(x_0^-)^2. \end{aligned} \tag{37}$$

We are now in the position to prove $u^*(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$. To do so, we will treat the standard case and the singular case separately.

For the standard case, denote by $c > 0$ such that $R \geq cI_m$. Then it follows from (37) that

$$cE\int_0^{\tau_n^*}|u^*(t)|^2dt \leq P_+(0)(x_0^+)^2 + P_-(0)(x_0^-)^2. \tag{38}$$

This implies that $u^*(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$.

For the singular case, construct a sequence of stopping times as follows

$$\theta_n := \inf\{t \geq 0 \Big| \int_0^t (|x^*(s)|^2 + |C(s)x^*(s) + D(s)u^*(s)|^2)ds \geq n\} \wedge T.$$

Again θ_n increasingly converge to T almost surely due to (36). Rewrite equation (1) under $u^*(\cdot)$ as a kind of BSDE with a random terminal time:

$$\begin{cases} dx^*(t) = [(A - B(D'D)^{-1}D'C)x^*(t) + B(D'D)^{-1}D'z(t)]dt + z(t)'dW(t), \quad t \in [0, \tau_n^* \wedge \theta_n], \\ x^*(\tau_n^* \wedge \theta_n) = x^*(\tau_n^* \wedge \theta_n), \end{cases} \tag{39}$$

where $z(t) := C(t)x^*(t) + D(t)u^*(t)$. Theorem 4.2 provides that there is a constant $c > 0$ such that for any $t \in [0, T]$, $P_+(t) \geq c$ and $P_-(t) \geq c$. Thus, equation (37) with τ_n^* replaced by $\tau_n^* \wedge \theta_n$ leads to

$$cE[x^*(\tau_n^* \wedge \theta_n)^2] \leq P_+(0)(x_0^+)^2 + P_-(0)(x_0^-)^2.$$

On the other hand, the standard estimate for the BSDE (39) gives that, for a constant $\tilde{c} > 0$,

$$\begin{aligned} E\int_0^{\tau_n^* \wedge \theta_n} (|x^*(s)|^2 + |z(s)|^2)ds &\leq \tilde{c}E[x^*(\tau_n^* \wedge \theta_n)^2] \\ &\leq \frac{\tilde{c}}{c}[P_+(0)(x_0^+)^2 + P_-(0)(x_0^-)^2]. \end{aligned}$$

Appealing to Fatou's lemma, we conclude that $x^*(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R})$ and $z(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^k)$. This in turn implies $u^*(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ as $u^*(t) = [D(t)'D(t)]^{-1}D(t)'[z(t) - C(t)x^*(t)]$. ■

Lemma 5.1 *Equation (35) has a unique continuous $\{\mathcal{F}_t\}_{t\geq 0}$-adapted solution.*

Proof: Setting

$$\begin{aligned} &B_+(t) := B(t)\xi_+(t, P_+(t), \Lambda_+(t)), \ B_-(t) := B(t)\xi_-(t, P_-(t), \Lambda_-(t)), \\ &D_+(t) := D(t)\xi_+(t, P_+(t), \Lambda_+(t)), \ D_-(t) := D(t)\xi_-(t, P_-(t), \Lambda_-(t)), \end{aligned}$$

then (35) becomes

$$\begin{cases} dx(t) = & [A(t)x(t) + B_+(t)x^+(t) + B_-(t)x^-(t)]dt \\ & +[x(t)C(t)' + x^+(t)D_+(t)' + x^-(t)D_-(t)']dW(t), \quad t \in [0,\, T], \\ x(0) = & x_0. \end{cases} \tag{40}$$

Consider the following two linear SDEs:

$$\begin{cases} dx_+(t) = & [A(t) + B_+(t)]x_+(t)dt + x_+(t)[C(t) + D_+(t)]'dW(t), \quad t \in [0,\, T], \\ x(0) = & x_0^+, \end{cases} \tag{41}$$

and

$$\begin{cases} dx_-(t) = & [A(t) - B_-(t)]x_-(t)dt + x_-(t)[C(t) - D_-(t)]'dW(t), \quad t \in [0,\, T], \\ x(0) = & x_0^-. \end{cases} \tag{42}$$

It is well known that both equations (41) and (42) have unique continuous $\{\mathcal{F}_t\}_{t\geq 0}$-adapted solutions which can be represented explicitly as

$$x_+(t) = x_0^+ \exp\left\{\int_0^t [A(s)+B_+(s)]ds + \int_0^t [C(s)+D_+(s)]'dW(s) - \frac{1}{2}\int_0^t |C(s)+D_+(s)|^2 ds\right\}$$

and

$$x_-(t) = x_0^- \exp\left\{\int_0^t [A(s)-B_-(s)]ds + \int_0^t [C(s)-D_-(s)]'dW(s) - \frac{1}{2}\int_0^t |C(s)-D_-(s)|^2 ds\right\}.$$

Define

$$x(t) := x_+(t) - x_-(t).$$

Since $x_+(t)\geq 0$, $x_-(t)\geq 0$, and $x_+(t)x_-(t)=0$, we conclude

$$x^+(t) = x_+(t), \quad x^-(t) = x_-(t), \quad \forall t\in[0,T].$$

Subtracting (42) from (41) we get that $x(\cdot)$ is a continuous adapted solution of (40).

Let us turn to the uniqueness of solution. Suppose that $x_1(\cdot)$ and $x_2(\cdot)$ are two continuous adapted solutions of equation (40). Put $\hat{x}(\cdot) := x_1(\cdot)-x_2(\cdot)$. We apply a linearization procedure as follows. Set

$$\alpha_+(t) := \frac{x_1^+(t)-x_2^+(t)}{x_1(t)-x_2(t)}1_{\{x_1(t)\neq x_2(t)\}}, \quad \alpha_-(t) := \frac{x_1^-(t)-x_2^-(t)}{x_1(t)-x_2(t)}1_{\{x_1(t)\neq x_2(t)\}}.$$

Then $\hat{x}(\cdot)$ is a continuous adapted solution of the following linear SDE:

$$\left\{\begin{array}{rl} d\hat{x}(t) = & [A(t)+B_+(t)\alpha_+(t)+B_-(t)\alpha_-(t)]\hat{x}(t)dt \\ & +\hat{x}(t)[C(t)+D_+(t)\alpha_+(t)+D_-(t)\alpha_-(t)]'dW(t), \quad t\in[0,\,T], \\ \hat{x}(0) = & 0. \end{array}\right. \tag{43}$$

Hence $\hat{x}(t)\equiv 0$ via a similar representation as (41) or (42), and the uniqueness of solution is proved. ∎

Let us conclude this section by noting that a by-product of Theorem 5.1 is the uniqueness of solution to the ESREs (8) and (9). Indeed, consider an LQ control problem in an interval $[s,T]$, with $s\in[0,T)$, where the system dynamics is (1) with initial time s and initial state $x(s)=x_s\in L^2_{\mathcal{F}_s}(\Omega;\mathbb{R})$, and the cost functional is

$$J_s(x_s,u(\cdot)) := E\left\{\int_s^T [Q(t)x(t)^2 + u(t)'R(t)u(t)]dt + Gx(T)^2 \Big| \mathcal{F}_s\right\}.$$

Let $(P_+,\,\Lambda_+)\in L^2_{\mathcal{F}}(\Omega;\,C(0,\,T;\mathbb{R}))\times L^2_{\mathcal{F}}(0,\,T;\mathbb{R}^k)$ and $(P_-,\,\Lambda_-)\in L^2_{\mathcal{F}}(\Omega;\,C(0,\,T;\mathbb{R}))\times L^2_{\mathcal{F}}(0,\,T;\mathbb{R}^k)$ be *any* bounded, nonnegative (in the standard case) or bounded, uniformly positive (in the singular case) solutions to (8) and (9), respectively. Then, going through the same analysis as in the proof of Theorem 5.1 we deduce that the optimal cost is

$$J_s^*(x_s) := \inf_{u(\cdot)\text{ admissible}} J_s(x_s,u(\cdot)) = P_+(s)(x_s^+)^2 + P_-(s)(x_s^-)^2. \tag{44}$$

This proves the following uniqueness result.

Theorem 5.2 *Each of the ESREs (8) and (9) admits at most one bounded, nonnegative (in the standard case) or bounded, uniformly positive (in the singular case) solution.*

6 Application to a mean–variance portfolio selection problem

Consider a financial market with $m+1$ securities, consisting of a bank account and m stocks. The value of the bank account, $S_0(t)$, satisfies an ordinary differential equation:

$$\begin{cases} dS_0(t) = r(t)\, S_0(t)\, dt, \quad t \in [0,\, T], \\ S_0(0) = s_0 > 0, \end{cases} \tag{45}$$

where the interest rate $r(t) > 0$ is a deterministic, uniformly bounded, scalar-valued function. The price of each of the stocks, $S_1(t), \cdots, S_m(t)$, satisfies the stochastic differential equation (SDE):

$$\begin{cases} dS_i(t) = S_i(t)\Big\{\mu_i(t)\, dt + \sum_{j=1}^m \sigma_{ij}(t)\, dW^j(t)\Big\}, \quad t \in [0,\, T], \\ S_i(0) = s_i > 0, \end{cases} \tag{46}$$

where $\mu_i(t) > 0$ and $\sigma_i(t) = [\sigma_{i1}, \cdots,\, \sigma_{im}(t)]$ are the appreciation rate and dispersion (or volatility) rate of the i^{th} stock. Here, $\mu_i(t)$ and $\sigma_{ij}(t)$ are scalar-valued, $\{\mathcal{F}_t\}_{t\geq 0}$-adapted, uniformly bounded *stochastic processes.* Denoting

$$\sigma(t) := \begin{bmatrix} \sigma_1(t) \\ \vdots \\ \sigma_m(t) \end{bmatrix} \in \mathbb{R}^{m\times m}, \tag{47}$$

we assume throughout that $\sigma(t)$ is uniformly non-degenerate: that is, there exists a deterministic $\delta > 0$ such that

$$\sigma(t)\, \sigma(t)' \geq \delta\, I_m, \quad \forall\, t \in [0,\, T],\ P - a.s.. \tag{48}$$

In particular, $\sigma(t)$ must be non-singular a.e. $t \in [0,\, T]$, P-a.s..

Suppose that the total wealth of an agent at time $t \geq 0$ is denoted by $x(t)$. If transaction costs and consumption are ignored, and share trading takes place in continuous time, then we have

$$\begin{cases} dx(t) = \Big\{r(t)\, x(t) + \sum_{i=1}^m [\mu_i(t) - r(t)]\, u_i(t)\Big\}\, dt \\ \qquad\qquad + \sum_{j=1}^m \sum_{i=1}^m \sigma_{ij}(t)\, u_i(t)\, dW^j(t), \quad t \in [0,\, T], \\ x(0) = x_0 > 0, \end{cases} \tag{49}$$

where $u_i(t)$ is the total market value of the agent's wealth in the i^{th} asset. We refer to $u(\cdot) := (u_1(\cdot), \cdots, u_m(\cdot))'$ as the *portfolio* of the agent. In our model, short-selling of the stocks is not allowed; hence we have the following constraints on a portfolio $u(\cdot) = (u_1(\cdot), \cdots, u_m(\cdot))'$:

$$u_i(t) \geq 0, \quad \forall t \in [0, T], \quad i = 1, \cdots, m. \tag{50}$$

Note that $u_0(\cdot)$ has been excluded from a portfolio since it is completely determined by the allocation of stocks and the total wealth $x(\cdot)$. Moreover, we do allow $u_0(t) < 0$, meaning that the agent is borrowing the amount $|u_0(t)|$ from the bank at rate $r(t)$.

Definition 6.1 A portfolio $u(\cdot)$ is said to be *admissible* if it is $\mathbb{R}^m$-valued, square-integrable (i.e., $E\int_0^T |u(t)|^2 dt < +\infty$), $\{\mathcal{F}_t\}_{t\geq 0}$-adapted, and satisfies (50). In this case, we refer to $(x(\cdot), u(\cdot))$ as an *admissible (wealth–portfolio) pair.*

In a mean–variance portfolio selection problem, an agent's objective is to find an admissible portfolio $u(\cdot)$ such that the expected terminal wealth satisfies $Ex(T) = z$, for some $z \geq x_0 e^{\int_0^T r(s)ds}$, while the risk measured by the variance of the terminal wealth

$$\text{Var } x(T) := E[x(T) - Ex(T)]^2 = E[x(T)]^2 - z^2 \tag{51}$$

is minimized. The restriction of the targeted payoff $z \geq x_0 e^{\int_0^T r(s)ds}$ is natural as the latter can always be achieved by putting all the money in the bank. Mathematically, it can be formulated as the following problem parameterized by $z \geq x_0 e^{\int_0^T r(s)ds}$:

$$\begin{cases} \text{Minimize} & J_{\text{MV}}(x_0, u(\cdot)) := E[x(T)]^2 - z^2, \\ \text{subject to:} & \begin{array}{l} Ex(T) = z, \\ (x(\cdot), u(\cdot)) \text{ is admissible for (49).} \end{array} \end{cases} \tag{52}$$

The above problem is called *feasible* if there is at least one portfolio satisfying the constraints of (52). Finally, an optimal portfolio to (52) is called an *efficient portfolio* corresponding to z, the corresponding $(\text{Var } x(T), z)$ is called an *efficient point*, whereas the set of all the efficient points, with $z \geq x_0 e^{\int_0^T r(s)ds}$, is called an *efficient frontier.*

Equation (49) can be rewritten as

$$\begin{cases} dx(t) = [r(t)x(t) + B(t)u(t)]dt + u(t)'\sigma(t)dW(t), \\ x(0) = x_0, \end{cases} \tag{53}$$

where

$$B(t) := (\mu_1(t) - r(t), \ldots, \mu_m(t) - r(t)). \tag{54}$$

Since the problem (52) involves a terminal constraint $Ex(T) = z$, we first investigate conditions under which the problem is feasible for *any* $z \in [x_0 e^{\int_0^T r(s)ds}, +\infty)$.

Theorem 6.1 *The mean-variance problem (52) is feasible for every* $z \in [x_0 e^{\int_0^T r(s)ds}, +\infty)$ *if and only if*

$$\sum_{i=1}^{m} E\int_0^T [\mu_i(t) - r(t)]^+ dt > 0. \tag{55}$$

Proof: We first prove the "if" part. Define

$$M_i := \{(t,\omega) : \mu_i(t,\omega) > r(t)\}, \ \ i = 1, 2, \cdots, m.$$

Condition (55) implies that at least one of the sets M_i has non-zero measure (in terms of the product of the Lebesgue measure and P). Suppose M_{i_0} has non-zero measure. Construct a family of admissible portfolios $u^{\beta}(\cdot) := \beta u(\cdot)$, where $\beta \geq 0$ and the components of $u(\cdot)$ are all zero except its i_0-th component which is defined to be

$$u_{i_0}(t,\omega) := \begin{cases} \mu_{i_0}(t,\omega) - r(t), & \text{if } (t,\omega) \in M_{i_0}, \\ 0, & \text{if } (t,\omega) \notin M_{i_0} \end{cases} \tag{56}$$

Let $x^{\beta}(\cdot)$ be the wealth process corresponding to $u^{\beta}(\cdot)$. By linearity of the wealth equation, we have $x^{\beta}(t) = x^0(t) + \beta x^1(t)$, where $x^0(t) := x_0 e^{\int_0^t r(s)ds}$ and $x^1(\cdot)$ is the solution to the following equation

$$\begin{cases} dx^1(t) = [r(t)x^1(t) + B(t)u(t)]dt + u(t)'\sigma(t)dW(t), \\ x^1(0) = 0. \end{cases} \tag{57}$$

Therefore, problem (52) is feasible for every $z \in [x_0 e^{\int_0^T r(s)ds}, +\infty)$ if there exists $\beta \geq 0$ such that $z = Ex^{\beta}(T) \equiv x^0(T) + \beta Ex^1(T)$. Equivalently, (52) is feasible for every $z \in [x_0 e^{\int_0^T r(s)ds}, +\infty)$ if $Ex^1(T) > 0$. However, applying Itô's formula we get

$$d[e^{\int_t^T r(s)ds} x^1(t)] = e^{\int_t^T r(s)ds} B(t)u(t)dt + \{\cdots\}dW(t).$$

Integrating from 0 to T and taking expectation we obtain

$$Ex^1(T) = E\int_0^T e^{\int_t^T r(s)ds} B(t)u(t)dt > 0, \tag{58}$$

due to the way $u(\cdot)$ was constructed. Consequently, (52) is feasible if (55) holds.

Conversely, suppose that problem (52) is feasible for every $z \in [x_0 e^{\int_0^T r(s)ds}, +\infty)$. Then for each z, there is an admissible portfolio $u(\cdot)$ so that $Ex(T) = z$. However, we can always decompose $x(t) = x^0(t) + x^1(t)$ where $x^1(\cdot)$ satisfies (57). This leads to $Ex^0(T) + Ex^1(T) = z$. Now, $Ex^0(T) \equiv z_0$ is independent of $u(\cdot)$; thus it is necessary that there is a $u(\cdot)$ with $Ex^1(T) > 0$. It follows then from (58) that (55) must be valid. ■

Now we are going to solve the optimization problem (52) under the feasibility assumption (55). To handle the constraint $Ex(T) = z$ we apply the Lagrange multiplier technique. Define

$$\begin{aligned} J(x_0, u(\cdot), \lambda) := & \ E\{x(T)^2 - z^2 - 2\lambda[x(T) - z]\} \\ = & \ E[|x(T) - \lambda|^2] - (\lambda - z)^2, \ \ \lambda \in \mathbb{R}. \end{aligned} \tag{59}$$

Since $J_{\rm MV}(x_0,u(\cdot))$ is strictly convex in $u(\cdot)$ and the constraint function $Ex(T)-z$ is affine in $u(\cdot)$, we can apply the well-known duality theorem (see, e.g., [19]). Based on this theorem, we may first solve the following unconstrained problem parameterized by the Lagrange multiplier $\lambda\in\mathbb{R}$:

$$\left\{\begin{array}{ll}\text{Minimize} & J(x_0,u(\cdot),\lambda):=E[|x(T)-\lambda|^2]-(\lambda-z)^2,\\ \text{subject to:} & (x(\cdot),u(\cdot)) \text{ is admissible for (53).}\end{array}\right. \tag{60}$$

This problem is exactly a singular case of the general LQ model solved in Section 5, with $Q(t)=R(t)=0$ and $\Gamma=\mathbb{R}^m_+$. Thus we will apply the general result to the problem. Let us first write down the specialization of the ESREs (8) and (9) as

$$\left\{\begin{array}{l} dP_+(t)=-[2r(t)P_+(t)+H_+^*(t,P_+(t),\Lambda_+(t))]dt+\Lambda_+(t)'dW(t),\quad t\in[0,\,T],\\ P_+(T)=1,\\ P_+(t)>0,\end{array}\right. \tag{61}$$

$$\left\{\begin{array}{l} dP_-(t)=-[2r(t)P_-(t)+H_-^*(t,P_-(t),\Lambda_-(t))]dt+\Lambda_-(t)'dW(t),\quad t\in[0,\,T],\\ P_-(T)=1,\\ P_-(t)>0,\end{array}\right. \tag{62}$$

where

$$\begin{array}{l} H_+^*(t,P,\Lambda):=\min_{v\in\mathbb{R}^m_+}\left\{v'P\sigma(t)\sigma(t)'v+2v'[B(t)'P+\sigma(t)\Lambda]\right\},\\ H_-^*(t,P,\Lambda):=\min_{v\in\mathbb{R}^m_+}\left\{v'P\sigma(t)\sigma(t)'v-2v'[B(t)'P+\sigma(t)\Lambda]\right\}.\end{array} \tag{63}$$

Also, define

$$\begin{array}{l} \xi_+(t,P,\Lambda):=\mathrm{argmin}_{v\in\mathbb{R}^m_+}H_+(t,v,P,\Lambda),\\ \xi_-(t,P,\Lambda):=\mathrm{argmin}_{v\in\mathbb{R}^m_+}H_-(t,v,P,\Lambda),\quad (t,P,\Lambda)\in[0,T]\times\mathbb{R}\times\mathbb{R}^k.\end{array} \tag{64}$$

Clearly, Theorems 4.2 and 5.2 apply to (61) and (62) ensuring that they admit unique bounded, uniformly positive solutions.

Lemma 6.1 *Assume that (55) holds, and let* $(P_+,\,\Lambda_+)\in L^2_{\mathcal{F}}(\Omega;\,C(0,\,T;\mathbb{R}))\times L^2_{\mathcal{F}}(0,\,T;\mathbb{R}^m)$ *and* $(P_-,\,\Lambda_-)\in L^2_{\mathcal{F}}(\Omega;\,C(0,\,T;\mathbb{R}))\times L^2_{\mathcal{F}}(0,\,T;\mathbb{R}^m)$ *be the unique bounded, uniformly positive solutions to the ESREs (61) and (62) respectively. Then it must hold that*

$$P_+(0)e^{-2\int_0^T r(s)ds}-1\le 0,\quad \textit{and}\quad P_-(0)e^{-2\int_0^T r(s)ds}-1<0. \tag{65}$$

Proof: Define $g(t):=P_-(t)e^{-2\int_t^T r(s)ds}$. Then it is straightforward that

$$dg(t)=-e^{-2\int_t^T r(s)ds}H_-^*(t,P_-(t),\Lambda_-(t))dt+e^{-2\int_t^T r(s)ds}\Lambda_-(t)'dW(t).$$

Integrating from 0 to T and taking expectation we have

$$1-g(0)=-E\int_0^T e^{-2\int_t^T r(s)ds}H_-^*(t,P_-(t),\Lambda_-(t))dt\ge 0, \tag{66}$$

since $H_-^*(t, P_-(t), \Lambda_-(t)) \leq 0$ by its very definition. Hence $P_-(0)e^{-2\int_0^T r(s)ds} \equiv g(0) \leq 1$. Similarly, we can prove that $P_+(0)e^{-2\int_0^T r(s)ds} - 1 \leq 0$.

It remains to prove the strict inequality $P_-(0)e^{-2\int_0^T r(s)ds} - 1 < 0$. In fact, if $P_-(0)e^{-2\int_0^T r(s)ds} - 1 = 0$, then it follows from (66) that $H_-^*(t, P_-(t), \Lambda_-(t)) = 0$, a.e.$t \in [0, T]$, P-a.s.. Thus we deduce, from the uniqueness of solution to the BSDE (62), that $P_-(t) = e^{2\int_t^T r(s)ds}$, and $\Lambda_-(t) = 0$. Consequently, $H_-^*(t, P_-(t), 0) = 0$.

On the other hand,

$$\begin{aligned} H_-^*(t, P_-(t), 0) &= \min_{v \in \mathbb{R}_+^m} P_-(t)[v'\sigma(t)\sigma(t)'v - 2v'B(t)'] \\ &\leq \min_{v \in \mathbb{R}_+^m} K_3[K_4|v|^2 - 2B(t)v], \end{aligned}$$

for some constants $K_3 > 0$ and $K_4 > 0$. Notice that the minimum value on the right hand side of the above is *strictly* negative whenever $B(t, \omega)^+ := (B_1(t, \omega)^+, \cdots, B_m(t, \omega)^+) \neq 0$. In view of the assumption (55), the set of (t, ω) on which $B(t, \omega)^+$ is non-zero has a non-zero measure. Thus we get a contradiction.

∎

Remark 6.1 One does not have the strict inequality $P_+(0)e^{-2\int_0^T r(s)ds} - 1 < 0$ since no information about $B(t, \omega)^- := (B_1(t, \omega)^-, \cdots, B_m(t, \omega)^-)$ is available. On the other hand, the inequality $P_-(0)e^{-2\int_0^T r(s)ds} - 1 < 0$ is exactly what is required in the sequel.

Theorem 6.2 *Let $(P_+,\, \Lambda_+) \in L^2_{\mathcal{F}}(\Omega;\, C(0,\, T; \mathbb{R})) \times L^2_{\mathcal{F}}(0,\, T; \mathbb{R}^m)$ and $(P_-,\, \Lambda_-) \in L^2_{\mathcal{F}}(\Omega;\, C(0,\, T; \mathbb{R})) \times L^2_{\mathcal{F}}(0,\, T; \mathbb{R}^m)$ be the unique bounded, uniformly positive solutions to the ESREs (61) and (62) respectively. Then the following state feedback control*

$$u^*(t) = \xi_+(t, P_+(t), \Lambda_+(t))\left(x(t) - \lambda e^{-\int_t^T r(s)ds}\right)^+ + \xi_-(t, P_-(t), \Lambda_-(t))\left(x(t) - \lambda e^{-\int_t^T r(s)ds}\right)^- \tag{67}$$

is optimal for the problem (60). Moreover, in this case the optimal cost is

$$\begin{aligned} &J^*(x_0, \lambda) := \inf_{u(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}_+^m)} J(x_0, u(\cdot), \lambda) \\ &= \begin{cases} [P_+(0)e^{-2\int_0^T r(s)ds} - 1]\lambda^2 - 2[x_0 P_+(0)e^{-\int_0^T r(s)ds} - z]\lambda + P_+(0)x_0^2 - z^2, \\ \qquad \text{if } x_0 > \lambda e^{-\int_0^T r(s)ds}, \\ [P_-(0)e^{-2\int_0^T r(s)ds} - 1]\lambda^2 - 2[x_0 P_-(0)e^{-\int_0^T r(s)ds} - z]\lambda + P_-(0)x_0^2 - z^2, \\ \qquad \text{if } x_0 \leq \lambda e^{-\int_0^T r(s)ds}. \end{cases} \end{aligned} \tag{68}$$

Proof: Set

$$y(t) := x(t) - \lambda e^{-\int_t^T r(s)ds}. \tag{69}$$

It turns out the wealth equation (53) in terms of $y(\cdot)$ has exactly the same form except the initial condition:

$$\begin{cases} dy(t) = [r(t)y(t) + B(t)u(t)]dt + u(t)'\sigma(t)dW(t), \\ y(0) = x_0 - \lambda e^{-\int_0^T r(s)ds}, \end{cases} \tag{70}$$

whereas the cost function (59) can be written as

$$J(y_0, u(\cdot), \lambda) = Ey(T)^2 - (\lambda - z)^2. \tag{71}$$

The above problem (70)–(71) is exactly a special case of the general problem we have solved in Section 5 (ignoring the constant term $-(\lambda - z)^2$ in (71)). Hence the optimal feedback control (67) follows from (19). Finally, the optimal cost is

$$J^*(x_0, \lambda) = P_+(0)[(x_0 - \lambda e^{-\int_0^T r(s)ds})^+]^2 + P_-(0)[(x_0 - \lambda e^{-\int_0^T r(s)ds})^-]^2 - (\lambda - z)^2$$

which equals the right hand side of (68) after some simple manipulations. ∎

Theorem 6.3 (**Efficient portfolios and efficient frontier**) *Assume that (55) holds. Then the efficient portfolio corresponding to $z \geq x_0 e^{\int_0^T r(s)ds}$, as a feedback of the wealth process, is*

$$u^*(t) = \xi_+(t, P_+(t), \Lambda_+(t)) \left(x^*(t) - \lambda^* e^{-\int_t^T r(s)ds}\right)^+ + \xi_-(t, P_-(t), \Lambda_-(t)) \left(x^*(t) - \lambda^* e^{-\int_t^T r(s)ds}\right)^-, \tag{72}$$

where

$$\lambda^* := \frac{z - x_0 P_-(0) e^{-\int_0^T r(s)ds}}{1 - P_-(0) e^{-2\int_0^T r(s)ds}}. \tag{73}$$

Moreover, the efficient frontier is

$$\mathrm{Var}\, x^*(T) = \frac{P_-(0) e^{-2\int_0^T r(s)ds}}{1 - P_-(0) e^{-2\int_0^T r(s)ds}} \left[Ex^*(T) - x_0 e^{\int_0^T r(s)ds}\right]^2, \quad Ex^*(T) \geq x_0 e^{\int_0^T r(s)ds}. \tag{74}$$

Proof: First note that λ^* in (73) is well defined thanks to Lemma 6.1. Now, if $z = x_0 e^{\int_0^T r(s)ds}$, then it is straightforward that the corresponding efficient portfolio is $u^*(t) \equiv 0$, meaning that all the wealth is to be put in the bank account. The resulting wealth process is $x^*(t) = x_0 e^{\int_0^t r(s)ds}$. On the other hand, in this case the associated $\lambda^* = x_0 e^{\int_0^T r(s)ds}$. Thus the portfolio given by (72) reduces to $u^*(t) \equiv 0$ with $x^*(t) = x_0 e^{\int_0^t r(s)ds}$. This implies that (72) is indeed the efficient portfolio when $z = x_0 e^{\int_0^T r(s)ds}$.

So now we need only to prove the theorem for any fixed $z > x_0 e^{\int_0^T r(s)ds}$. Applying the duality theorem (see, e.g., [19, p.224, Theorem 1][1]) we have

$$J^*_{\mathrm{MV}}(x_0) := \inf_{u(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m_+)} J_{\mathrm{MV}}(x_0, u(\cdot)) = \sup_{\lambda \in \mathbb{R}} \inf_{u(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m_+)} J(x_0, u(\cdot), \lambda) > -\infty, \tag{75}$$

[1] To be precise, one should apply [19, p. 236, Problem 7] together with the proof of [19, p.224, Theorem 1] in our case, as there is an *equality* constraint in (52). To be able to use the result there, one needs to check a condition posed in [19, p. 236, Problem 7], namely, 0 is an interior point of the set $\mathcal{T} := \{Ex(T) - z \,|\, x(\cdot)$ is the wealth process of an admissible portfolio $u(\cdot)$ with $x(0) = x_0\}$. In view of Theorem 6.1 and the assumption (55), we have $[x_0 e^{\int_0^T r(s)ds} - z, +\infty) \subset \mathcal{T}$. Hence 0 is an interior point of $\mathcal{T}$ because $x_0 e^{\int_0^T r(s)ds} - z < 0$.

and the optimal feedback control for (52) is (72), due to Theorem 6.2, with λ replaced by λ^* which maximizes $J^*(x_0,\lambda)$ $(=\inf_{u(\cdot)\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m_+)} J(x_0,u(\cdot),\lambda))$ over $\lambda\in\mathbb{R}$.

If $\lambda\in(-\infty, x_0 e^{\int_0^T r(s)ds})$, then the expression (68) gives, taking into consideration (65) and the fact that $z\geq x_0 e^{\int_0^T r(s)ds}$,

$$\begin{aligned}\frac{\partial}{\partial\lambda}J^*(x_0,\lambda) &= 2[P^+(0)e^{-2\int_0^T r(s)ds}-1]\lambda-2[x_0P_+(0)e^{-\int_0^T r(s)ds}-z]\\ &\geq 2[P^+(0)e^{-2\int_0^T r(s)ds}-1]x_0e^{\int_0^T r(s)ds}-2[x_0P_+(0)e^{-\int_0^T r(s)ds}-x_0e^{\int_0^T r(s)ds}]\\ &= 0.\end{aligned}$$

Hence,

$$\sup_{\lambda\in\mathbb{R}}J^*(x_0,\lambda)=\sup_{\lambda\in[x_0e^{\int_0^T r(s)ds},+\infty)}J^*(x_0,\lambda).$$

But for $\lambda\in[x_0e^{\int_0^T r(s)ds},+\infty)$, it follows from (68) that $J^*(x_0,\lambda)$ is a quadratic function in λ whose maximizer is given by (73) (noticing Lemma 6.1), whereas

$$\begin{aligned}J^*_{\rm MV}(x_0) &= \sup_{\lambda\in\mathbb{R}}\ \inf_{u(\cdot)\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m_+)}J^*(x_0,\lambda)\\ &= \sup_{\lambda\in\mathbb{R}}\left\{[P_-(0)e^{-2\int_0^T r(s)ds}-1]\lambda^2-2[x_0P_-(0)e^{-\int_0^T r(s)ds}-z]\lambda+P_-(0)x_0^2-z^2\right\}\\ &= \frac{P_-(0)e^{-2\int_t^T r(s)ds}}{1-P_-(0)e^{-2\int_0^T r(s)ds}}\left[z-x_0e^{\int_0^T r(s)ds}\right]^2,\quad z\geq x_0e^{\int_0^T r(s)ds}.\end{aligned}\tag{76}$$

This proves (74), noting that $Ex^*(T)=z$. ∎

Corollary 6.1 *Assume that (55) holds. Then the efficient portfolio (72) can be rewritten as*

$$u^*(t)=\xi_-(t,P_-(t),\Lambda_-(t))\left(\lambda^*e^{-\int_t^T r(s)ds}-x^*(t)\right).\tag{77}$$

Proof: It suffices to prove that under the feedback policy (72) the corresponding wealth trajectory $x^*(\cdot)$ satisfies

$$x^*(t)-\lambda^*e^{-\int_t^T r(s)ds}\leq 0.\tag{78}$$

To this end, write $y(t):=x^*(t)-\lambda^*e^{-\int_t^T r(s)ds}$. Then it is immediate from (53) that $y(\cdot)$ follows

$$\begin{cases} dy(t)=[r(t)y(t)+B_+(t)y^+(t)+B_-(t)y^-(t)]dt+[D_+(t)y^+(t)+D_-(t)y^-(t)]'dW(t),\\ y(0)=x_0-\lambda^*e^{-\int_0^T r(s)ds},\end{cases}\tag{79}$$

where

$$\begin{aligned}&B_+(t):=B(t)\xi_+(t,P_+(t),\Lambda_+(t)),\ \ B_-(t):=B(t)\xi_-(t,P_-(t),\Lambda_-(t)),\\ &D_+(t):=\sigma(t)'\xi_+(t,P_+(t),\Lambda_+(t)),\ \ D_-(t):=\sigma(t)'\xi_-(t,P_-(t),\Lambda_-(t)).\end{aligned}$$

Note that $y(0)=x_0-\lambda^*e^{-\int_0^T r(s)ds}\leq 0$ by virtue of (73) and the fact that $z\geq x_0e^{\int_0^T r(s)ds}$. Hence the proof of Lemma 5.1 yields that $y^+(t)=y(0)^+\exp\{\cdots\}=0$, which proves that $y(t)\leq 0$. ∎

It is interesting to note that, as indicated by the preceding theorem and corollary, after all only one of the two Riccati equations, (62), is necessary in the final solution to the portfolio selection problem. This is essentially due to the fact that one is only interested in the "non-satiation portion" (i.e., that corresponds to $z \geq x_0 e^{\int_0^T r(s)ds}$) of the entire variance-minimizing boundary. Because of this, the form of the efficient portfolio (77) turns out to be strikingly similar to its shorting-permitted counterpart [28]. In particular, if shorting is allowed, then the definition (64) should be modified so that $\xi_-(t, P, \Lambda)$ is the minimum point of $H_-(t, v, P, \Lambda)$ over $v \in \mathbb{R}^m$, or $\xi_-(t, P, \Lambda) = (\sigma(t)\sigma(t)')^{-1}\left[B(t)' + \sigma(t)\frac{\Lambda}{P}\right]$. If furthermore all the market coefficients are deterministic, then $\Lambda(t) \equiv 0$ and the result of [28] is recovered.

Also, it follows from (78) that the wealth trajectory under the efficient portfolio is capped almost surely, at any time, by the present value of a *deterministic* constant λ^*.

Finally, we remark that in the portfolio selection application the control (portfolio) constraint is taken to be $\Gamma = \mathbb{R}_+^m$, for it has significant financial interpretation (no-shorting). We can easily cope with other forms of constrained portfolio thanks to the general results established in Sections 4–5. An example is, in the case of two stocks, $\Gamma = \{(u_1, u_2) \in \mathbb{R}^2 | u_1 \leq 2u_2\}$. Such a constraint can be interpreted as maintaining certain weights on different stocks. Note that if we deal with a general portfolio constraint, then explicit characterization of the feasibility such as (55) may no longer be possible. However, it is still possible to obtain certain implicit feasibility condition based on the dual of the constraint cone, Γ. Details are left to the interested reader.

7 Concluding Remarks

In this paper, we have solved explicitly a stochastic LQ control problem where the control is constrained by a cone and all the coefficients are random. The solution is heavily dependent on the two nonlinear BSDEs which are introduced in this paper for the first time. The study on these two equations is interesting from the point view of BSDE theory. A continuous-time mean–variance portfolio selection problem has been then solved as a special case of the general constrained stochastic LQ model.

A major assumption of the paper is that the state variable be one-dimensional. Although this assumption is valid in many interesting applications including the financial one, it is a very challenging open problem to obtain explicit solution to a multi-dimensional problem and study the possible associated BSDEs.

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