

# A Note on Semivariance

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## Abstract

In a recent paper (Jin, Yan and Zhou 2004) it is proved that efficient strategies of the continuous-time mean–semivariance portfolio selection model are in general *never* achieved save for a trivial case. In this note, we show that the mean–semivariance efficient strategies in a single period are *always* attained irrespective of the market condition or the security return distribution. Further, for the below-target semivariance model the attainability is established under the arbitrage-free condition. Finally, we extend the results to problems with general downside risk measures.

KEY WORDS: single period, semivariance, below-mean, below-target, downside risk, coercivity

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# 1 Introduction

Mean–variance analysis would be the most widely used form of risk–return analysis in financial practice, except that it is questionable to use variance as a measure of risk. Variance is a measure of volatility since it penalizes upside deviations from the mean as much as downside deviations. Markowitz (1959) Chapter 9 presents semivariance as an alternate measure. The two forms of semivariance proposed there are the expected squared negative part of (a) the deviation from expected value, also known as the *below-mean semivariance*, and (b) the deviation from some fixed value such as zero return, or the *below-target semivariance*. The latter, in particular, includes expected squared loss. Markowitz (1959) presented a computing procedure for tracing out the set of single-period mean–semivariance (in either sense) efficient sets subject to a bounded polyhedral constraint set, assuming that the joint distribution of returns is a finite sample space. But Markowitz does not establish even the existence of such an efficient set when the constraint set is unbounded. The research on semivariance abounds in literature, see, e.g., Quirk and Saposnik (1962), Mao (1970), Hogan and Warren (1972), Klemkosky (1973), Porter (1974), Nantell and Price (1979), as well as a survey paper Nawrocki (1999). Since there is no closed-form solution to the mean–semivariance models (as opposed to the mean–variance; see Merton 1972 for a mean–variance analytical solution where shorting is allowed), all these works seem to have focused on the comparison between semivariance and variance or the numerical computation of the semivariance efficient frontier, whereas the very existence of such an efficient frontier has not been established to our best knowledge (Lemma 1.23 of Steinbach 2001 states such an existence; unfortunately its proof is incorrect<sup>1</sup>).

The present paper shows the existence of the mean–semivariance efficient set for any (generally *unbounded*) closed subset of  $\mathbb{R}^n$ , provided only that finite semivariance exists on the set. This result was needed, and is important, for three reasons: (1) completeness; (2) CAPM constraint sets are unbounded; and (3) recent results on semivariance efficiency in continuous time (Jin, Yan and Zhou 2004) show that efficient strategies are *never* achieved except for a trivial case. Thus the results for the single-period case reported here contrasts with the results for the continuous-time case. These results also contrast with those for mean–variance efficiency in which optimal strategies are achieved both in the continuous-time case (Zhou and Li 2000) and in the single-period case (Markowitz 1959 and Merton 1972).

Technically, the existence of a mean–variance efficient portfolio follows from the coer-

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<sup>1</sup>In the proof of Lemma 1.23 of Steinbach (2001), the following assertion is used: a convex function attains its infimum over a non-empty closed convex set, which is not correct in general; see, e.g., Section 27 of Rockafellar 1970 for an extensive discussion on this issue.

civity of the variance function<sup>2</sup> if the covariance matrix is non-singular. In this case, the corresponding optimization problem is effectively one of minimizing a continuous function over a closed *bounded* region, and hence the attainability of optimal solutions. When the covariance matrix is singular, the above argument does no longer apply, yet the celebrated Frank–Wolfe theorem (Frank and Wolfe 1956) ensures the existence.<sup>3</sup> Clearly, in the realm of semivariance one needs to resort to a different approach to establish the existence. In this paper, we will study both below-mean and below-target semivariance models in a single period. For the former, we prove the attainability almost *unconditionally*, even when there are arbitrage opportunities in the market. For the latter, one has to assume the arbitrage-free condition, and a counter example is given if the condition is violated.

The remainder of the paper is organized as follows. In section 2 we introduce the models under consideration. Section 3 is devoted to two technical lemmas that are key to this paper. The main results are derived in section 4. Extensions to more general downside-risk models, including the *lower partial moment model* (Bawa 1975 and Fishburn 1977), are presented in section 5.

## 2 Models

Suppose there are  $n$  ( $n \geq 2$ ) securities available in the market and consider a single investment period. The market uncertainty is described by a probability space  $(\Omega, \mathcal{F}, P)$  where the mathematical expectation is denoted by  $E(\cdot)$ . The total return of the  $i^{\text{th}}$  security during the period is a random variable  $\xi_i$ , meaning that the payoff of one unit investment in security  $i$  is  $\xi_i$  units,  $i = 1, 2, \dots, n$ . Suppose  $E\xi_i = r_i$  and  $\text{Var}(\xi_i) < +\infty$ .

We consider two portfolio selection models. The first one is the mean–semivariance portfolio selection modelled as follows:

$$\begin{aligned} & \text{minimize} && E\left[\left(\sum_{i=1}^n x_i \xi_i - E\left(\sum_{i=1}^n x_i \xi_i\right)\right)^-\right]^2, \\ & \text{subject to} && \begin{cases} \sum_{i=1}^n x_i = a, \\ \sum_{i=1}^n x_i r_i = z, \end{cases} \end{aligned} \tag{1}$$

where  $x_i \in \mathbb{R}$  represents the capital amount invested in the  $i^{\text{th}}$  security,  $i = 1, 2, \dots, n$  (hence  $x := (x_1, \dots, x_n)$  is a *portfolio*),  $a \in \mathbb{R}$  is the initial budget of the investor, and  $z \in \mathbb{R}$  a pre-determined expected payoff. Here  $x^- := \max(-x, 0)$  for any real number  $x$ . This problem is

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<sup>2</sup>A function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is called *coercive* if  $\lim_{|x| \rightarrow +\infty} f(x) = +\infty$ .

<sup>3</sup>The Frank–Wolfe theorem asserts that any *quadratic* function bounded below on a nonempty polyhedron must achieve its infimum on the polyhedron.

also referred to as below-mean semivariance model. In contrast, the second problem, termed below-target semivariance model, is the following

$$\begin{aligned} & \text{minimize} && E[(\sum_{i=1}^n x_i \xi_i - b)^-]^2, \\ & \text{subject to} && \sum_{i=1}^n x_i = a, \end{aligned} \tag{2}$$

where  $b \in \mathbb{R}$  represents a pre-specified target.

### 3 Two Lemmas

In this section we present two technical lemmas which are key to the proofs of the main results. The lemmas are on the following optimization problem

$$\min_{x \in \mathbb{R}^m} E[(A + B'x)^-]^2, \tag{3}$$

where  $B \equiv (B_1, \dots, B_m)'$  ( $'$  is the matrix transpose), and  $A, B_i, i = 1, \dots, m$ , are random variables with  $EA^2 < +\infty, EB_i^2 < +\infty, i = 1, \dots, m$ .

**Lemma 3.1** *If  $EB_i = 0, i = 1, \dots, m$ , then problem (3) admits optimal solutions.*

*Proof.* First we assume that  $B_1, \dots, B_m$  are linearly independent, namely,  $\alpha_1 = \dots = \alpha_m = 0$  whenever  $P(\sum_{i=1}^m \alpha_i B_i = 0) = 1$  for real numbers  $\alpha_1, \dots, \alpha_m$ .

Define  $S := \{(k, y) \in \mathbb{R}^{m+1} : 0 \leq k \leq 1, |y| = 1\}$ ,  $l := \inf_{(k,y) \in S} E[(kA + B'y)^-]^2$ . When  $(k, y) \in S$ ,  $E[(kA + B'y)^-]^2 \leq E(kA + B'y)^2 \leq 2(EA^2 + E|B|^2)$ . By virtue of the dominated convergence theorem, we conclude that  $E[(kA + B'y)^-]^2$  is continuous in  $(k, y) \in S$ . Since  $S$  is compact, there exists  $(k^*, y^*) \in S$  such that  $l = E[(k^*A + B'y^*)^-]^2$ .

If  $l = 0$ , then we claim  $k^* > 0$ . In fact, if  $k^* = 0$ , then  $E[(B'y^*)^-]^2 = 0$  which yields  $P(B'y^* \geq 0) = 1$ . However,  $E[B'y^*] = E[B']y^* = 0$ ; hence  $P(B'y^* = 0) = 1$ , violating the assumption that  $B_1, \dots, B_m$  are linearly independent. Therefore,  $k^* > 0$ . As a result,  $x^* := y^*/k^*$  is an optimal solution for (3) since it achieves the zero value of the objective function (recall that  $l = 0$ ).

If  $l > 0$ , then for any  $x \in \mathbb{R}^m$  with  $|x| \geq 1$ , we have

$$E[(A + B'x)^-]^2 = |x|^2 E\left[\left(\frac{A}{|x|} + \frac{x'}{|x|}B\right)^-\right]^2 \geq l|x|^2.$$

This shows that the function to be minimized in (3) is coercive. Furthermore, the objective function in (3) is continuous in  $x$ , hence (3) must admit optimal solutions.

Now we remove the assumption that  $B_1, \dots, B_m$  are linearly independent. If  $P(B = 0) = 1$ , then every  $x \in \mathbb{R}^m$  is optimal. If  $P(B \neq 0) > 0$ , then there is a subset of

$\{B_1, \dots, B_m\}$  whose elements are linearly independent, and every element not in this subset is a linear combination (with deterministic linear coefficients) of the elements in the subset. Without loss of generality, suppose this subset is  $\{B_1, \dots, B_k\}$  with  $k \leq m$ . Denote  $\tilde{B} := (B_1, \dots, B_k)'$ . By the preceding proof, the problem

$$\min_{y \in \mathbb{R}^k} E[(A + \tilde{B}'y)^-]^2 \quad (4)$$

admits optimal solutions, with the same optimal value as that of problem (3). Let  $y^* \in \mathbb{R}^k$  be an optimal solution to (4). Then  $x^* := ((y^*)', 0)'$  is optimal for problem (3).  $\square$

The assumption that each  $B_i$  has zero mean is crucial in Lemma 3.1 and cannot be removed in general. The following is a counter-example.

**Example 3.1** Let  $A = -1$ ,  $B = (e^{W_1}, \dots, e^{W_m})$ , where  $(W_1, \dots, W_m)$  follows  $N(0, I_m)$ . For any  $0 \neq x \in \mathbb{R}_+^m$ , it follows from the dominated convergence theorem that  $\lim_{\alpha \rightarrow +\infty} E[(A + B'(\alpha x))^-]^2 = 0$ . This implies that the optimal value of (3) is zero. However, this value cannot be achieved since  $E[(A + x'B)^-]^2 > 0$  for any  $x \in \mathbb{R}^m$ .

Notwithstanding the above example, the zero-mean condition of Lemma 3.1 can be removed if replaced by other proper conditions.

**Lemma 3.2** *Assume that  $P(B'y \geq 0) < 1$  for any  $y \in \mathbb{R}^m$  with  $|y| = 1$ , then problem (3) admits optimal solutions.*

*Proof.* We only need to consider the case  $l = 0$  in the proof in Lemma 3.1. If  $k^* = 0$ , then  $E[(B'y^*)^-]^2 = 0$  leading to  $P(B'y^* \geq 0) = 1$ . But this contradicts to the assumption. So it must hold that  $k^* \neq 0$ . The rest of the proof is the same as that of Lemma 3.1.  $\square$

**Remark 3.1** The assumption of Lemma 3.2 is mild and includes many meaningful cases. For example, it is satisfied when  $B_1, \dots, B_m$  have multivariate (nondegenerate) normal distributions (i.e., there is a finite set of mutually independent normal random variables such that each  $B_i$  is a linear combination of these normal random variables, and none of them degenerates to be non-random). More importantly, as we will see in the subsequent section, the assumption translates into the no-arbitrage condition when applied to portfolio selection.

## 4 Main Results

We first deal with the below-mean semivariance model (1).

**Theorem 4.1** For any  $a \in \mathbb{R}$  and  $z \in \mathbb{R}$ , problem (1) admits optimal solutions if and only if it admits feasible solutions.

*Proof.* Set  $R_i := \xi_i - r_i$ ,  $i = 1, 2, \dots, n$ . Then  $ER_i = 0$  and we can rewrite the problem (1) as

$$\begin{aligned} & \text{minimize} && E[(\sum_{i=1}^n x_i R_i)^-]^2, \\ & \text{subject to} && \begin{cases} \sum_{i=1}^n x_i = a, \\ \sum_{i=1}^n x_i r_i = z. \end{cases} \end{aligned} \quad (5)$$

Eliminating  $x_1$  by replacing it by  $a - \sum_{i=2}^n x_i$ , we get that problem (5) is equivalent to

$$\begin{aligned} & \text{minimize} && E[(aR_1 + \sum_{i=2}^n x_i (R_i - R_1))^+ ]^2, \\ & \text{subject to} && \sum_{i=2}^n x_i (r_i - r_1) = z - ar_1. \end{aligned} \quad (6)$$

We now consider two cases. The first case is when  $r_i = r_1$  for all  $i$ . In this case, if  $z \neq ar_1$ , then problem (1) clearly admits no feasible solution. So we assume  $z = ar_1$ . Problem (6), with its constraint satisfied automatically, becomes

$$\min_{(x_2, \dots, x_n) \in \mathbb{R}^{n-1}} E[(aR_1 + \sum_{i=2}^n x_i (R_i - R_1))^+ ]^2, \quad (7)$$

which admits optimal solutions by Lemma 3.1.

The second case is when there exists  $i$  so that  $r_i \neq r_1$ . Without loss of generality, we suppose  $r_2 \neq r_1$ . In this case, we replace  $x_2$  by  $x_2 = \frac{z - ar_1}{r_2 - r_1} - \sum_{i=3}^n x_i \frac{r_i - r_1}{r_2 - r_1}$  in problem (6) to get the following equivalent problem:

$$\min_{(x_3, \dots, x_n) \in \mathbb{R}^{n-2}} E \left[ \left( aR_1 + \frac{z - ar_1}{r_2 - r_1} (R_2 - R_1) + \sum_{i=3}^n x_i (R_i - R_1 - (r_i - r_1) \frac{R_2 - R_1}{r_2 - r_1}) \right)^+ \right]^2. \quad (8)$$

Again, the existence of optimal solutions follows immediately from Lemma 3.1.  $\square$

**Remark 4.1** The main idea of the proof of the preceding theorem is that, save for some trivial cases, the objective function of (1) is essentially coercive. So the optimization problem is equivalent to one with a bounded closed feasible region which leads to the existence of optimal solutions. Consequently, the result still holds if there are additional constraints so long as the corresponding constraint sets are closed. These include the no-shorting constraint or, more generally, polyhedral constraints.

Next we move on to the below-target semivariance model (2). For this model we need to impose the following assumption on the underlying market.

**Assumption (A)** There is no arbitrage opportunity in the market, namely, there is no  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  such that  $\sum_{i=1}^n x_i = 0$ ,  $P(\sum_{i=1}^n x_i \xi_i \geq 0) = 1$  and  $P(\sum_{i=1}^n x_i \xi_i > 0) > 0$ .

**Theorem 4.2** *Under Assumption (A), problem (2) admits optimal solutions for any  $a \in \mathbb{R}$ .*

*Proof.* First we assume that  $\xi_1, \dots, \xi_n$  are linearly independent. Eliminating  $x_1$  from the constraint of (2) we get the following equivalent problem:

$$\min_{(x_2, \dots, x_n) \in \mathbb{R}^{n-1}} E \left[ \left( a\xi_1 - b + \sum_{i=2}^n x_i (\xi_i - \xi_1) \right)^- \right]^2. \quad (9)$$

For any  $(x_2, \dots, x_n) \in \mathbb{R}^{n-1} \setminus \{0\}$ , with a slight abuse of notation define  $x_1 := -\sum_{i=2}^n x_i$ . Then by Assumption (A), either  $P(\sum_{i=1}^n x_i \xi_i \geq 0) < 1$  or  $P(\sum_{i=1}^n x_i \xi_i = 0) = 1$  holds. Since  $\xi_1, \dots, \xi_n$  are linearly independent, the latter is impossible. Hence  $P(\sum_{i=1}^n x_i \xi_i \geq 0) < 1$ , which can be rewritten as  $P(\sum_{i=2}^n x_i (\xi_i - \xi_1) \geq 0) < 1$ . It then follows from Lemma 3.2 that problem (9) admits optimal solutions.

When  $\xi_1, \dots, \xi_n$  are not linearly independent, a similar argument as in the proof of Lemma 3.1 can be established to prove the existence.  $\square$

**Remark 4.2** It is interesting to note that the attainability of optimal solutions is *unconditional* for the below-mean model (1), even when the market provides arbitrage opportunities, whereas the result for the below-target counterpart (2) requires the arbitrage-free condition. Assumption (A) can *not* be removed from Theorem 4.2. Indeed, take a market where  $\xi_1 = (b-1)/a$  (i.e., it corresponds to a fixed-income security) and  $\xi_i - \xi_1 = e^{W_i}$ ,  $i = 2, \dots, n$ , then Example 3.1 reveals that there is no optimal solution. Clearly this market is not arbitrage-free since one security deterministically dominates another one.

## 5 Extensions to General Downside Risk

In this section, we generalize the risk measure from semivariance to a general downside-risk function. We call a measurable function  $f : \mathbb{R} \rightarrow \mathbb{R}_+$  a *downside risk function* if  $f(x) = 0$  when  $x \geq 0$ , and  $f(x) > 0$  when  $x < 0$ . A typical downside risk function is  $f(x) = (x^-)^p$ ,  $p > 0$ , corresponding to the so-called lower partial moment risk measure.

Parallel to the semivariance case, we study the following two models with downside-risk measure respectively:

$$\begin{aligned} & \text{Minimize} && Ef\left(\sum_{i=1}^n x_i \xi_i - E(\sum_{i=1}^n x_i \xi_i)\right), \\ & \text{subject to} && \begin{cases} \sum_{i=1}^n x_i = a, \\ \sum_{i=1}^n x_i r_i = z, \end{cases} \end{aligned} \quad (10)$$

and

$$\begin{aligned} & \text{minimize} && Ef(\sum_{i=1}^n x_i \xi_i - b), \\ & \text{subject to} && \sum_{i=1}^n x_i = a. \end{aligned} \quad (11)$$

As before we first consider the following optimization problem:

$$\min_{x \in \mathbb{R}^m} Ef(A + B'x), \quad (12)$$

where  $B \equiv (B_1, \dots, B_m)'$ , and  $A, B_i, i = 1, \dots, m$ , are random variables. This problem is called feasible if there exists  $x_0 \in \mathbb{R}^m$  so that  $Ef(A + B'x_0) < +\infty$ .

**Lemma 5.1** *Assume*

- (i)  $EB_i = 0, i = 1, \dots, m$ ;
- (ii)  $f(\cdot)$  is lower semi-continuous;
- (iii) there exists a measurable function  $g : \mathbb{R} \rightarrow \mathbb{R}_+$  with  $\lim_{x \rightarrow +\infty} g(x) = +\infty$  such that  $f(kx) \geq g(k)f(x)$  for any  $k, x \in \mathbb{R}$ .

Then problem (12) admits optimal solutions if and only if it is feasible.

*Proof.* Suppose the problem is feasible. As before we need only to prove the existence of optimal solutions assuming that  $B_1, \dots, B_m$  are linearly independent. Define  $S := \{(k, y) \in \mathbb{R}^{m+1} : 0 \leq k \leq 1, |y| = 1\}$ ,  $l := \inf_{(k,y) \in S} Ef(kA + B'y) < +\infty$ .

Now we prove that there exists a  $(k^*, y^*) \in S$  such that  $l = Ef(k^*A + B'y^*)$ . Suppose  $\{(k_i, y_i) : i = 1, 2, \dots\}$  is a minimizing sequence for  $\inf_{(k,y) \in S} Ef(kA + B'y)$ . Since  $S$  is compact, there exists a convergent subsequence of  $\{(k_i, y_i) : i = 1, 2, \dots\}$ . Without loss of generality, suppose  $\lim_{i \rightarrow +\infty} (k_i, y_i) = (k^*, y^*) \in S$ . Then

$$\begin{aligned} Ef(k^*A + B'y^*) &= Ef(\lim_{i \rightarrow +\infty} (k_i A + B'y_i)) \\ &\leq E \underline{\lim}_{i \rightarrow +\infty} f(k_i A + B'y_i) \\ &\leq \underline{\lim}_{i \rightarrow +\infty} Ef(k_i A + B'y_i) \\ &= l, \end{aligned} \quad (13)$$



where the last inequality is due to Fatou's lemma. This shows  $l = Ef(k^*A + B'y^*)$ .

If  $l = 0$ , then  $Ef(k^*A + B'y^*) = 0$  leading to  $P(k^*A + B'y^* \geq 0) = 1$ . On the other hand, an argument exactly the same as that in the proof of Lemma 3.1 yields  $k^* > 0$ . Thus  $P(A + B'\frac{y^*}{k^*} \geq 0) = 1$  which results in  $Ef(A + B'\frac{y^*}{k^*}) = 0 = l$ . This suggests that  $x^* := \frac{y^*}{k^*}$  is an optimal solution for (12) since it achieves the zero (optimal) value of the objective function.

If  $l > 0$ , then for any  $x \in \mathbb{R}^m \setminus \{0\}$ , we have

$$Ef(A + B'x) \geq g(|x|)Ef\left(\frac{A}{|x|} + \frac{x'}{|x|}B\right) \geq l \cdot g(|x|).$$

This shows that  $Ef(A + B'x)$  is coercive, and hence problem (12) is equivalent to one with a closed bounded feasible region. A similar argument as in (13) then establishes the desired existence.  $\square$

**Lemma 5.2** *In Lemma 5.1, the condition (i) can be replaced by*

(i')  $P(B'y \geq 0) < 1$  for any  $y \in \mathbb{R}^m$  with  $|y| = 1$ .

*Proof.* It can be proved in the same way as that of Lemma 3.2.  $\square$

**Theorem 5.1** *Assume*

(i)  $f(\cdot)$  is lower semi-continuous;

(ii) there exists a measurable function  $g : \mathbb{R} \rightarrow \mathbb{R}_+$  with  $\lim_{x \rightarrow +\infty} g(x) = +\infty$  such that  $f(kx) \geq g(k)f(x)$  for any  $k, x \in \mathbb{R}$ .

*Then for any  $a \in \mathbb{R}$  and  $z \in \mathbb{R}$ , problem (10) admits optimal solutions if and only if it admits feasible solutions.*

*Proof.* Given Lemma 5.1, the proof is the same as that of Theorem 4.1.  $\square$

**Remark 5.1** Both conditions (i) and (ii) imposed on the risk function  $f$  in the preceding theorem are rather mild. For example, the lower partial moment risk  $f(x) = (x^-)^p$ ,  $p > 0$ , satisfies these conditions as long as the  $p$ -th moment of each return  $\xi_i$  is finite. Note that the conditions also imply that  $\lim_{x \uparrow 0} f(x) = 0$ ,  $\lim_{x \rightarrow -\infty} f(x) = +\infty$ .

In a similar fashion, we have the following result concerning the below-target downside risk model (5.2).

**Theorem 5.2** *Under Assumption (A) and the same conditions (i) and (ii) of Theorem 5.1, problem (5.2) admits optimal solutions.*

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