# Multiple-objective risk-sensitive control

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## Abstract

This paper is concerned with a (minimizing) multiple-objective risk-sensitive control problem. Asymptotic analysis leads to the introduction of a new class of two-player, zero-sum, deterministic differential games. The distinguishing feature of this class of games is that the cost functional is multiple-objective in nature, being composed of the risk-neutral integral costs associated with the original risk-sensitive problem. More precisely, the opposing player in such a game seeks to maximize the most 'vulnerable' member of a given set of cost functionals while the original controller seeks to minimize the worst 'damage' that the opponent can do over this set. It is then shown that the problem of finding an efficient risk-sensitive controller is equivalent, asymptotically, to solving this differential game. Surprisingly, this differential game is proved to be *independent* of the weights on the different objectives in the original multiple-objective risk-sensitive problem. As a by-product, our results generalize the existing results for the single-objective risk-sensitive control problem to a substantially larger class of nonlinear systems, including those with control-dependent diffusion terms.

# 1 Introduction

The distinguishing feature of risk-sensitive control problems is that cost functionals involve the expectation of an exponential where the exponent of this exponential is the cost functional of a standard (risk-neutral) stochastic control problem. One consequence of the exponential term is that larger values of the exponent are weighted more heavily. For this reason, robust (or risk-averse) controllers can be obtained by minimizing the risk-sensitive cost. Another important property of the risk-sensitive control problem is its relationship with the class of two-player, zero-sum, deterministic differential games associated with the so-called $H_\infty$ control problem. In this setting, the controller (for the risk-sensitive problem) takes the part of the minimizing player in the differential game while the opponent may be interpreted as a worst case disturbance. As a consequence, robustness issues for linear, nonlinear and stochastic systems can be studied in the framework of risk-sensitive control as well as differential games. For further details of such interpretations of the $H_\infty$ control problem, we refer the reader to [4, 11, 18].

In this paper, we study (minimizing) risk-sensitive control problems with multiple objectives. Multi-objective optimization problems arise naturally in practice. For example, the classical Markowitz mean–variance model for portfolio selection [25] is a typical multi-objective optimization problem, with two competing objectives: return and risk. A single-objective optimization problem with several constraints may also be formulated as a multi-objective problem, with the constraints functions regarded as objective functions. Here, we are particularly interested in the relationship between a given class of multiple-objective risk-sensitive control problems and a new class of deterministic differential games, which is to be introduced in this paper. To make it precise, asymptotic analysis leads to the introduction of a class of two-player, zero-sum, deterministic differential games. The distinguishing feature of this new class of problems is that its cost is multiple-objective in nature, being comprised of the risk-neutral integral costs associated with the original risk-sensitive problem. Specifically, in this differential game a set of finitely many cost functionals is given. Rather than seeking to maximize a single cost functional (as in the usual differential game), the opponent seeks to maximize the largest member of this set of cost functionals - that is, the opponent attacks the most vulnerable member of the set, and disregards the others. On the other hand, the original controller tries to minimize the worst damage that the opponent can do. Naturally, this general situation arises in many applications. We show that this problem, from the viewpoint of the original controller, is a multiple-objective optimization problem. Furthermore, as the small noise parameter $\epsilon \to 0$, the value function for the multiple-objective risk-sensitive problem converges to the upper value associated with the aforementioned differential game. More interestingly, we show that finding efficient solutions for *different* performance tradeoffs

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is equivalent, asymptotically, to solving the *same* deterministic differential game.

We also point out that our results apply to a large class of nonlinear stochastic systems under reasonably mild assumptions. In particular, our work incorporates some fairly recent advances on stochastic control in which the diffusion term may depend on the state and/or the control. For a complete account of the latest results on this topic, the reader is directed to the book [26]. We highlight that control dependence of the diffusion term gives rise to interesting phenomena which reveals some essential differences between deterministic and stochastic control problems; for instance, in the stochastic linear quadratic problem, the control weighting matrix may be negative definite, but the problem remain well posed [7]. In addition, stochastic optimal control problems with control dependent diffusions arise naturally in many finance applications; see, for example, [5, 13, 23, 26]. Also, unlike the analysis in [19] for the single-objective problem, we need not assume that the control enters linearly in the drift (or diffusion) and certain smoothness assumptions on the value function are not required in the analysis. In fact, one consequence of our analysis is that many of the results in [19] for the single objective problem can be generalized to a substantially larger class of nonlinear systems under milder assumptions.

The paper is organized as follows. In Section 2, we formulate multiple-objective risk-sensitive control problems, and in Section 3, we introduce a new class of deterministic differential games that are closely related to the multiple-objective risk-sensitive problems. In Section 4, a transformation that is required in the subsequent asymptotic analysis is discussed. In Section 5, we study the asymptotic properties of the value function associated with the multiple-objective risk-sensitive problem and show that it converges to the value function of the deterministic differential game that was introduced in Section 3. Finally, Section 6 gives some concluding remarks. The Appendix highlights results from nonsmooth analysis and the viscosity solution theory that are used in the paper.

# 2 Multiple-objective risk-sensitive control

## 2.1 Multiple-objective optimization

A fundamental problem in many applications is finding a control input that meets the requirements of several competing objectives, simultaneously. Suppose that each of $l + 1$ objectives in a certain application is represented by an objective function $J_i(\cdot)$, $i = 0, \cdots, l$. Suppose that by lowering the value of a particular objective functional, the corresponding objective is more successfully met. An input $u$ is said to be *efficient* if none of the objective functionals $J_0(u), \cdots, J_l(u)$ can be decreased any further without increasing one of the other cost functionals. The vector of objective functionals resulting from an efficient input is said to belong

to the *efficient frontier.*

Two major issues in multiple-objective optimization are:

1. Constructing the efficient frontier (useful for determining which trade-offs are possible);

2. Finding the efficient inputs which correspond to points on the efficient frontier.

Given a constraint set $\Omega$ and objective functionals $J_i : \Omega \to \mathbb{R}$, for $i = 0, \cdots, l$, the problem of finding an efficient input is commonly expressed in the following way:

$$\min_{u \in \Omega} \{ J_0(u), \cdots, J_l(u) \}. \tag{1}$$

In the case when the objective functionals are convex and $\Omega$ is convex, efficient inputs and points on the efficient frontier are *completely* parametrized as follows: $u^* \in \Omega$ is efficient if and only if there exists $\lambda = (\lambda_0, \cdots, \lambda_l) \in \mathbb{R}^{l+1}$, $\lambda \geq 0$, $\lambda \neq 0$ such that $u^*$ is the optimal solution of the single-objective problem

$$\min_{u \in \Omega} L(\lambda; u) := \sum_{i=0}^{l} \lambda_i J_i(u). \tag{2}$$

On the other hand, when the problem is not convex, the optimization problem (2) only characterizes a *subset* of the efficient frontier. In this case, to identify the entire efficient frontier, one needs to resort to different approaches, such as transforming to a single-objective minimax problem or a single-objective problem with constraints. However, the equivalent new problems are hard to solve due to the non-convex nature of the problem. For a detailed discussion, the reader may consult [6, 27].

**Remark 2.1** Our focus throughout this paper is finding the efficient inputs for (1) which can be parametrized according to (2); that is, solving (2) for a given and fixed $\lambda \geq 0$, $\lambda \neq 0$. In this situation, we may assume, without loss of generality, that $\lambda_i > 0$ for every $i = 0, \cdots, l$. More specifically, having $\lambda_k = 0$ for some $k$ is the same as removing $J_k(\cdot)$ from the objective set and solving (2) for a problem with fewer objectives but with $\lambda_i > 0$ for each one of them. Hence, we shall assume for the remainder of this paper that $\lambda_i > 0$ for every $i$.

## 2.2   Statement of multi-objective risk-sensitive problem

In this subsection we formulate the multiple-objective risk-sensitive control problem under consideration in this paper. Since we shall be using dynamic programming, we shall work within the framework of the weak formulation for stochastic control (see [26]) which we shall summarize later for reader's convenience. Let $(s, x) \in [0, T) \times \mathbb{R}^n$ be given and fixed. Consider the following controlled SDE:

$$\begin{cases} dx(t) = b(t, x(t), u(t))\, dt + \sqrt{\frac{\epsilon}{2\gamma^2}}\, \sigma(t, x(t), u(t))\, dB(t) \\ \\ x(s) = x \end{cases} \tag{3}$$

where $\epsilon > 0$ and $\gamma > 0$ are parameters and $u(\cdot)$ is the $U$-valued control input. Note in particular that unlike the model studied in [19], the drift may be nonlinear in $u$, and the diffusion may depend on $u$. The Brownian motion $B(t)$ is assumed to be scalar-valued throughout this paper. Note, however, that this is only for the sake of simplicity and a multi-dimensional Brownian motion renders no essential difficulty.

For every $i = 0, \cdots, l$, we have an associated risk-sensitive cost functional, which is defined as follows:

$$J_i^\epsilon(s, x; u(\cdot)) = E\left\{\exp\frac{1}{\epsilon}\left[\int_s^T f^i(t, x(t), u(t))\, dt + g^i(x(T))\right]\right\}. \tag{4}$$

We introduce the following assumptions:

**Assumption:**

($\mathbf{A1}$) $U \subseteq \mathbb{R}^m$ is compact, and $T > 0$.

($\mathbf{A2}$) The maps $b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $f^i : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $g^i : \mathbb{R}^n \to \mathbb{R}$, $i = 0, \cdots, l$, are uniformly continuous and bounded. Also, there exists a constant $L > 0$ such that for $\varphi(t, x, u) = b(t, x, u)$, $\sigma(t, x, u)$, $f^i(t, x, u)$, $g^i(x)$,

$$|\varphi(t, x, u) - \varphi(t, y, u) \le L\,|x - y|, \quad \forall t \in [0, T], \ x, y \in \mathbb{R}^n, \ u \in U. \tag{5}$$

Our aim is to find a so-called *efficient control* associated with (3)-(4). The class of admissible controls (in the weak formulation) is as follows. (For more details, the reader is directed to the books [21, 26]). For every fixed $s \in [0, T)$, let $\mathcal{U}[s, T]$ denote the set of 5-tuples $(\Omega, \mathcal{F}, P, B(\cdot), u(\cdot))$ which satisfy the following properties:

1. $(\Omega, \mathcal{F}, P)$ is a complete probability space;

2. $\{B(t)\}_{t \ge s}$ is a 1-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$ over $[s, T]$, and $\mathcal{F}_t^s = \sigma\{B(r)\,|\,s \le r \le t\}$, augmented by all the $P$-null sets in $\mathcal{F}$;

3. $u : [s, T] \times \Omega \to U$ is an $\{F_t^s\}_{t \ge s}$-progressively measurable process on $(\Omega, \mathcal{F}, P)$;

4. Under $u(\cdot)$, for any initial condition $x \in \mathbb{R}^n$, the SDE (3) admits a unique solution $x(\cdot)$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t^s\}_{t \ge s}, P)$, and the integrals in the cost functionals (4) are well-defined with this solution.

If it is clear from the context what $(\Omega, \mathcal{F}, P)$ and $B(\cdot)$ are, we will write $u(\cdot) \in \mathcal{U}[s, T]$ as shorthand for 5-tuple $(\Omega, \mathcal{F}, P, B(\cdot), u(\cdot)) \in \mathcal{U}[s, T]$.

Let $\lambda \in \mathbb{R}^{l+1}$, $\lambda_i > 0$ be given and fixed. For every such $\lambda$, consider the cost functional

$$L^\epsilon(s,\, x,\, \lambda;\, u(\cdot)) = \sum_{i=0}^l \lambda_i\, J_i^\epsilon(s,\, x;\, u(\cdot)). \tag{6}$$

In this paper we shall focus on solving problems of the following form:

**Problem:** For a given $(s, x) \in [0, T) \times \mathbb{R}^n$ and $\lambda \in \mathbb{R}^{l+1}$, $\lambda_i > 0$, find a 5-tuple $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \bar{B}(\cdot), \bar{u}(\cdot)) \in \mathcal{U}[s, T]$ such that:

$$L^\epsilon(s, x, \lambda; \bar{u}(\cdot)) = \min_{u(\cdot) \in \mathcal{U}[s,T]} L^\epsilon(s, x, \lambda; u(\cdot)). \tag{7}$$

It should be noted that the original multiple-objective optimal control problem associated with (3)-(4) is to find, for every given $(s, x) \in [0, T) \times \mathbb{R}^n$, a 5-tuple $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \bar{B}(\cdot), \bar{u}(\cdot)) \in \mathcal{U}[s, T]$ which is efficient for $J_0^\epsilon(s, x; u(\cdot)), \cdots, J_l^\epsilon(s, x; u(\cdot))$. As mentioned earlier, in the case when the cost functionals $J_i^\epsilon(s, x; u(\cdot))$ are convex in $u(\cdot)$ and the SDE (3) is linear in $x(\cdot)$ and $u(\cdot)$ (a typical such case is the so-called linear quadratic control), the entire efficient frontier associated with (3)-(4) can be traced out by solving the optimal control problem (7) with different $\lambda \in \mathbb{R}^{l+1}$, $\lambda \geq 0$, $\lambda \neq 0$. On the other hand, if the problem is not convex, the optimal controls to (7) will generally only give rise to a subset of the efficient frontier. However, (7) still plays a key role in the multiple-objective problem, and we shall focus our attention on problems of this form.

In the papers [22, 24], the multiple-objective linear quadratic (LQ) control problem is studied. Since a linear combination of convex quadratic functionals is also a convex quadratic functional, the Lagrangian $L^\epsilon(s, x, \lambda; u(\cdot))$ resulting from integral quadratic functionals $J_i^\epsilon(s, x; u(\cdot))$ is also a convex integral quadratic functional. Therefore, in the LQ case, (7) is a standard LQ problem, and efficient controls are completely obtained by solving LQ problems. On the other hand, a linear combination of risk-sensitive cost functionals $J_i^\epsilon(s, x; u(\cdot))$ can not be reduced to something simpler. It seems that elegant solutions can not be obtained, even in the simplest cases. This may account for the fact that (to our knowledge) no earlier work has been done on this problem.

## 2.3    Hamilton-Jacobi-Bellman equation

In this subsection, we present the Hamilton-Jacobi-Bellman (HJB) equation associated with the problem consisting of (3) and (7).

For technical reasons (the requirement of bounded terminal conditions in the theory of viscosity solutions), it is convenient to replace the cost functional (4) with an equivalent expression which we now introduce. By (**A2**), the functionals $f^i$ are uniformly bounded. Hence, there is a constant $K < \infty$ such that

$$\left| \int_s^T f^i(t, x(t), u(t)) \, dt \right| \leq K, \quad \forall (s, x) \in [0, T) \times \mathbb{R}^n, \ u(\cdot) \in \mathcal{U}[s, T]. \tag{8}$$

Let $\Psi : \mathbb{R} \to \mathbb{R}$ be any smooth, uniformly Lipschitz functional such that:

$$\Psi(y) = \begin{cases} 2K, & y \in (2K, \infty), \\ y, & y \in [-K, K], \\ -2K, & y \in (-\infty, -2K). \end{cases} \tag{9}$$

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Due to the uniform bound (8), we may replace (4) by:

$$J_i^\epsilon(s, x; u(\cdot)) = E\left\{\exp\frac{1}{\epsilon}\left[\Psi\left(\int_s^T f^i(t, x(t), u(t))\, dt\right) + g^i(x(T))\right]\right\}. \tag{10}$$

In order to study (7) (with (4) replaced by (10)) using dynamic programming, we make the following problem transformation. For convenience, we shall denote $y(t) = [y_0(t), \cdots, y_l(t)]'$ with $f(t, x, u)$ defined similarly. For every $(s, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{l+1}$, consider the optimal control problem with dynamics:

$$\begin{cases} dx(t) = b(t, x(t), u(t))\, dt + \sqrt{\frac{\epsilon}{2\gamma^2}}\, \sigma(t, x(t), u(t))\, dB(t) \\[2mm] dy(t) = f(t, x(t), u(t))\, dt \\[2mm] x(s) = x, \; y(s) = y \end{cases} \tag{11}$$

and cost:

$$L^\epsilon(s, x, y, \lambda; u(\cdot)) = E\left\{\sum_{i=0}^l \lambda_i \exp\frac{1}{\epsilon}\left[g^i(x(T)) + \Psi(y_i(T))\right]\right\}. \tag{12}$$

The HJB equation associated with (11)-(12) is:

$$\begin{cases} v_t^\epsilon(t, x, y) + \inf_u\left\{v_x^\epsilon(t, x, y)\cdot b(t, x, u) + \sum_{i=0}^l v_{y_i}^\epsilon(t, x, y)\cdot f^i(t, x, u)\right. \\[2mm] \left. + \frac{\epsilon}{4\gamma^2}\, \mathrm{tr}\left[\sigma(t, x, u)\, \sigma(t, x, u)'\, v_{xx}^\epsilon(t, x, y)\right]\right\} = 0, \quad (t, x, y) \in [0, T) \times \mathbb{R}^n \times \mathbb{R}^{l+1} \\[2mm] v^\epsilon(T, x, y) = \sum_{i=0}^l \lambda_i \exp\frac{1}{\epsilon}\left[g^i(x) + \Psi(y_i)\right], \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^{l+1}. \end{cases} \tag{13}$$

We have the following result.

**Theorem 2.1** *Suppose that* (**A1**) *and* (**A2**) *hold. Then*

$$v^\epsilon(s, x, y) = \inf_{u(\cdot) \in \mathcal{U}[s, T]} L^\epsilon(s, x, y, \lambda; u(\cdot)), \quad (s, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{l+1} \tag{14}$$

*is the unique bounded viscosity solution of (13), where $L^\epsilon(s, x, y, \lambda; u(\cdot))$ is given by (12) and $(x(\cdot), y(\cdot))$ is the unique solution of (11) corresponding to $u(\cdot) \in \mathcal{U}[s, T]$.*

**Proof:** Since (**A1**) and (**A2**) hold and the terminal cost (12) is uniformly bounded, from [26, Theorems 5.2 and 6.1] the result follows immediately. ∎

Throughout this paper, for any given and fixed $\lambda \in \mathbb{R}^{l+1}$ such that $\lambda_i > 0$, we shall refer to $v^\epsilon(s, x, y)$ as defined by (14) as the *value function* associated with (11)-(12).

# 3   A differential game

In this section, we introduce a class of two-player, zero-sum, deterministic differential games that are closely related to the multiple-objective risk-sensitive problems formulated in the previous section. We shall adopt the Elliott-Kalton formulation for this problem; see [12, 16], for example. As will be shown later this family of differential games is closely related to the multiple-objective risk sensitive problem, introduced in Section 2.2, and is characterized by a non-standard cost functional ((16) below). We assume throughout that the functions $b$, $\sigma$ and $f^i$ satisfy (**A2**).

Suppose that the system dynamics are governed by the ordinary differential equation (ODE):

$$\begin{cases} \dot{x}(t) = b(t, x(t), u(t)) + \sigma(t, x(t), u(t))\, w(t) \\[2mm] x(s) = x, \end{cases} \tag{15}$$

where $u(\cdot)$ is the $U$-valued input of player 1 (the control player), and $w(\cdot)$ is the $W$-valued input of player 2 (the disturbance player, or opponent). Let $\lambda \in \mathbb{R}^{l+1}$, $\lambda_i > 0$ be given and fixed (see Remark 2.1). The cost associated with inputs $u(\cdot)$ and $w(\cdot)$ is given by:

$$\begin{aligned} J&(s, x; u(\cdot), w(\cdot)) \\ &= \max_i \left\{ \int_s^T f^i(t, x(t), u(t))\, dt + g^i(x(T)) - \gamma^2 \int_s^T |w(t)|^2\, dt \right\}. \end{aligned} \tag{16}$$

The set of *admissible controls/disturbances* for players 1 and 2 are:

$$\mathcal{U}_d[s, T] = \left\{ u(\cdot) : [s, T] \to U \,\Big|\, u(\cdot) \text{ is } \mathcal{B}[s, T] - \text{measurable} \right\}, \tag{17}$$

$$\mathcal{W}_d[s, T] = \left\{ w(\cdot) : [s, T] \to W \,\Big|\, w(\cdot) \text{ is } \mathcal{B}[s, T] - \text{measurable} \right\}, \tag{18}$$

respectively, where $\mathcal{B}[s, T]$ denotes the Borel $\sigma$-algebra on $[s, T]$. We make the following assumption:

**Assumption:**

(**A1**)$'$ $U \subseteq \mathbb{R}^m$ and $W \subseteq \mathbb{R}$ are compact, and $T > 0$.

In addition, for any $t \in (s, T]$, two admissible inputs $u_1(\cdot)$, $u_2(\cdot) \in \mathcal{U}_d[s, T]$, are said to be *equivalent* on $[s, T]$ if $u_1(\cdot) = u_2(\cdot)$ on $[s, t]$. We shall denote this by $u_1(\cdot) \approx u_2(\cdot)$ on $[s, t]$. (Similarly for disturbances). We note that under (**A1**)$'$ and (**A2**), the ODE (15) has a unique solution $x(\cdot)$ for every $u(\cdot) \in \mathcal{U}_d[s, T]$ and $w(\cdot) \in \mathcal{W}_d[s, T]$.

We are interested in the *upper differential game* associated with (15)-(16), which calls for the definition of a *strategy* for player 2. The class of *admissible strategies* for player 2 is:

$$\begin{aligned} \Gamma_d[s, T] = \Big\{ \alpha : \mathcal{U}_d[s, T] \to \mathcal{W}_d[s, T] \,\Big|\, &\text{for every } t \in [s, T] \\ u_1(\cdot) \approx u_2(\cdot) \text{ on } [s, t] \Rightarrow &\alpha[u_1(\cdot)] \approx \alpha[u_2(\cdot)] \text{ on } [s, t] \Big\}. \end{aligned} \tag{19}$$

The *upper differential game* associated with (15)-(16) can be stated as follows: Find $(\bar{u}(\cdot), \bar{w}(\cdot)) \in \mathcal{U}_d[s, T] \times \mathcal{W}_d[s, T]$ such that

$$V(s, x) := J(s, x; \bar{u}(\cdot), \bar{w}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}_d[s,T]} \sup_{\alpha[\cdot] \in \Gamma_d[s,T]} J(s, x; u(\cdot), \alpha[u(\cdot)]). \qquad (20)$$

$V$ is commonly referred to as the *upper value* of the differential game (15)-(16).

As in the risk-sensitive case, the boundedness of $f^i$ in (**A2**) allows us to replace (16) with an equivalent expression involving the function $\Psi(\cdot)$ as defined by (9). To employ the dynamic programming approach to the differential game just introduced, let $y \in \mathbb{R}^{l+1}$ be given and consider the ODE:

$$\begin{cases} \dot{x}(t) = b(t, x(t), u(t)) + \sigma(t, x(t), u(t)) \, w(t) \\[2mm] \dot{y}(t) = f(t, \, x(t), \, u(t)) \\[2mm] x(s) = x, \; y(s) = y \end{cases} \qquad (21)$$

and cost:

$$J(s, \, x, \, y; u(\cdot), w(\cdot)) = \max_i \left\{ \Psi(y_i(T)) + g^i(x(T)) - \gamma^2 \int_s^T |w(t)|^2 \, dt \right\}. \qquad (22)$$

Clearly, (15)-(16) corresponds to the special case $y = 0$ in (21)-(22). The *upper Isaacs equation* associated with (21)-(22) is:

$$\begin{cases} V_t + \inf_{u \in U} \sup_{w \in W} \left\{ (b + \sigma \, w) \cdot V_x + \sum_{i=0}^l f^i \, V_{y_i} - \gamma^2 \, |w|^2 \right\} = 0, \\[2mm] \quad (t, \, x, \, y) \in [0, T) \times \mathbb{R}^n \times \mathbb{R}^{l+1} \\[2mm] V(T, x, y) = \max_i \left\{ g^i(x) + \Psi(y_i) \right\}, \quad (x, \, y) \in \mathbb{R}^n \times \mathbb{R}^{l+1}. \end{cases} \qquad (23)$$

We have the following characterization of the solution of (23).

**Theorem 3.1** *Under* (**A1**)$'$ *and* (**A2**),

$$V(s, x, y) = \inf_{u(\cdot) \in \mathcal{U}_d[s,T]} \sup_{\alpha[\cdot] \in \Gamma_d[s,T]} J(s, \, x, \, y; u(\cdot), \, \alpha[u(\cdot)]), \quad (s, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{l+1} \qquad (24)$$

*is the unique viscosity solution of (23), where $J(s, \, x, \, y; u(\cdot), \, w(\cdot))$ is given by (22) and $(x(\cdot), \, y(\cdot))$ is the solution of (21) corresponding to $(u(\cdot), \, w(\cdot)) \in \mathcal{U}_d[s, T] \times \mathcal{W}_d[s, T]$.*

We note that (15)-(16) (similar comments apply to (21)-(22)) is a differential game problem in which player 2 chooses an input $w(\cdot) \in \mathcal{W}_d[s, T]$ to maximize the cost $J(s, x; u(\cdot), w(\cdot))$ by making the largest of the terms

$$J_i(s, x; u(\cdot), w(\cdot)) = \int_s^T f^i(t, x(t), u(t)) \, dt + g^i(x(T)) - \gamma^2 \int_s^T |w(t)|^2 dt$$

as large as possible; that is, intuitively, $w(\cdot)$ inflicts damage by attacking the cost functional $J_i(s, x; u(\cdot), w(\cdot))$ that it can do most harm to (the most vulnerable one), and ignores the others. On the other hand, player 1 chooses the input $u(\cdot) \in \mathcal{U}_d[s, T]$ which minimizes the worst damage that $w(\cdot)$ can do over all these $J_i(s, x; u(\cdot), w(\cdot))$.

One important observation is that the optimal input $\bar{u}(\cdot) \in \mathcal{U}[s, T]$ for player 1 is also an efficient input for a certain multiple-objective control problem. To see this, consider the following. For every $u(\cdot) \in \mathcal{U}[s, T]$, let

$$J_i^*(s, x; u(\cdot)) := \sup_{\alpha[\cdot] \in \Gamma_d[s, T]} J_i(s, x; u(\cdot), \alpha[u(\cdot)]).$$

Then the upper value of the game (15)-(16) is

$$\inf_{u(\cdot) \in \mathcal{U}_d[s, T]} \sup_{\alpha[\cdot] \in \Gamma_d[s, T]} J(s, x; u(\cdot), \alpha[u(\cdot)])$$

$$= \inf_{u(\cdot) \in \mathcal{U}_d[s, T]} \sup_{\alpha[\cdot] \in \Gamma_d[s, T]} \max_i \left\{ J_i(s, x; u(\cdot), \alpha[u(\cdot)]) \right\}$$

$$= \inf_{u(\cdot) \in \mathcal{U}_d[s, T]} \max_i \left\{ \sup_{\alpha[\cdot] \in \Gamma_d[s, T]} J_i(s, x; u(\cdot), \alpha[u(\cdot)]) \right\}$$

$$= \inf_{u(\cdot) \in \mathcal{U}_d[s, T]} \max_i \left\{ J_i^*(s, x; u(\cdot)) \right\}. \tag{25}$$

It is easy to show that the optimal control $u^*(\cdot)$ for the min-max problem (25) is efficient for the multiple-objective problem with objectives $J_i^*(s, x; u(\cdot))$. For this reason, the deterministic differential games (15)-(16) and (21)-(22) are multiple-objective in nature.

# 4  Transformation

To show the relationship between the multiple-objective risk-sensitive problem (11)-(12) and the differential game (21)-(22), we introduce the following transformation:

$$V^\epsilon(t, x, y) = \epsilon \ln v^\epsilon(t, x, y). \tag{26}$$

In this section, we prove several results on the properties of $V^\epsilon$. These are required in our study of the asymptotic behavior of $V^\epsilon$ and the correspondence between the multiple-objective risk-sensitive problem and the differential games. The first result shows that the super- and sub-differentials of $V^\epsilon$ are uniformly bounded. However, the proof of this result requires the following additional assumption.

**Assumption:**

(**A3**) $\sigma(t, x, u) = \sigma(t, u)$, and $b(t, x, u)$, $f^i(t, x, u)$ $i = 0, \cdots, l$ are differentiable in $x$.

**Proposition 4.1** *Suppose that Assumptions* (**A1**), (**A2**) *and* (**A3**) *hold. Let* $V^\epsilon(s, x, y)$ *be defined by (26), and* $D_{t,(x,y)}^{1,2,+} V^\epsilon(s, x, y)$ *and* $D_{t,(x,y)}^{1,2,-} V^\epsilon(s, x, y)$, *the super- and sub-differentials,*

*be defined as in (51) and (52) in the Appendix. Then there exists $K < \infty$, independent of $(s, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{l+1}$ and $\epsilon > 0$, such that $|p| \leq K$, for all $(q, p, P) \in D^{1,2,+}_{t,(x,y)} V^\epsilon(s, x, y) \cup D^{1,2,-}_{t,(x,y)} V^\epsilon(s, x, y)$.*

**Proof:** Let $\epsilon > 0$ and $(s, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{l+1}$ be given, and $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{u}(\cdot))$ be an admissible triple for (11) with the initial condition $(s, x, y)$. For any $z \in \mathbb{R}^n$, let $(x(\cdot), y(\cdot), \bar{u}(\cdot))$ be an admissible triple for (11), corresponding to the initial condition $(s, x + z, y)$. It follows that:

$$x(t) - \bar{x}(t) = \Gamma(t)\, z, \tag{27}$$

where the $\mathbb{R}^{n \times n}$-valued process $\Gamma(\cdot)$ is the solution of the SDE:

$$\begin{cases} d\Gamma(t) = [b_x(t, \bar{x}(t), \bar{u}(t)) + \Sigma_1(t)]'\, \Gamma(t)\, dt \\ \Gamma(s) = I \end{cases} \tag{28}$$

and

$$\Sigma_1(t) = \int_s^t [b_x(r, \bar{x}(r) + \alpha\, (x(r) - \bar{x}(t)), \bar{u}(t)) - b_x(r, \bar{x}(r), \bar{u}(t))]\, dr.$$

In addition,

$$y(t) - \bar{y}(t) = \Big( \int_s^t [f_x(r, \bar{x}(r), \bar{u}(r)) + \Sigma_2(r)]\, \Gamma(r)\, dr \Big)'\, z = \Phi(t)\, z \tag{29}$$

where $\Sigma_2(\cdot)$ is defined like $\Sigma_1(\cdot)$, but with $f_x$ replacing $b_x$. Since $b$ is Lipschitz continuous in $x$, uniformly in $(t, u) \in [0, T] \times U$, it follows from Gronwall's inequality that there exists a constant $K > 0$ (which is independent of $(s, x, y)$ and $\epsilon > 0$) such that:

$$|\Phi(t)| \leq K, \quad |\Gamma(t)| \leq K, \quad \forall t \in [s, T], \quad P - a.s. \tag{30}$$

In particular, the bound (30) (together with the uniform Lipschitz continuity of $f$ in $x$) implies that

$$|x(T) - \bar{x}(T)| \leq K\, |z|, \quad |y(T) - \bar{y}(T)| \leq K\, |z|. \tag{31}$$

Now, fix $(s, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{l+1}$, $z \in \mathbb{R}^n$ and $\epsilon > 0$. For any $\delta > 0$, there exists an admissible triple $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{u}(\cdot))$, possibly depending on $(s, x, y)$, $z$, $\epsilon$, and $\delta$, such that:

$$V^\epsilon(s, x, y) \geq J^\epsilon(s, x, y; \bar{u}(\cdot)) - \delta,$$

where

$$J^\epsilon(s, x, y; u(\cdot)) = \epsilon \ln E \left\{ \sum_{i=0}^l \lambda_i \exp \frac{1}{\epsilon}\, [g^i(x(T)) + \Psi(y_i(T))] \right\}$$

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and $(x(\cdot), y(\cdot))$ is the state process obtained from (11) when the input is $u(\cdot)$. Therefore,

$$V^{\epsilon}(s, x + z, y) - V^{\epsilon}(s, x, y) \leq J^{\epsilon}(s, x + z, y; \bar{u}(\cdot)) - J^{\epsilon}(s, x, y; \bar{u}(\cdot)) + \delta. \qquad (32)$$

It is easy to show that:

$$
\begin{aligned}
&J^{\epsilon}(s, x + z, y; \bar{u}(\cdot)) - J^{\epsilon}(s, x, y; \bar{u}(\cdot)) \\
&= \frac{1}{E\left\{\sum_{i=0}^{l} \lambda_i \exp \frac{1}{\epsilon}[g^i(x(T)) + \Psi(y_i(T))]\right\}} \\
&\quad \times E\left[\left\{\sum_{i=0}^{l} \lambda_i \exp \frac{1}{\epsilon}[g^i(x(T)) + \Psi(y_i(T))]\begin{bmatrix} g_x^i(\bar{x}(T)) \\ \Psi_y(\bar{y}_i(T)) \end{bmatrix}\right\}' \begin{bmatrix} x(T) - \bar{x}(T) \\ y(T) - \bar{y}(T) \end{bmatrix}\right] \\
&\quad + o(|x(T) - \bar{x}(T)| + |y(T) - \bar{y}(T)|).
\end{aligned}
$$

Substituting (27), (29) into this expression, and noting (30) as well as the Lipschitz continuity of $g^i$ and $\Psi$, it follows that:

$$
\begin{aligned}
&V^{\epsilon}(s, x + z, y) - V^{\epsilon}(s, x, y) \\
&\leq \quad J^{\epsilon}(s, x + z, y; \bar{u}(\cdot)) - J^{\epsilon}(s, x, y; \bar{u}(\cdot)) + \delta \leq K|z| + o(|z|) + \delta \qquad (33)
\end{aligned}
$$

for some constant $K < \infty$, which is independent of $\bar{u}(\cdot)$, $(s, x, y)$, $z$, $\epsilon > 0$ and $\delta > 0$. Since this is true for all $\delta > 0$, we can let $\delta \to 0$ in (33). In a similar way, the reverse inequality can be obtained. Hence, it follows from (33) and (59) (see Appendix) that $|p| \leq K$ for all $p \in \partial_x V^{\epsilon}(s, x, y)$, where $\partial_x V^{\epsilon}$ denotes the partial generalized gradient of $V^{\epsilon}$ with respect to $x$. Our result is then an immediate consequence of:

$$(q, p, P) \in D_{t, (x, y)}^{1, 2 +} V^{\epsilon}(t, x, y) \cup D_{t, (x, y)}^{1, 2 -} V^{\epsilon}(t, x, y) \Rightarrow p \in \partial_x V^{\epsilon}(t, x, y).$$

(This follows from (56)-(57) and (60)). ∎

We point out that Proposition 4.1 is a generalization of Lemma 2.1 in [19] (which is proved using ideas from [14]). However, certain key differences between our proof of Proposition 4.1 and that of Lemma 2.1 in [19] should be noted. First, we have not assumed the existence of an optimal control $u(\cdot)$ and an optimal disturbance strategy $\alpha[\cdot]$ in our proof. Second, unlike Lemma 2.1, the diffusion term in Proposition 4.1 may depend on the control $u(\cdot)$. Finally, we have not assumed that $V^{\epsilon}$ is smooth.

**Proposition 4.2** *Suppose that (A1), (A2) and (A3) hold. Let $\epsilon > 0$ and $\lambda \in \mathbb{R}^{l+1}$, $\lambda_i > 0$ be given and fixed. Let $v^{\epsilon}$ be the value function of the risk-sensitive problem (11)-(12) and $V^{\epsilon} = \epsilon \ln v^{\epsilon}$. Then there exists a compact subset $W \subseteq \mathbb{R}$ which is independent of $\epsilon > 0$ such*

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*that $V^\epsilon$ is the unique viscosity solution of the PDE:*

$$
\begin{cases}
V_t^\epsilon(t,x,y) + \inf_{u \in U} \sup_{w \in W} \Big\{ V_x^\epsilon(t,x,y) \cdot (b(t,x,u) + \sigma(t,u)\,w) \\[2mm]
\quad + \sum_{i=0}^l V_{y_i}^\epsilon(t,x,y) \cdot f^i(t,x,u) - \gamma^2 \,|w|^2 + \frac{\epsilon}{4\gamma^2} tr\,[\,\sigma(t,u)\,\sigma(t,u)'\,V_{xx}^\epsilon(t,x,y)]\Big\} = 0, \\[2mm]
(t,\,x,\,y) \in [0,T) \times \mathbb{R}^n \times \mathbb{R}^{l+1}, \\[2mm]
V^\epsilon(T,x,y) = \epsilon \ln\Big\{ \sum_{i=0}^l \lambda_i\, \exp \frac{1}{\epsilon}\,\big[g^i(x) + \Psi(y_i)\big]\Big\}, \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}^{l+1}.
\end{cases}
\tag{34}
$$

**Proof:** It follows immediately from Theorem 3.1 that $V^\epsilon$ is the unique bounded viscosity solution of the PDE (34) but with $W = \mathbb{R}$. To see that $W$ may be replaced by an $\epsilon$-independent compact subset of $\mathbb{R}$, consider the following. For every $(q,p,P) \in D_{t,(x,y)}^{1,2,+} V^\epsilon(s,x,y)$ where $p = (p^{(1)}, p^{(2)}) \in \mathbb{R}^n \times \mathbb{R}^{l+1}$, we have:

$$
q + \inf_{u \in U} \sup_{w \in W} \Big\{ p^{(1)} \cdot (b(t,x,u) + \sigma(t,u)\,w)
$$
$$
+ \sum_{i=0}^l p_i^{(2)} \cdot f^i(t,x,u) - \gamma^2\,|w|^2 + \frac{\epsilon}{4\gamma^2}\sigma(t,u)'\,P\,\sigma(t,u)\Big\} \geq 0,
\tag{35}
$$

where the maximizing $w$ in (35) is given by

$$
w = \frac{1}{2\gamma^2}\,p^{(1)} \cdot \sigma(t,\,u).
\tag{36}
$$

Since $\sigma(t,\,u)$ is uniformly bounded on $[0,\,T] \times U$ (Assumption (**A2**)), and $p$ is bounded (uniformly in $(t,x,y)$ and $\epsilon > 0$) for every $(q,\,p,\,P) \in D_{t,(x,y)}^{1,2,+} V^\epsilon(t,x,y) \cup D_{t,(x,y)}^{1,2,-} V^\epsilon(t,x,y)$ (Proposition 4.1), it follows that

$$
w = \frac{1}{2\gamma^2}\,p^{(1)} \cdot \sigma(t,u) \in \bar{W}, \quad \forall (t,x,y) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^{l+1},\ u \in U
$$

for some compact $\bar{W} \in \mathbb{R}$, which is independent of $\epsilon > 0$. Therefore, we may replace $W = \mathbb{R}$ with and compact subset $W \subseteq \mathbb{R}$ (with $\bar{W} \subseteq W$), and the maximizing $w$ in (35) will still be given by (36). It is easy to see that the same argument applies for the case of sub-solutions. ∎

# 5   Asymptotic results: $\epsilon \downarrow 0$

Clearly, it follows from (**A2**) that the terminal values $V(T,\,x,\,y)$ of (23) and $V^\epsilon(T,\,x,\,y)$ of (34) are continuous. The following uniform convergence result is required in the proof of Theorem 5.1.

**Lemma 5.1** *Suppose that* (**A2**) *holds. Let* $\lambda \in \mathbb{R}^{l+1}$ *be given such that* $\lambda_i > 0$. *Then*

$$
\epsilon \ln\Big\{ \sum_{i=0}^l \lambda_i\, \exp \frac{1}{\epsilon}\,\big[g^i(x) + \Psi(y_i)\big]\Big\} \to \max_i \big\{ g^i(x) + \Psi(y_i)\big\} \quad as\ \epsilon \downarrow 0,
\tag{37}
$$

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*uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{R}^{l+1}$.*

**Proof:** In fact, we show that convergence is uniform on the whole space $\mathbb{R}^n \times \mathbb{R}^{l+1}$. First, note that there is a constant $-\infty < K < \infty$ such that $\ln \lambda_i \geq K$ for every $\lambda_i > 0$. Also, let us denote

$$i(x) := \arg \max_i \left\{ g^i(x) + \Psi(y_i) \right\}.$$

Since

$$\epsilon \ln \left( \sum_{i=0}^{l} \lambda_i \right) + \max_i \left\{ g^i(x) + \Psi(y_i) \right\}$$
$$\geq \quad \epsilon \ln \left\{ \sum_{i=0}^{l} \lambda_i \, \exp \frac{1}{\epsilon} \left[ g^i(x) + \Psi(y_i) \right] \right\}$$
$$\geq \quad \epsilon \ln \lambda_{i(x)} + \max_i \left\{ g^i(x) + \Psi(y_i) \right\},$$

it follows that

$$\epsilon \ln \left( \sum_{i=0}^{l} \lambda_i \right) \geq \epsilon \ln \left\{ \sum_{i=0}^{l} \lambda_i \, \exp \frac{1}{\epsilon} \left[ g^i(x) + \Psi(y_i) \right] \right\} - \max_i \left\{ g^i(x) + \Psi(y_i) \right\} \geq \epsilon K$$

which implies the result. ∎

It is interesting to note that the limit in (37) does not depend on the values of the parameters $\lambda$, which accounts for the seemingly surprising fact that the small noise limit of the multiple-objective risk-sensitive problems is *independent* of the weights on the different objectives (see Theorem 5.1 below).

The following asymptotic result shows the relationship between the risk-sensitive problem (11)-(12) and the differential game (21)-(22). The convergence proof follows the general methods of Barles and Perthame [2]. In particular, the notion of solution that is used in this approach (and in the proof of Theorem 5.1) is the generalized definition of a discontinuous viscosity solution. This is required since the functions (39) and (43) below are only semi-continuous in general. In addition, the proof uses a comparison theorem for semi-continuous viscosity sub- and super-solutions. The definition of a discontinuous viscosity solution is quite similar to that of a continuous solution. The reader should refer to [17, Chapter VII] for a detailed description of the Barles and Perthame method. The definition of a discontinuous viscosity solution as well as the comparison theorem for semi-continuous sub- and super-solutions can also be found there.

**Theorem 5.1** *Assume that* (**A1**), (**A2**) *and* (**A3**) *hold. For every* $\epsilon > 0$, *let* $v^\epsilon$ *be the value function of the multiple-objective risk-sensitive problem (11)-(12), as defined by (14), and* $V^\epsilon = \epsilon \ln v^\epsilon$. *Let* $W \subseteq \mathbb{R}$ *be the compact set as determined in Proposition 4.2, and* $V$ *be the*

*upper value of the associated deterministic differential game (21)-(22). Then $V$ and $V^\epsilon$ are the unique viscosity solutions of (23) and (34), respectively. Moreover,*

$$\lim_{\epsilon \downarrow 0} V^\epsilon(t, x, y) = V(t, x, y), \tag{38}$$

*uniformly on compact subsets.*

**Proof:** Let $W \subseteq \mathbb{R}$ be as stated in the Theorem. By Theorem 3.1, $V$ is the unique viscosity solutions of (23) and the upper value of (21)-(22). Also, by Proposition 4.2, $V^\epsilon$ is the unique viscosity solution of (34).

To prove (38), for every $(t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{l+1}$, define

$$\bar{V}(t, x, y) = \limsup_{\epsilon \downarrow 0, \, s \to t, \, \bar{x} \to x, \, \bar{y} \to y} V^\epsilon(s, \bar{x}, \bar{y}). \tag{39}$$

Since $V^\epsilon(s, x, y)$ is uniformly bounded on compact subsets, it follows that $\bar{V}$ is well defined and upper-semi-continuous. We now prove that $\bar{V}$ is a viscosity sub-solution of (23). First, since $V^\epsilon(T, x, y) \to V(t, x, y)$ uniformly on compact subsets, see Lemma 5.1, it follows from [17, Proposition VII.5.1] that

$$\bar{V}(T, x, y) = \max_i \left\{ g^i(x) + \Psi(y_i) \right\}. \tag{40}$$

Let $\psi \in C^\infty((0, T) \times \mathbb{R}^n \times \mathbb{R}^{l+1})$. Suppose that $\bar{V} - \psi$ has a local maximum at $(t^0, x^0, y^0) \in (0, T) \times \mathbb{R}^n \times \mathbb{R}^{l+1}$. It follows from (39) that there exists a sequence $\{(t^\epsilon, x^\epsilon, y^\epsilon)\}$ such that $V^\epsilon - \psi$ has a local maximum at $(t^\epsilon, x^\epsilon, y^\epsilon)$, $(t^\epsilon, x^\epsilon, y^\epsilon) \to (t^0, x^0, y^0)$ and $V^\epsilon(t^\epsilon, x^\epsilon, y^\epsilon) \to \bar{V}(t^0, x^0, y^0)$. Since $V^\epsilon$ is a viscosity sub-solution of (34), it follows that at every $(t^\epsilon, x^\epsilon, y^\epsilon)$ (denoting $\psi^\epsilon = \psi(t^\epsilon, x^\epsilon, y^\epsilon)$ etc.), we have:

$$\psi_t^\epsilon + \inf_{u \in U} \sup_{w \in W} \left\{ \frac{\epsilon}{4\gamma^2} \operatorname{tr}(\sigma \sigma' \psi_{xx}^\epsilon) + (b + \sigma w) \cdot \psi_x^\epsilon + \sum_{i=0}^{l} f^i \psi_{y_i}^\epsilon - \gamma^2 |w|^2 \right\} \geq 0 \tag{41}$$

for every $\epsilon > 0$. Hence, by the continuity properties of $\psi$, the convergence properties of $(t^\epsilon, x^\epsilon, y^\epsilon)$ and Assumptions $(\mathbf{A1})'$-$(\mathbf{A2})$, we can let $\epsilon \to 0$ which gives:

$$\psi_t + \inf_{u \in U} \sup_{w \in W} \left\{ (b + \sigma w) \cdot \psi_x + \sum_{i=0}^{l} f^i \psi_{y_i} - \gamma^2 |w|^2 \right\} \geq 0 \tag{42}$$

at $(t^0, x^0, y^0)$. Therefore, it follows from (39) and (42) that $\bar{V}$ is a viscosity sub-solution of (23). In the same way, the function

$$\underline{V}(t, x, y) = \liminf_{\epsilon \downarrow 0, \, s \to t, \, \bar{x} \to x, \, \bar{y} \to y} V^\epsilon(s, \bar{x}, \bar{y}) \tag{43}$$

is well defined, lower semi-continuous, satisfies the terminal condition

$$\underline{V}(T, x, y) = \max_i \left\{ g^i(x) + \Psi(y_i) \right\},$$

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(using Lemma 5.1 and [17, Proposition VII.5.1] as above) and is a viscosity super-solution of (23). By the definition of $\bar{V}$ and $\underline{V}$, we have $\bar{V} \geq \underline{V}$. On the other hand, the comparison theorem for semi-continuous viscosity sub- and super-solutions, see [17, Theorem VII.8.1]), gives $\bar{V} \leq \underline{V}$. Therefore, $\tilde{V} := \bar{V} = \underline{V}$ is continuous on $[0, T] \times \mathbb{R}^n \times \mathbb{R}^{l+1}$, and

$$\tilde{V}(t, x, y) = \lim_{\epsilon \downarrow 0} V^\epsilon(t, x, y)$$

uniformly on compact subsets. Finally, $\tilde{V} \equiv V$ by the uniqueness of viscosity solution to (23). This completes the proof. ∎

It is interesting that $V$ is independent of the actual *value* of $\lambda_i$. Note that in the setting of the multiple-objective risk-sensitive problem (11)-(12), $\lambda_i$ can be viewed as a weight for the $i^{th}$ criteria: the higher the value of $\lambda_i$ the more important the $i^{th}$ criteria becomes with different values of $\lambda_i$ representing different performance trade-offs between the $l+1$ criteria. Theorem 5.1 suggests that various different performance tradeoffs for the multiple-objective risk-sensitive problem are equivalent, asymptotically, to the *same* limiting deterministic differential game. This seems to be quite surprising and counter-intuitive. However, the multiple-objective nature of the original problem is indeed captured by the cost functional (22) associated with the limiting differential game; see also the discussion in the last paragraph of Section 3.

The proof of Theorem 5.1 is analogous to its single-objective counterpart, namely, Theorem 4.1 in [19]. One difference, however, should be noted. In [19], it is assumed that $V^\epsilon$ is a *classical* solution of the associated PDE. This fact is then used in the convergence proof to establish the parallel inequality to (41). However, it can be seen from the proof of Theorem 5.1 that smoothness of $V^\epsilon$, which as well known is not a reasonable assumption, is not essential for the result to hold. In particular, we have only made assumptions which guarantee that $V^\epsilon$ is a viscosity solution of (34).

The (upper) differential game (15)-(16) plays an important role in robust control. (For a detailed account on the role of differential games in robust control, the reader is referred to the book [4]). The relationship between the (single-objective) risk-sensitive control problem and the upper value of the differential game (15)-(16) was first established in [19]. The single-objective problem is a special case of the problem we are studying. One contribution of this paper is to show that the results in [19] hold under milder conditions for a significantly larger class of systems. In particular, the diffusion term may depend on the control. Also, we need not assume that the drift term in (3) is linear in the control.

# 6 Conclusion

In this paper, we have studied the relationship between multiple-objective risk-sensitive control problems and a class of new deterministic differential games introduced in this paper. More

specifically, we have shown that the problem of finding an efficient risk-sensitive controller is equivalent, asymptotically, to solving this problem. In addition, we have proved that this limiting differential game is independent of the size of the weights $\lambda_i > 0$ on the competing objectives in the risk-sensitive problem. That is, various different performance tradeoffs for the risk-sensitive problem are equivalent, asymptotically, to the *same* game problem.

This class of deterministic differential games is characterized by a cost functional which is multiple-objective in nature, being composed of the risk-neutral integral costs associated with the original risk-sensitive problem. In this game, the opposing player seeks to attack the most vulnerable member in the given finite set of cost functionals (corresponding to the exponents of the risk-sensitive costs), whereas the controller tries to minimize the worst damage that the opposing player can do over this set. It appears that many practical problems can be modeled as a differential game of this sort. To our best knowledge, this paper is also the first one that examines the connection between multiple-objective risk-sensitive control and differential games.

Our results apply to a large class of nonlinear stochastic systems, under reasonably mild assumptions. In particular, our work incorporates some fairly recent advances on the control of systems in which the diffusion term may depend on the state and/or the control. (See the book [26] for a comprehensive treatment of stochastic control problems with control dependent diffusions). In particular, optimal control problems with control dependent diffusions arise frequently in finance applications.

Although we have focused on multiple-objective problems, we have also been able to generalize some known results for the single-objective case. In particular, we have shown that the main results in [19] are true under milder conditions for a significantly larger class of stochastic systems, as discussed above. On the other hand, unlike the standard (risk-neutral) stochastic linear quadratic case [7], when the diffusion depends on the control, the linear quadratic risk-sensitive problem and the linear quadratic stochastic and deterministic differential games have not been solved, even in the single-objective case. They are interesting open problems.

# Appendix

We present here some basic definitions and results from the theory of viscosity solutions and nonsmooth analysis which are referred to in the paper. For a detailed discussion of viscosity solutions, the reader is referred to the papers [9, 10] as well as the books [1, 15, 26]. For a discussion of nonsmooth analysis, we recommend the book [8]. A proof of the relationship between sub/super-differentials and Clarke's generalized gradient can be found in [26, 28].

# Viscosity solutions

Consider the following nonlinear, scalar, first-order PDE

$$\begin{cases} v_t + H(t, x, v_x) = 0, & (t, x) \in [0, T] \times \mathbb{R}^n \\ v(T, x) = g(x) \end{cases} \tag{44}$$

(a special case of which is (23)), and the nonlinear, scalar second-order PDE

$$\begin{cases} v_t + H(t, x, v_x, v_{xx}) = 0, & (t, x) \in [0, T] \times \mathbb{R}^n \\ v(T, x) = g(x). \end{cases} \tag{45}$$

(see (13) or (34), for example). It is well known that the PDEs (23), (13) and (34) do not, in general, have classical (smooth) solutions. A generalized concept of solution, called a viscosity solution, is introduced in [10]. The main result in [10] is that under certain mild conditions, there exists a unique viscosity solution of (44). In the second-order case, uniqueness is proven in [20]. The definition of a viscosity solution of the first-order PDE (44) is as follows:

**Definition 6.1** *Let $v \in C([0, T] \times \mathbb{R}^n)$ and $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$. Then the first order super-differential of $v$ at $(t_0, x_0)$ is given by*

$$D_{t, x}^{1, +} v(t_0, x_0) = \Big\{ (\psi_t(t_0, x_0), \psi_x(t_0, x_0)) \Big|$$
$$\psi \in C^\infty((0, T) \times \mathbb{R}^n) \text{ and } v - \psi \text{ has a local maximum at } (t_0, x_0) \Big\} \tag{46}$$

*and the first order sub-differential by*

$$D_{t, x}^{1, -} v(t_0, x_0) = \Big\{ (\psi_t(t_0, x_0), \psi_x(t_0, x_0)) \Big|$$
$$\psi \in C^\infty((0, T) \times \mathbb{R}^n) \text{ and } v - \psi \text{ has a local minimum at } (t_0, x_0) \Big\}. \tag{47}$$

*Moreover, $v$ is a viscosity solution of (44) if*

$$v(T, x) = g(x), \quad \forall x \in \mathbb{R}^n \tag{48}$$

*and*

$$q + H(t, x, p) \geq 0, \quad \forall (q, p) \in D_{t, x}^{1, +} v(t, x) \tag{49}$$
$$q + H(t, x, p) \leq 0, \quad \forall (q, p) \in D_{t, x}^{1, +} v(t, x) \tag{50}$$

*for all $(t, x) \in [0, T] \times \mathbb{R}^n$.*

In particular, $v$ is called a *viscosity sub-solution* if it satisfies (48)-(49), and a *viscosity super-solution* if it satisfies (48) and (50). Also, for any $v \in C([0, T] \times \mathbb{R}^n)$, we can define *partial*

*super/sub-differentials* of $v$ with respect to $x$ at $(t_0, x_0)$ (which we denote by $D_x^{1,+} v(t, x)$ and $D_{t,x}^{1,-} v(t, x)$, respectively) by keeping $t = t_0$ fixed, and calculating the super/sub-differentials of $v(t_0, x)$ in the $x$ variable.

For the second-order case, we have the following:

**Definition 6.2** *Let $v \in C([0, T] \times \mathbb{R}^n)$ and $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$. Then the second-order super-differential of $v$ at $(t_0, x_0)$ is defined by*

$$D_{t,x}^{1,2,+} v(t_0, x_0) = \Big\{ (\psi_t(t_0, x_0), \psi_x(t_0, x_0)), \psi_{xx}(t_0, x_0) \Big|$$
$$\psi \in C^\infty((0, T) \times \mathbb{R}^n) \text{ and } v - \psi \text{ has a local maximum at } (t_0, x_0) \Big\}, \quad (51)$$

*and the second order sub-differential of $v$ is defined by*

$$D_{t,x}^{1,2,-} v(t_0, x_0) = \Big\{ (\psi_t(t_0, x_0), \psi_x(t_0, x_0)), \psi_{xx}(t_0, x_0) \Big|$$
$$\psi \in C^\infty((0, T) \times \mathbb{R}^n) \text{ and } v - \psi \text{ has a local minimum at } (t_0, x_0) \Big\}. \quad (52)$$

*Moreover, $v$ is a viscosity solution of (45) if*

$$v(T, x) = g(x), \quad \forall\, x \in \mathbb{R}^n \quad (53)$$

*and*

$$q + H(t, x, p, P) \geq 0, \quad \forall (q, p) \in D_{t,x}^{1,2,+} v(t, x) \quad (54)$$
$$q + H(t, x, p, P) \leq 0, \quad \forall (q, p) \in D_{t,x}^{1,2,-} v(t, x) \quad (55)$$

*for all $(t, x) \in [0, T] \times \mathbb{R}^n$.*

As in the first-order case, $v$ is called a *viscosity sub-solution* of (45) if (53)-(54) are satisfied, and a *viscosity super-solution* if (53) and (55) are satisfied. Clearly,

$$(q, p, P) \in D_{t,x}^{1,2,+} v(t, x) \Rightarrow (q, p) \in D_{t,x}^{1,+} v(t, x) \Rightarrow p \in D_x^{1,+} v(t, x) \quad (56)$$

and

$$(q, p, P) \in D_{t,x}^{1,2,-} v(t, x) \Rightarrow (q, p) \in D_{t,x}^{1,-} v(t, x) \Rightarrow p \in D_x^{1,-} v(t, x). \quad (57)$$

## Nonsmooth analysis

The following results from nonsmooth analysis are used in our proof of Proposition 4.1. For an in depth discussion, we recommend the book [8].

We begin with a definition of the *generalized gradient*.

**Definition 6.3 (Generalized gradient)** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The generalized gradient of $f$ at $x \in \mathbb{R}^n$ is*

$$\partial f(x) = \left\{ \xi \in \mathbb{R}^n \;\middle|\; \langle \xi, \, y \rangle \leq \limsup_{z \to x, \, h \downarrow 0} \frac{f(z + h \, y) - f(z)}{|h|} \right\}. \tag{58}$$

If $f : \mathbb{R}^n \times U \to \mathbb{R}$ for some subset $U$ of $\mathbb{R}^m$, the *partial generalized gradient* of $f$ at $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times U$ is obtained by fixing $u = \bar{u}$, and calculating the generalized gradient by treating $f(x, \bar{u})$ as a function of $x$.

An alternative characterization of $\partial f$ is obtained from the following well known result: If $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, then $f$ is differentiable almost everywhere (Rademacher's Theorem [8]). Let $\Omega_f$ denote the set of all points at which $f$ is not differentiable. Then we have the following result. (See Theorem 2.5.1, p. 63 of [8]).

**Theorem 6.1** *Let $f$ satisfy the conditions in Theorem 6.3 and suppose that $S$ is any set of Lebesgue measure 0. Then*

$$\partial f(x) = \mathrm{co} \left\{ \lim_{i \to \infty} \nabla f(x_i) \;\middle|\; x_i \to x, \, x_i \notin S, \, x_i \notin \Omega_f \right\}. \tag{59}$$

*where* co *denotes the convex hull.*

The following result is used in the proof of Proposition 4.1. (See [26, 28]).

**Proposition 6.1** *If $v$ is locally Lipschitz in $(t, \, x)$, then*

$$D_x^{1, \, +} v(t, x) \cup D_x^{1, \, -} v(t, \, x) \subseteq \partial_x v(t, x). \tag{60}$$

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