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ON THE EXISTENCE OF OPTIMAL RELAXED CONTROLS OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS*

XUN YU ZHOU†

Abstract. This paper is concerned with control problem of systems governed by stochastic partial differential equations, the drift and diffusion terms of which are second- and first-order differential operators, respectively. The existence of an optimal relaxed control is studied in both cases where the systems are degenerate and nondegenerate. It is shown that the higher regularity conditions on the initial state, as required in the existing results, can be dispensed with if the Wiener process is one-dimensional. Some special cases of multidimensional Wiener process are also discussed, which in particular leads to an improvement of a recent result of Bensoussan and Nisio. The method is based on an analysis of the group generated by the first-order differential operator. As an application, an existence theorem of the optimal relaxed control is proved for partially observed diffusions with correlation between the controlled states and the observation noises.

Key words. stochastic partial differential equation (SPDE), optimal relaxed control, partially observed diffusion, group of operators

AMS(MOS) subject classifications. 60H15, 93E20, 49A60, 93E11

1. Introduction. In this paper we consider an optimal control problem of the following kind of stochastic partial differential equations (SPDE):

$$
\begin{aligned}
dq(t,x) = {} & [\partial_i(a^{ij}(x,u(t))\partial_j q(t,x)) + b^i(x,u(t))\partial_i q(t,x) \\
& + c(x,u(t))q(t,x) + f(x,u(t))]\,dt \\
& + [\sigma^i(x)\partial_i q(t,x) + h(x)q(t,x) + g(x)]\,dW(t), \\
& \qquad x\in R^d, \quad t\in[0,T],
\end{aligned} \tag{1.1}
$$

$$q(0,x) = q_0(x), \quad x\in R^d,$$

where W is a one-dimensional Wiener process with $W(0)=0$, $\{u(t): 0\leqq t\leqq T\}$ is an admissible control (in the usual sense), and $\partial_i := \partial/\partial x_i$, $i=1,2,\cdots,d$. Note that here and in the following we always use the conventional repeated indices for summation.

The optimal control problem is to minimize a given cost functional over the totality of admissible controls. SPDE (1.1) occurs in many areas of science, especially in physics (see [8]-[10] and [13] and the references therein). From the mathematical point of view, the most important example is perhaps the filtering problem. More precisely, the control problem of partially observed diffusions can be reduced to the control problem of SPDE (1.1) (Zakaï's equation), with the σ^i in (1.1) corresponding to the correlation between the controlled state and the observation noises ([12], [13], and [15]).

The existence of an optimal control in the usual sense seems to be a very difficult problem and remains open in general. Many authors turned to studying the relaxed control problem ([1], [4], [6] and [13]). For system (1.1), when $\sigma^i=0$, Bensoussan and Nisio [1] have proved the existence of an optimal relaxed control, assuming (a^{ij}) is uniformly positive definite and $q_0\in H^2(R^d)$ ($H^k(R^d) :=$ Sobolev space $W_2^k(R^d)$). When $\sigma^i\neq 0$, the situation is much more complicated. For this case, the recent delicate

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† Department of Mathematics, Faculty of Science, Kobe University, Rokko, Kobe 657, Japan.

work of Nagase and Nisio [13] shows that the existence theorem still holds when $(a^{ij}-3/2\sigma^i\sigma^j)$ is nonnegative definite and $q_0 \in H^3(R^d)$. Both [1] and [13] have applied a similar method to that of Nagase [12], the key point of which is to employ a compact embedding lemma, see [12, Remark 3.1]. These results, however, require the higher regularity on the initial point q_0.

The purpose of the present paper is to establish the existence of an optimal relaxed control of SPDE (1.1), with some natural regularity conditions on the initial state q_0. Different from that of [1], [12], and [13], our method depends on an analysis of the *group* generated by the first-order differential operator, inspired by Da Prato, Iannelli, and Tubaro [3]. This frees us from the set-up of [1], [12], and [13], allowing us to eliminate the higher regularity restriction on q_0. The main results of this paper, roughly speaking, are as follows: Either

(i) $(a^{ij}-1/2\sigma^i\sigma^j)$ is nonnegative definite and $q_0 \in H^1(R^d)$, or

(ii) $(a^{ij}-1/2\sigma^i\sigma^j)$ is uniformly positive definite and $q_0 \in H^0(R^d)(=L^2(R^d))$

will ensure the existence of an optimal relaxed control.

An apparent defect of our method, however, is that the Wiener process W is required to be one-dimensional. For multidimensional cases, the method applies only to some special cases. In particular, we will show that our method is effective to the setting of Bensoussan and Nisio [1]; therefore, their result may be considerably improved.

It is also worth noting that this work benefits so much by the delicate and deep results of Krylov and Rozovskii [8]-[10] concerning the SPDE theory, and in particular, the a priori estimate of differential operators (Lemma 3.1 below).

The paper is organized as follows: In § 2, we will give a precise definition of the relaxed system of (1.1) as well as some basic notation and facts. Sections 3 and 4 are the main parts of the paper, in which the existence theorems are proved for degenerate case (i.e., $(a^{ij}-1/2\sigma^i\sigma^j)$ is nonnegative definite) and nondegenerate case (i.e., $(a^{ij}-1/2\sigma^i\sigma^j)$ is uniformly positive definite), respectively. In § 5, we discuss the cases when the Wiener process is multidimensional, as well as the application of the main results to the partially observed diffusions.

2. Preliminaries. We define a family of second-order differential operators $\{A(u): u \in \Gamma \subset R^m\}$ and a first-order differential operator M by

$$A(u)\phi(x) := \partial_i(a^{ij}(x,u)\partial_j\phi(x)) + b^i(x,u)\partial_i\phi(x) + c(x,u)\phi(x), \tag{2.1}$$

$$M\phi(x) := \sigma^i(x)\partial_i\phi(x) + h(x)\phi(x), \quad \text{for } x \in R^d, \tag{2.2}$$

where a^{ij}, b^i, c, σ^i, and h are real-valued functions for $i, j = 1, 2, \cdots, d$.

Let H^k be the Sobolev space $W_2^k(R^d)$ with the norm $\|\cdot\|_k$ $(k = 0, \pm 1, \pm 2, \cdots)$. $\langle\cdot,\cdot\rangle_k$ denotes the duality pairing between H^{k-1} and H^{k+1} under $(H^k)^* = H^k$, and $(\cdot,\cdot)_k$ is the inner product in H^k. For $r \geqq 0$, define

$$L_r^2 := \{\phi : \phi \text{ is a real-valued Borel function on } R^d, \text{ and } (1+|\cdot|^2)^{r/2}\phi(\cdot) \in H^0\},$$

with the norm $\|\phi\|_{0,r} := (\int_{R^d} |(1+|x|^2)^{r/2}\phi(x)|^2\, dx)^{1/2}$.

Let H_r^k be the subspace of L_r^2 consisting of functions whose generalized derivatives up to the order k belong to L_r^2. It becomes a Hilbert space with the norm

$$\|\phi\|_{k,r} := \left(\sum_{|\alpha| \leqq k} \frac{|\alpha|!}{\alpha^1! \cdots \alpha^d!} \|D^\alpha \phi\|_{0,r}^2\right)^{1/2},$$

see [10, § 2].

For any second-order differential operator L, when we write $\langle L\phi, \psi\rangle_k$, then L is understood to be an operator from H^{k+1} to H^{k-1} by formally using Green's formula. For example, for the operators $A(u)$ defined by (2.1), we have

$$\langle A(u)\phi, \psi\rangle_k := -(a^{ij}(\cdot, u)\partial_j\phi, \partial_i\psi)_k + (b^i(\cdot, u)\partial_i\phi, \psi)_k + (c(\cdot, u)\phi, \psi)_k \quad \text{for } \phi, \psi \in H^{k+1}. \tag{2.3}$$

Now we recall the definition of relaxed controls, according to [1], [4], and [13]. By Λ we denote the set of all measures λ on $[0, T]\times\Gamma$ such that

$$\lambda([0, s]\times\Gamma) = s, \quad \text{for } s \leqq T. \tag{2.4}$$

If Γ is a compact set, then Λ is compact when endowed with the weak convergence topology ([1] and [13]).

Set $\sigma_t(\Lambda)$:= the σ-field generated by $\{\lambda: \lambda([0, s]\times A) \in \mathscr{B}(R^+),\ s \leqq t,\ A \in \mathscr{B}(\Gamma)\}$ and $\sigma(\Lambda) := \sigma_T(\Lambda)$. Let $\mathscr{P} := \mathscr{P}(\Lambda)$ be the space of probabilities on $(\Lambda, \sigma(\Lambda))$, then Prohorov's theorem yields that $\mathscr{P}$ is a compact metric space when endowed with the weak convergence topology.

By (2.4), λ is represented by $\lambda(dt, du) = \lambda'(t, du)\,dt$, where $\lambda'(t, \cdot)$ is a probability on Γ for almost all t and determined uniquely expect t-null set. For any bounded and uniformly continuous function ρ on $R^d \times \Gamma$, set $\tilde{\rho}(t, x, \lambda) := \int_\Gamma \rho(x, u)\lambda'(t, du)$. Define a family of operators $\{\tilde{A}(t, \lambda): t \in [0, T],\ \lambda \in \Lambda\}$:

$$\tilde{A}(t, \lambda)\phi(x) := \partial_i(\tilde{a}^{ij}(t, x, \lambda)\partial_j\phi(x)) + \tilde{b}^i(t, x, \lambda)\partial_i\phi(x) + \tilde{c}(t, x, \lambda)\phi(x), \quad \text{for } x \in R^d. \tag{2.5}$$

Now we introduce the relaxed system.

Definition 2.1. $\mathscr{R} = (\Omega, \mathscr{F}, P, \mathscr{F}_t, W, \mu)$ is called a relaxed system if

(2.6) $(\Omega, \mathscr{F}, P, \mathscr{F}_t)$ is a standard probability space with filteration $\{\mathscr{F}_t : 0 \leqq t \leqq T\}$;

(2.7) W is an $\mathscr{F}_t$-adapted one-dimensional Wiener process with $W(0) = 0$;

(2.8) μ is an $\mathscr{F}_t$-adapted Λ-valued random variable (Λ-r.v.), i.e., $\mu(B_1 \times B_2)$ is $\mathscr{F}_t$-measurable whenever $B_1 \in \mathscr{B}([0, t])$ and $B_2 \in \mathscr{B}(\Gamma)$.

For simplicity, we put $\mathscr{R} = (W, \mu)$ if no confusion arises, and sometimes we simply call μ a relaxed control.

R denotes the totality of relaxed controls. For $\mathscr{R} = (W, \mu)$, $\pi(\mathscr{R})$ denotes the image measure of (W, μ) on $C(0, T; R^1)\times\Lambda$. Again, by endowing the space $\Pi := \{\pi(\mathscr{R}): \mathscr{R} \in \mathrm{R}\}$ with the weak convergence topology, we have the following proposition [1], [13].

Proposition 2.1. *Π is a compact metric space.*

Definition 2.2. We say $\mathscr{R}_n$ converges to $\mathscr{R}$, writing $\mathscr{R}_n \to \mathscr{R}$, if $\pi(\mathscr{R}_n) \to \pi(\mathscr{R})$ weakly.

Given $\mathscr{R} = (\Omega, \mathscr{F}, P, \mathscr{F}_t, W, \mu)$, consider the following SPDE:

$$dq(t) = (\tilde{A}(t, \mu)q(t) + \tilde{f}(t, \mu))\,dt + (Mq(t) + g)\,dW(t), \qquad q(0) = q_0. \tag{2.9}$$

An H^1-valued $\mathscr{F}_t$-adapted process $q = q^{\mathscr{R}}(\cdot, q_0)$ is called a solution of (2.9) or a response for the relaxed control $\mathscr{R}$ if

$$E\int_0^T \|q(t)\|_1^2\,dt < +\infty, \tag{2.10}$$

and for any $\eta \in C_0^\infty(R^d)$ (smooth function on R^d with compact support) and almost all $(t, \omega) \in [0, T] \times \Omega$,

$$(q(t), \eta)_0 = (q_0, \eta)_0 + \int_0^t \langle \tilde{A}(s, \mu) q(s), \eta \rangle_0 \, ds + \int_0^t (\tilde{f}(s, \mu), \eta)_0 \, ds$$
$$+ \int_0^t (Mq(s) + g, \eta)_0 \, dW(s). \tag{2.11}$$

For each initial q_0, we are given a cost functional

$$J(q_0, \mathcal{R}) := E\{F(q^{\mathcal{R}}(\cdot, q_0)) + G(q^{\mathcal{R}}(T, q_0))\}, \qquad \mathcal{R} \in \mathsf{R}. \tag{2.12}$$

The optimal relaxed control problem is to minimize $J(q_0, \cdot)$ over R, for each q_0.

Remark 2.1. Since we will mainly consider the relaxed control in the following, we will simply write $A(t, \mu) = \tilde{A}(t, \mu)$, etc., if no confusion arises.

3. Degenerate case. Let us fix a positive constant K. We introduce the following conditions on the functions appearing in (1.1):

(A1) a^{ij}, b^i, c: $R^d \times \Gamma \to R^1$, σ^i, h: $R^d \to R^1$ are continuous functions; these functions and their derivatives in x up to second order do not exceed K in absolute value;

(A2) $a^{ij} = a^{ji}$, $i, j = 1, 2, \cdots, d$, and $(a^{ij} - \frac{1}{2}\sigma^i\sigma^j)_{ij}$ is a nonnegative definite matrix;

(A3) $f(\cdot, u) \in H^1$, $g \in H^2$; the absolute values of f, g together with the H^1-norm of $f(\cdot, u)$ and the H^2-norm of g do not exceed K;

(A3)$_r$ For some $r > 0$, $f(\cdot, u) \in L^2_r$, $g \in H^1_r$, the absolute values of f, g together with the L^2_r-norm of $f(\cdot, u)$ and the H^1_r-norm of g do not exceed K.

The following lemma of a prior estimate is a special case of [9, Lemma 2.1].

LEMMA 3.1. *Let $\hat{A}$ and $\hat{M}$ be any second-order and first-order differential operators that have the forms of* (2.1) *and* (2.2), *respectively, and whose coefficients satisfy* (A1) *and* (A2). *Then there exists a constant N depending only on K in* (A1) *such that*

$$2\{\langle \hat{A}\phi, \phi \rangle_k + (\hat{f}, \phi)_k\} + \|\hat{M}\phi + \hat{g}\|_k^2 \leqq N(\|\phi\|_k^2 + \|\hat{f}\|_k^2 + \|\hat{g}\|_{k+1}^2), \tag{3.1}$$
$$\textit{for any } \phi \in H^{k+1}, \hat{f} \in H^k, \hat{g} \in H^{k+1}; \ k = 0, 1.$$

COROLLARY 3.1. *Let $\hat{M}$ be a first-order differential operator that has the form of* (2.2) *and whose coefficients satisfy* (A1). *Then there exists a constant N depending only on K such that*

$$|(\hat{M}\phi + \hat{g}, \phi)_k| \leqq N(\|\phi\|_k^2 + \|\hat{g}\|_k^2), \quad \textit{for any } \phi \in H^{k+1}, \hat{g} \in H^k; \ k = 0, 1. \tag{3.2}$$

The following result concerning the solution of SPDE is known from Krylov and Rozovskii [9] and [10].

PROPOSITION 3.1. *Assume* (A1)–(A3) *and $q_0 \in H^1$; then for any $\mathcal{R} \in \mathsf{R}$,* (2.9) *has a unique solution $q^{\mathcal{R}}(\cdot, q_0) \in L^2([0, T] \times \Omega; H^1) \cap L^2(\Omega; C(0, T; H^0))$ and there exists a constant C, depending only on K and T, such that*

$$\sup_{0 \leqq t \leqq T} E\|q^{\mathcal{R}}(t, q_0)\|_k^2 \leqq CE\left\{\|q_0\|_k^2 + \int_0^T [\|f(s, \mu)\|_k^2 + \|g\|_{k+1}^2] \, ds\right\}, \quad k = 0, 1. \tag{3.3}$$

Moreover, for any $p \geqq 2$, there exists a constant $C(p)$ such that

$$\sup_{0 \leqq t \leqq T} E\|q^{\mathcal{R}}(t, q_0)\|_k^p \leqq C(p)E\left\{\|q_0\|_k^p + \int_0^T [\|f(s, \mu)\|_k^p + \|g\|_{k+1}^p] \, ds\right\}, \tag{3.4}$$
$$k = 0, 1.$$

The main idea of the present paper is based on the fact that the first-order differential operator M generates a strongly continuous *group* on H^0. The following proposition states the detailed properties of M, with the proof provided in the Appendix.

PROPOSITION 3.2. *On the Hilbert space H^0, define an operator M by (2.2) with the domain $D(M) := H^1$, then*

(i) *M can be extended to a closed operator (still denoted by M) that generates a strongly continuous group $\{e^{Mt}: -\infty < t < +\infty\}$ on H^0. Moreover, H^1 is an invariant subspace of e^{Mt} for each t, and there exists a constant N (the same as the constant in (3.2)), such that*

$$\|e^{Mt}\|_{L(H^0 \to H^0)} \leqq e^{N|t|}, \tag{3.5}$$

$$\|e^{Mt}\|_{L(H^1 \to H^1)} \leqq e^{N|t|}, \quad \text{for any } t \in (-\infty, +\infty); \tag{3.6}$$

(ii) *Denote by M^* the adjoint operator of M on H^0, then $H^1 \subset D(M^*)$ and M^* also generates a strongly continuous group $\{e^{M^*t} = (e^{Mt})^*: -\infty < t < +\infty\}$ on H^0. Moreover, H^1 is an invariant subspace of e^{M^*t} for each t, and with the same constant N, we have*

$$\|e^{M^*t}\|_{L(H^0 \to H^0)} \leqq e^{N|t|}, \tag{3.7}$$

$$\|e^{M^*t}\|_{L(H^1 \to H^1)} \leqq e^{N|t|}, \quad \text{for any } t \in (-\infty, +\infty); \tag{3.8}$$

(iii) *Define two operators M^2, M^{*2} from H^1 to H^{-1} by the following formula:*

$$\langle M^2\phi, \psi\rangle_0 = (M\phi, M^*\psi)_0 = \langle \phi, M^{*2}\psi\rangle_0, \quad \text{for } \phi, \psi \in H^1, \tag{3.9}$$

then M^2 and M^{*2} are bounded linear operators from H^1 to H^{-1}.

From now on, when we write M, M^*, M^2, and M^{*2}, it is always understood to be in the sense of that in Proposition 3.2.

The following lemma will play an essential role in this paper.

LEMMA 3.2. *Let D be a set in R^d such that*

$$D \text{ is bounded, open, and with smooth boundary.} \tag{3.10}$$

Define $W_D[0, T] := \{\phi: \phi \in L^2(0, T; H^1(D)), d\phi/dt \in L^2(0, T; H^{-1}(D))\}$ with the norm

$$\|\phi\|_{W_D[0,T]} := \left(\int_0^T \|\phi(t)\|_{1,D}^2\, dt + \int_0^T \|d\phi(t)/dt\|_{-1,D}^2\, dt\right)^{1/2}, \tag{3.11}$$

where $H^k(D)$ is the Sobolev space $W_2^k(D)$ with the Sobolev norm $\|\cdot\|_{k,D}$. Then the embedding $W_D[0, T] \to L^2(0, T; H^0(D))$ is compact.

Proof. Since the embedding $H^1(D) \to H^0(D)$ is compact, the result follows from [11, Thm. 5.1, p. 58]. □

PROPOSITION 3.3. *Assume (A1)-(A3), $q_0 \in H^1$, and $\mathscr{R} \in \mathrm{R}$. Let $q(\cdot) = q^{\mathscr{R}}(\cdot, q_0)$ be the solution of (2.9). Set $p(t) := e^{-MW(t)}q(t)$, then p satisfies*

$$\begin{aligned} d(p(t), \phi)_0 = \{&\langle (A(t, \mu) - \tfrac{1}{2}M^2)q(t), e^{-M^*W(t)}\phi\rangle_0 \\ &+ (e^{-MW(t)}(f(t, \mu) - Mg), \phi)_0\}\, dt + (e^{-MW(t)}g, \phi)_0\, dW(t), \end{aligned} \tag{3.12}$$

for any $\phi \in H^1$.

Proof. Take $\phi, \psi \in H^1$. Define $\rho(t) := (e^{-M^*t}\phi, \psi)_0$. By Proposition 3.2,

$$d\rho(t)/dt = -(M^* e^{-M^*t}\phi, \psi)_0 = -(e^{-M^*t}\phi, M\psi)_0,$$

$$d^2\rho(t)/dt^2 = (M^* e^{-M^*t}\phi, M\psi)_0 = \langle M^{*2} e^{-M^*t}\phi, \psi\rangle_0.$$

Hence by Itô's formula, we have

$$d\rho(W(t)) = \tfrac{1}{2}\langle M^{*2} e^{-M^*W(t)}\phi, \psi\rangle_0\, dt - (M^* e^{-M^*W(t)}\phi, \psi)_0\, dW(t); \tag{3.13}$$

namely, we have the following formula in the space H^{-1}:

$$d(e^{-M^*W(t)}\phi) = \tfrac{1}{2}M^{*2} e^{-M^*W(t)}\phi\, dt - M^* e^{-M^*W(t)}\phi\, dW(t). \tag{3.14}$$

Therefore, again by Itô's formula,

$$
\begin{aligned}
d(p(t), \phi)_0 &= d(q(t), e^{-M^* W(t)}\phi)_0 \\
&= \langle A(t,\mu)q(t)+f(t,\mu), e^{-M^* W(t)}\phi\rangle_0\, dt \\
&\quad + (Mq(t)+g, e^{-M^* W(t)}\phi)_0\, dW(t) \\
&\quad + \langle q(t), \tfrac{1}{2}M^{*2} e^{-M^* W(t)}\phi\rangle_0\, dt - (q(t), M^* e^{-M^* W(t)}\phi)_0\, dW(t) \\
&\quad - (Mq(t)+g, M^* e^{-M^* W(t)}\phi)_0\, dt \\
&= \{\langle (A(t,\mu)-\tfrac{1}{2}M^2)q(t), e^{-M^* W(t)}\phi\rangle_0 \\
&\quad + (e^{-MW(t)}(f(t,\mu)-Mg), \phi)_0\}\, dt \\
&\quad + (e^{-MW(t)}g, \phi)_0\, dW(t).
\end{aligned} \tag{3.15}
$$

This proves (3.12). □

We need an additional lemma for technical reasons.

LEMMA 3.3. *For some $r>0$, we assume* (A1)-(A3), $(A3)_r$, *and $q_0 \in H^1 \cap L^2_r$. For $\mathcal{R} \in \mathrm{R}$, let $q(t,x) = q^{\mathcal{R}}(t,x,q_0)$ be the solution of* (2.9), *then there exists a constant C, independent of $\mathcal{R}$, such that*

$$
E\int_{|x|>\rho} |q(t,x)|^2\, dx \le C/(1+\rho^2)^r, \quad \textit{for any } \rho>0. \tag{3.16}
$$

Proof. By Krylov and Rozovskii [10], we have a constant C' such that

$$
\sup_{0\le t\le T} E\|q(t)\|^2_{0,r} \le C'E\left\{\|q_0\|^2_{0,r} + \int_0^T (\|f(t,\mu)\|^2_{0,r} + \|g\|^2_{1,r})\, dr\right\} \le C;
$$

hence

$$
E\int_{|x|>\rho} |q(t,x)|^2\, dx \le E\|q(t)\|^2_{0,r}/(1+\rho^2)^r \le C/(1+\rho^2)^r. \qquad \square
$$

THEOREM 3.1. *Assume* (A1)-(A3), $(A3)_r$, *and $q_0 \in H^1 \cap L^2_r$ for some $r>0$. Denote by $q^{\mathcal{R}}(\cdot, q_0)$ the solution of* (2.9) *for $\mathcal{R} \in \mathrm{R}$. If $\mathcal{R}_n \to \mathcal{R}$, then*

$$
q^{\mathcal{R}_n}(\cdot, q_0) \to q^{\mathcal{R}}(\cdot, q_0) \textit{ in law, as } L^2(0,T;H^0)\textit{-r.v.}; \tag{3.17}
$$

$$
q^{\mathcal{R}_n}(T, q_0) \to q^{\mathcal{R}}(T, q_0) \textit{ in law, as } H^0\textit{-r.v.} \tag{3.18}
$$

Proof. Suppose $\mathcal{R}_n = (W_n, \mu_n)$ and $\mathcal{R} = (W, \mu)$. We write $q_n(\cdot) := q^{\mathcal{R}_n}(\cdot, q_0)$ for simplicity. Define $p_n(t) := e^{-MW_n(t)}q_n(t)$, $p_{n,1}(t) := \int_0^t e^{-MW_n(s)}g\, dW_n(s)$, and $p_{n,2}(t) := p_n(t) - p_{n,1}(t)$. Then, by Propositions 3.1 and 3.2, we have

$$
\begin{aligned}
E\int_0^T \|p_n(t)\|_1^2\, dt &\le E\int_0^T e^{2N|W_n(t)|}\|q_n(t)\|_1^2\, dt \\
&\le E\left(\sup_{0\le t\le T} e^{2N|W_n(t)|}\int_0^T \|q_n(t)\|_1^2\, dt\right) \\
&\le \left(E\sup_{0\le t\le T} e^{4N|W_n(t)|}\right)^{1/2}\left(E\int_0^T \|q_n(t)\|_1^4\, dt\right)^{1/2} \cdot T^{1/2}.
\end{aligned} \tag{3.19}
$$

It is known from Fernique's lemma [5] that $\sup_n (E \sup_{0\le t\le T} e^{4N|W_n(t)|}) < +\infty$; hence (3.19) yields

$$
\sup_n E\int_0^T \|p_n(t)\|_1^2\, dt < +\infty. \tag{3.20}
$$

Similarly, we have

$$\sup_n E\int_0^T \|p_{n,1}(t)\|_1^2\,dt \leqq \sup_n E\int_0^T\int_0^t e^{2N|W_n(s)|}\|g\|_1^2\,ds\,dt < +\infty. \tag{3.21}$$

Combining (3.20) and (3.21) gives

$$\sup_n E\int_0^T \|p_{n,2}(t)\|_1^2\,dt < +\infty. \tag{3.22}$$

On the other hand, by virtue of Proposition 3.3,

$$\begin{aligned} d(p_{n,2}(t), \phi)_0 = \{&\langle (A(t,\mu_n) - \tfrac{1}{2}M^2)q_n(t), e^{-M^*W_n(t)}\phi\rangle_0 \\ &+ (e^{-MW_n(t)}(f(t,\mu_n) - Mg), \phi)_0\}\,dt, \end{aligned} \tag{3.23}$$

for any $\phi \in H^1$.

Hence

$$\begin{aligned} |\langle dp_{n,2}(t)/dt, \phi\rangle_0| &\leqq \|(A(t,\mu_n) - \tfrac{1}{2}M^2)q_n(t)\|_{-1}\|e^{-M^*W_n(t)}\phi\|_1 \\ &\quad + e^{N|W_n(t)|}\|f(t,\mu_n) - Mg\|_0\|\phi\|_0 \\ &\leqq \text{const } e^{N|W_n(t)|}(\|q_n(t)\|_1 + 1)\|\phi\|_1, \end{aligned}$$

for any $\phi \in H^1$.

This yields

$$\begin{aligned} &\sup_n E\int_0^T \|dp_{n,2}(t)/dt\|_{-1}^2\,dt \\ &\leqq \text{const} \sup_n E\int_0^T e^{2N|W_n(t)|}(\|q_n(t)\|_1^2 + 1)\,dt \\ &< +\infty. \end{aligned} \tag{3.24}$$

Equations (3.22) and (3.24) imply that there exists a constant C_1 that is independent of n such that, for any $D \subset R^d$ with the property (3.10),

$$E\|p_{n,2}\|^2_{W_D[0,T]} \leqq C_1. \tag{3.25}$$

Let $D_k := \{x \in R^d : |x| < k\}$ for $k = 1, 2, \cdots$. Define a metric $\bar{d}$ on $L^2(0, T; H^0)$ by

$$\bar{d}(\phi, \psi) := \sum_{k=1}^{\infty} \frac{1}{2^k} \min\left\{1, \left(\int_0^T \|\phi(t) - \psi(t)\|^2_{0,D_k}\,dt\right)^{1/2}\right\}. \tag{3.26}$$

We denote by $\bar{L}^2(0, T; H^0)$ the completion of $L^2(0, T; H^0)$ by $\bar{d}$. For $\lambda > 0$,

$$B_\lambda := \{\phi \in \bar{L}^2(0, T; H^0) : \|\phi\|_{W_{D_k}[0,T]} \leqq (2^k\lambda)^{1/2}, \quad k = 1, 2, \cdots\}$$

is compact in $\bar{L}^2(0, T; H^0)$ due to Lemma 3.2. Now (3.25) yields

$$P(p_{n,2} \in B_\lambda) \leqq \sum_{k=1}^{\infty} \frac{1}{2^k\lambda} C_1 \leqq C_1/\lambda, \quad \text{for any } \lambda > 0; \tag{3.27}$$

hence, $\{p_{n,2}\}$ is tight as $\bar{L}^2(0, T; H^0)$-r.v. (cf. [7, Def. 2.2, p. 7]). Noting the compactness of Λ, $\{(W_n, \mu_n, p_{n,2})\}$ is tight as $C(0, T; R^1) \times \Lambda \times \bar{L}^2(0, T; H^0)$-r.v. Hence by Skorohod's theorem, we can choose a subsequence $\{n'\}$ and have $(\hat{W}_{n'}, \hat{\mu}_{n'}, \hat{p}_{n',2})$, $(\hat{W}, \hat{\mu}, \hat{p}_2)$ on a suitable probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$, such that

$$\text{law of } (\hat{W}_{n'}, \hat{\mu}_{n'}, \hat{p}_{n',2}) = \text{law of } (W_{n'}, \mu_{n'}, p_{n',2}), \tag{3.28}$$

and $\hat{P}$-almost surely

$$\hat{W}_{n'} \to \hat{W} \quad \text{uniformly on } [0, T], \tag{3.29}$$

$$\hat{\mu}_{n'} \to \hat{\mu} \quad \text{weakly on } \Lambda, \tag{3.30}$$

$$\hat{p}_{n',2} \to \hat{p}_2 \quad \text{in } \bar{L}^2(0, T; H^0), \text{ as } n' \to +\infty. \tag{3.31}$$

Define

$$\hat{p}_{n',1}(t) := \int_0^t e^{-M\hat{W}_{n'}(s)} g \, d\hat{W}_{n'}(s), \qquad \hat{p}_1(t) := \int_0^t e^{-M\hat{W}(s)} g \, d\hat{W}(s).$$

By virtue of (3.29) and the strong continuity of the group $\{e^{Mt}\}$, we have for fixed $(s, \hat{\omega})$, $\|e^{-M\hat{W}_{n'}(s)}g - e^{-M\hat{W}(s)}g\|_0^2 \to 0$ as $n' \to \infty$. By Fernique's lemma, it is easy to check that

$$\sup_{n'} \hat{E} \int_0^T \|e^{-M\hat{W}_{n'}(s)}g - e^{-M\hat{W}(s)}g\|_0^4 \, ds < +\infty,$$

which means that $\{\|e^{-M\hat{W}_{n'}(s)}g - e^{-M\hat{W}(s)}g\|_0^2\}$ is uniformly integrable on $[0, T] \times \hat{\Omega}$; hence

$$e^{-M\hat{W}_{n'}(s)}g \to e^{-M\hat{W}(s)}g \quad \text{in } L^2([0, T] \times \hat{\Omega}; H^0), \quad \text{as } n' \to \infty. \tag{3.32}$$

Combining (3.32) with (3.29), by a similar argument to that in [12, Lemma 3.3], we get

$$\hat{E} \int_0^T \|\hat{p}_{n',1}(t) - \hat{p}_1(t)\|_0^2 \, dt \to 0, \quad \text{as } n' \to \infty. \tag{3.33}$$

Define

$$\hat{p}_{n'}(t) := \hat{p}_{n',1}(t) + \hat{p}_{n',2}(t), \qquad \hat{p}(t) := \hat{p}_1(t) + \hat{p}_2(t);$$

then (3.31) and (3.33) yields and there exists a subsequence of $\{n'\}$ (still denoted by $\{n'\}$) such that

$$\hat{p}_{n'} \to \hat{p} \quad \text{in } \bar{L}^2(0, T; H^0), \quad \text{as } n' \to \infty, \qquad \hat{P}\text{-a.s.} \tag{3.34}$$

Define

$$\hat{q}_{n'}(t) := e^{M\hat{W}_{n'}(t)} \hat{p}_{n'}(t), \qquad \hat{q}(t) := e^{M\hat{W}(t)} \hat{p}(t).$$

Observing (3.28), we have

$$\text{law of } (\hat{W}_{n'}, \hat{\mu}_{n'}, \hat{q}_{n'}) = \text{law of } (W_{n'}, \mu_{n'}, q_{n'}). \tag{3.35}$$

Now we want to show

$$\hat{E} \int_0^T \|\hat{q}_{n'}(t) - \hat{q}(t)\|_0^2 \, dt \to 0, \quad \text{as } n' \to +\infty. \tag{3.36}$$

Indeed, for any $D \subset R^d$ that satisfies (3.10), we have

$$\begin{aligned}
\hat{E} \int_0^T &\|\hat{q}_{n'}(t) - \hat{q}(t)\|_{0,D}^2 \, dt \\
&= \hat{E} \int_0^T \|e^{M\hat{W}_{n'}(t)} \hat{p}_{n'}(t) - e^{M\hat{W}(t)} \hat{p}(t)\|_{0,D}^2 \, dt \\
&\leq 2\hat{E} \int_0^T \|e^{M\hat{W}_{n'}(t)} (\hat{p}_{n'}(t) - \hat{p}(t))\|_{0,D}^2 \, dt \\
&\quad + 2\hat{E} \int_0^T \|(e^{M\hat{W}_{n'}(t)} - e^{M\hat{W}(t)}) \hat{p}(t)\|_{0,D}^2 \, dt = I_3 + I_4, \quad \text{say},
\end{aligned} \tag{3.37}$$

$$I_3 \leqq 2\hat{E}\int_0^T e^{2N|\hat{W}_{n'}(t)|}\|\hat{p}_{n'}(t)-\hat{p}(t)\|_{0,D}^2\,dt$$
$$\leqq 2\hat{E}\left(\sup_{0\leqq t\leqq T} e^{2N|\hat{W}_{n'}(t)|}\int_0^T \|\hat{p}_{n'}(t)-\hat{p}(t)\|_{0,D}^2\,dt\right)$$
$$\leqq \text{const}\left\{\hat{E}\left(\int_0^T \|\hat{p}_{n'}(t)-\hat{p}(t)\|_{0,D}^2\,dt\right)^2\right\}^{1/2}.$$

By (3.4), it is easy to see that $\{(\int_0^T \|\hat{p}_{n'}(t)-\hat{p}(t)\|_{0,D}^2\,dt)^2\}$ is uniformly integrable on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$. Thus (3.34) derives $I_3 \to 0$ as $n' \to \infty$. Moreover, since for fixed $(t, \hat{\omega})$, $\|(e^{M\hat{W}_{n'}(t)} - e^{M\hat{W}(t)})\hat{p}(t)\|_{0,D}^2 \to 0$ as $n' \to \infty$, so by the uniformly integrability of $\{\|(e^{M\hat{W}_{n'}(t)} - e^{M\hat{W}(t)})\hat{p}(t)\|_{0,D}^2\}$ on $[0, T]\times\hat{\Omega}$, we have $I_4 \to 0$, as $n' \to +\infty$. Now we arrive at

$$\hat{E}\|\hat{q}_{n'} - \hat{q}\|_{L^2(0,T;H^0)} \to 0, \quad \text{as } n' \to +\infty. \tag{3.38}$$

By (3.38) and (3.16), we can write

$$\hat{E}\int_0^T \int_{|x|>\rho} |\hat{q}(t,x)|^2\,dx\,dt = \lim_{k\to\infty}\lim_{n'\to\infty} \hat{E}\int_0^T \int_{\rho<|x|<k} |\hat{q}_{n'}(t,x)|^2\,dx\,dt \tag{3.39}$$
$$\leqq CT/(1+\rho^2)^r \to 0, \quad \text{as } \rho \to +\infty.$$

Hence $\hat{q} \in L^2([0, T]\times\hat{\Omega}; H^0)$, and (3.36) follows from (3.38), (3.39), and (3.16).

By virtue of (3.38), we can prove that $\hat{q}$ is just the response for $\hat{\mathcal{R}} := (\hat{W}, \hat{\mu})$ by a standard argument (cf. [1], [12], and [13]). Noting (3.28)-(3.36), we have proved (3.17). For (3.18), we can apply entirely the same argument, observing the compactness of the embedding: $H^1(D) \to H^0(D)$. The proof is now complete. □

Remark 3.1. Condition (A3), and $q_0 \in L_r^2$ are required to ensure (3.36) from (3.38). If we drop these conditions, (3.38) still holds, and therefore $\hat{q}$ is still the response of $\hat{\mathcal{R}}$ by the above proof. Thus we have the following corollary.

COROLLARY 3.2. *If we only assume* (A1)-(A3) *and* $q_0 \in H^1$, *then whenever* $\mathcal{R}_n \to \mathcal{R}$, *we have*

$$q^{\mathcal{R}_n}(\cdot, q_0) \to q^{\mathcal{R}}(\cdot, q_0) \text{ in law}, \quad \text{as } w-L^2(0, T; H^1)\text{-r.v.}; \tag{3.40}$$

$$q^{\mathcal{R}_n}(T, q_0) \to q^{\mathcal{R}}(T, q_0) \text{ in law}, \quad \text{as } w-H^1\text{-r.v.}, \tag{3.41}$$

where "$w-X$" *denotes the space* X *endowed with the weak topology.*

Now we are in the position to prove the existence of optimal relaxed control for system (2.9) with the cost functional (2.12).

THEOREM 3.2. *In addition to the same assumptions as in Theorem* 3.1, *we assume*

(3.42) *F, G are continuous mappings from $L^2(0, T; H^0)$ and H^0 to R^1, respectively, and there exists $K > 0$, such that*

$$|F(\phi)| \leqq K(1+\|\phi\|_{L^2(0,T;H^0)}); \qquad |G(\psi)| \leqq K(1+\|\psi\|_0).$$

Then there exists an optimal relaxed control for the system (2.9) *with the cost functional* (2.12).

Proof. By Theorem 3.1, $J(q_0, \cdot)$ is continuous. However, Π is a compact metric space, which yields the existence of an optimal relaxed control. □

COROLLARY 3.3. *In addition to the assumptions of Corollary* 3.2, *we assume*

(3.43) *F, G are weakly continuous mappings from $L^2(0, T; H^1)$ and H^1 to R^1, respectively, and satisfy linear growth conditions as in* (3.42).

Then there exists an optimal relaxed control.

Proof. The result is a direct consequence of Corollary 3.2. □

4. Nondegenerate case. In this section we consider the optimal relaxed control problem (2.9) and (2.12) under the assumption that $(a^{ij}-\frac{1}{2}\sigma^i\sigma^j)_{ij}$ is uniformly positive definite. It will be proved that the existence of an optimal relaxed control still holds even when $q_0 \in H^0$.

The following assumptions remain in force throughout this section:

(B1) Same as (A1) in § 3;

(B2) $a^{ij}=a^{ji}$, $i, j=1, 2, \cdots, d$, and there exists $\alpha>0$ such that

$$(a^{ij}(x,u)-\tfrac{1}{2}\sigma^i(x)\sigma^j(x))\xi_i\xi_j \geqq \alpha|\xi|^2, \quad \text{for any } (x, u, \xi)\in R^d\times\Gamma\times R^d;$$

(B3) $f(\cdot, u)\in H^{-1}$, $g\in H^0$; the absolute values of f, g together with the H^{-1}-norm of $f(\cdot, u)$ and the H^0-norm of g do not exceed K.

The following proposition is an easy variant of the results in Krylov and Rozovskii [8] and Pardoux [14].

PROPOSITION 4.1. *Suppose $q_0\in H^0$, then for any $\mathcal{R}\in$ R, (2.9) has a unique solution $q^{\mathcal{R}}(\cdot, q_0)\in L^2([0, T]\times\Omega; H^1)\cap L^2(\Omega; C(0, T; H^0))$ and there exists a constant C, depending only on K, T, and α, such that*

$$\sup_{0\leqq t\leqq T} E\|q^{\mathcal{R}}(t, q_0)\|_0^2 \leqq CE\left\{\|q_0\|_0^2+\int_0^T [\|f(s,\mu)\|_{-1}^2+\|g\|_0^2]\,ds\right\}, \tag{4.1}$$

$$E\left(\int_0^T \|q^{\mathcal{R}}(t, q_0)\|_1^2\,dt\right)^2 \leqq CE\left\{\|q_0\|_0^4+\int_0^T [\|f(s,\mu)\|_{-1}^4+\|g\|_0^4]\,ds\right\}. \tag{4.2}$$

Moreover, for any $p\geqq 2$, there exists a constant $C(p)$ such that

$$\sup_{0\leqq t\leqq T} E\|q^{\mathcal{R}}(t, q_0)\|_0^p \leqq C(p)E\left\{\|q_0\|_0^p+\int_0^T [\|f(s,\mu)\|_{-1}^p+\|g\|_0^p]\,ds\right\}. \tag{4.3}$$

THEOREM 4.1. *Assume $q_0\in H^0$. Denote by $q^{\mathcal{R}}(\cdot, q_0)$ the response for $\mathcal{R}=(W,\mu)$ of the system. Then whenever $\mathcal{R}_n\to\mathcal{R}$, we have*

$$q^{\mathcal{R}_n}(\cdot, q_0)\to q^{\mathcal{R}}(\cdot, q_0) \text{ in law}, \quad \text{as } w-L^2(0,T;H^1)\text{-r.v.}; \tag{4.4}$$

$$q^{\mathcal{R}_n}(T, q_0)\to q^{\mathcal{R}}(T, q_0) \text{ in law}, \quad \text{as } w-H^0\text{-r.v.} \tag{4.5}$$

Proof. For simplicity, we denote $q_n(\cdot):=q^{\mathcal{R}_n}(\cdot, q_0)$, $q(\cdot):=q^{\mathcal{R}}(\cdot, q_0)$, for $\mathcal{R}_n=(W_n, \mu_n)$, $\mathcal{R}=(W,\mu)$.

Define $p_n(t):=e^{-MW_n(t)}q_n(t)$, then

$$\begin{aligned} E\int_0^T \|p_n(t)\|_1^2\,dt &\leqq E\int_0^T e^{2N|W_n(t)|}\|q_n(t)\|_1^2\,dt \\ &\leqq E\left(\sup_{0\leqq t\leqq T} e^{2N|W_n(t)|}\int_0^T \|q_n(t)\|_1^2\,dt\right) \\ &\leqq \left(E\sup_{0\leqq t\leqq T} e^{4N|W_n(t)|}\right)^{1/2}\left\{E\left(\int_0^T \|q_n(t)\|_1^2\,dt\right)^2\right\}^{1/2}. \end{aligned} \tag{4.6}$$

So by (4.2), $\sup_n E\int_0^T \|p_n(t)\|_1^2\,dt<+\infty$. Now by applying entirely the same argument as that in the proof of Theorem 3.1 and Corollary 3.2, we obtain (4.4). Noting (4.1), we also get (4.5). (But we no longer have $q_n(T)\to q(T)$ as $w-H^1$-random variable.) The proof is complete. □

Now we establish the existence of an optimal relaxed control for the system (2.9) with the cost functional (2.12).

THEOREM 4.2. *Suppose* $q_0 \in H^0$ *and* F, G *are weakly continuous mappings from* $L^2(0, T; H^1)$ *and* H^0, *respectively, to* R^1, *and they satisfy linear growth conditions. Then there exists an optimal relaxed control.*

5. Discussions and applications.

5.1. Multidimensional Wiener process cases. The method employed in the previous sections is somewhat similar to the *time change* technique in stochastic analysis. This method, however, fails to be effective in general for the system as follows:

$$\begin{aligned} &dq(t) = (A(t,\mu)q(t) + f(t,\mu))\,dt + \sum_{k=1}^{d'} (M_k q(t) + g_k)\,dW^k(t), \\ &q(0) = q_0, \end{aligned} \tag{5.1}$$

where $W := (W^1, \cdots, W^{d'})$ is a d'-dimensional Wiener process, and A, M_k have the same forms as (2.1) and (2.2).

However, in some special cases, we can still treat (5.1) by a similar argument to the one-dimensional Wiener process cases.

THEOREM 5.1. *Assume the coefficients of* A, M_k $(k = 1, 2, \cdots, d')$ *satisfy* (A1) *and* (A2) *of* § 3 (*respectively,* (B1) *and* (B2) *of* § 4). *Assume further that*

$$M_k M_j = M_j M_k, \quad \text{for } k \neq j, \tag{5.2}$$

then Theorem 3.2 *and Corollary* 3.3 (*respectively, Theorem* 4.2) *hold for the system* (5.1).

Proof. We will show this for $d' = 2$ for simplicity. By the proofs in the previous sections, it suffices to prove that we can construct a transformation $q :\to p$, such that p satisfies an SPDE whose diffusion term is independent of the state (refer to Proposition 3.3). To this end, define $p_1(t) := e^{-M_1 W^1(t)} q(t)$, then a similar calculation to that of Proposition 3.3 gives that p_1 satisfies the following differential formula in H^{-1} space:

$$\begin{aligned} dp_1(t) = {} & e^{-M_1 W^1(t)}\{(A(t,\mu) - \tfrac{1}{2}M_1^2)q(t) + f(t,\mu) - M_1 g_1\}\,dt \\ & + e^{-M_1 W^1(t)} g_1\,dW^1(t) + \{e^{-M_1(t) W^1(t)}(M_2 q(t) + g_2)\}\,dW^2(t). \end{aligned} \tag{5.3}$$

Put $p(t) := e^{-M_2 W^2(t)} p_1(t)$, then, for any $\phi \in H^1$,

$$\begin{aligned} d(p(t), \phi)_0 &= d(p_1(t), e^{-M_2^* W^2(t)}\phi)_0 \\ &= \langle e^{-M_1 W^1(t)}\{(A(t,\mu) - \tfrac{1}{2}M_1^2)q(t) + f(t,\mu) - M_1 g_1\}, e^{-M_2^* W^2(t)}\phi\rangle_0\,dt \\ &\quad + (e^{-M_1 W^1(t)} g_1, e^{-M_2^* W^2(t)}\phi)_0\,dW^1(t) \\ &\quad + (e^{-M_1 W^1(t)}(M_2 q(t) + g_2), e^{-M_2^* W^2(t)}\phi)_0\,dW^2(t) \\ &\quad + (p_1(t), \tfrac{1}{2}M_2^{*2} e^{-M_2^* W^2(t)}\phi)_0\,dt - (p_1(t), M_2^* e^{-M_2^* W^2(t)}\phi)_0\,dW^2(t) \\ &\quad + (e^{-M_1 W^1(t)}(M_2 q(t) + g_2), -M_2^* e^{-M_2^* W^2(t)}\phi)_0\,dt \\ &= \langle (A(t,\mu) - \tfrac{1}{2}M_1^2 - \tfrac{1}{2}M_2^2)q(t), e^{-M_1^* W^1(t)} e^{-M_2^* W^2(t)}\phi\rangle_0\,dt \\ &\quad + (e^{-M_2 W^2(t)} e^{-M_1 W^1(t)}(f(t,\mu) - M_1 g_1 - M_2 g_2), \phi)_0\,dt \\ &\quad + (e^{-M_2 W^2(t)} e^{-M_1 W^1(t)} g_1, \phi)_0\,dW^1(t) \\ &\quad + (e^{-M_2 W^2(t)} e^{-M_1 W^1(t)} g_2, \phi)_0\,dW^2(t). \end{aligned} \tag{5.4}$$

Note that in the above calculation, we have repeatedly employed the fact that $e^{M_1 t} M_2|_{H^1} = M_2 e^{M_1 t}|_{H^1}$, which is an easy consequence of the fact that $M_1 M_2 = M_2 M_1$. Now (5.4) has a similar form to (3.12), which allows us to apply the same argument as those in §§ 3 and 4 to obtain the desired results. □

Remark 5.1. Equation (5.2) holds when the coefficients of $\{M_k\}$ are constants or satisfy some linear dependence conditions.

Remark 5.2. In particular, (5.2) holds for the following kind of systems:

$$
\begin{aligned}
&dq(t) = (A(t,\mu)q(t) + f(t,\mu))\,dt + \sum_{k=1}^{d'} (h_k q(t) + g_k)\,dW^k(t), \\
&q(0) = q_0,
\end{aligned} \tag{5.5}
$$

where $h_k : R^d \to R^1$. The above system has been studied by Bensoussan and Nisio [1] (with $f = g_k = 0$). Their result is [1, Thm. 5.2]: if (a^{ij}) is uniformly positive definite and $q_0 \in H^2(R^d)$, then there exists an optimal relaxed control. Now we know the condition "$q_0 \in H^2(R^d)$" can be weakened to "$q_0 \in L^2(R^d)$"; on the other hand, if we assume $q_0 \in H^1(R^d)$, then the result is valid even if (a^{ij}) is degenerate.

5.2. Stochastic control with partial observation. First, let us remark that the results in §§ 3 and 4 are easily extended to the following kind of system:

$$
\begin{aligned}
&dq(t) = (\tilde{A}(t, W(t), \mu)q(t) + \tilde{f}(t, W(t), \mu))\,dt + (Mq(t) + g(W(t)))\,dW(t), \\
&q(0) = q_0,
\end{aligned} \tag{5.6}
$$

where

$$
\begin{aligned}
\tilde{A}(t, w, \mu)\phi(x) &:= \partial_i(\tilde{a}^{ij}(t, x, w, \mu)\partial_j\phi(x)) + \tilde{b}^i(t, x, w, \mu)\partial_i\phi(x) \\
&\quad + \tilde{c}(t, x, w, \mu)\phi(x), \\
M\phi(x) &:= \sigma^i(x)\partial_i\phi(x) + h(x)\phi(x), \\
\tilde{a}^{ij}(t, x, w, \mu) &:= \int_\Gamma a^{ij}(x, w, u)\mu'(t, du), \quad \text{etc.}
\end{aligned}
$$

Let B and $\hat{B}$ be independent Wiener processes on a probability space $(\Omega, \mathcal{F}, P)$, with values in R^1 and R^d, respectively. Consider the following SPDE in R^d:

$$
\begin{aligned}
&dX(t) = \gamma(X(t), Y(t), U(t))\,dt + \alpha(X(t), Y(t), U(t))\,d\hat{B}(t) + \sigma(X(t))\,dB(t), \\
&X(0) = \xi,
\end{aligned} \tag{5.7}
$$

with the observation

$$
dY(t) = \kappa(X(t))\,dt + dB(t), \qquad Y(0) = 0, \tag{5.8}
$$

where U is an admissible control. Note that σ is the correlation between the state and the observation.

Let Ξ and $L : R^d \times R^1 \to R^1$. The problem is to minimize the cost functional defined by

$$
J(U) := E\left\{\int_0^T \Xi(X(t), Y(t))\,dt + L(X(t), Y(t))\right\}, \tag{5.9}
$$

over the totality of admissible controls.

By the well-known relationship between the control problem for partially observed diffusions and that for SPDEs (cf. [12]–[15]), we may interpret the problem (5.7)–(5.9) to the one we have treated in the previous sections. Thus we have the following theorem, appealing to Corollary 3.3 and Theorem 4.2.

THEOREM 5.2. *We make the following assumptions*:

(C1) $\alpha : R^d \times R^1 \times \Gamma \to R^{d\times d}$, $\sigma : R^d \to R^d$, $\gamma : R^d \times R^1 \times \Gamma \to R^d$, $\kappa : R^d \to R^1$ *are continuous functions.* α, σ *and their derivatives in x up to fourth order do not exceed a constant K in absolute value*; γ, κ, *and their derivatives in x up to third order do not exceed K in absolute value*;

(C2) Ξ, $L: R^d \times R^1 \to R^1$ *are Lipschitz continuous and bounded by* K.

Then there exists an optimal relaxed control of the problem (5.7)-(5.9), *if we assume, in addition, either of the following two conditions*:

(i) ξ *has a density* $q_0 \in H^1$;

(ii) ξ *has a density* $q_0 \in H^0$, *and* $\alpha\alpha^*(x, y, u)$ *is uniformly positive definite.*

Remark 5.3. In [12] and [13], in addition to the higher regularity condition on q_0, it is required that $(\alpha\alpha^* - 2\sigma\sigma^*)$ be nonnegative definite, which means the correlation cannot be "too large." In this paper, this restriction is removed.

Let us conclude the paper by two remarks.

Remark 5.4. The relaxed controls turn out to be (usual) admissible controls when assuming some convex conditions (Roxin's conditions) on the coefficients a^{ij}, etc. The reader may refer to [1], [13] for details.

Remark 5.5. The aim of this paper is to reduce the regularity on the initial state. In the viewpoint of filtering problems, it is more natural to consider the Dirac initial state, although it seems to be a rather difficult problem since the SPDE theory itself with the Dirac initial condition has not been well established.

Appendix. In this appendix, we prove Proposition 3.2.

On the space H^0, consider the following densely defined operator:

$$(A.1)\qquad M: \begin{cases} D(M) \quad (=\text{the domain of } M) := H^1 \\ M\phi(x) := \sigma^i(x)\partial_i\phi(x) + h(x)\phi(x), \quad \text{for } \phi \in H^1, \qquad x \in R^d. \end{cases}$$

M thus defined is not a closed operator, but it is clearly closable. Denote by $\bar{M}$ the closed extension of M (i.e., the graph of $\bar{M}$ is the closure of the graph of M). On the other hand, M^*, the adjoint operator of M on H^0, is given by

$$(A.2)\; M^*: \begin{cases} D(M^*) = \{\phi \in H^0 : -\partial_i(\sigma^i\phi) + h\phi \in H^0\}, \\ M^*\phi(x) = -\partial_i(\sigma^i(x)\phi(x)) + h(x)\phi(x), \quad \text{for } \phi \in D(M^*), \qquad x \in R^d. \end{cases}$$

Since H^0 is reflexive, so $\bar{M} \equiv M^{**}$, which is given by

$$(A.3)\qquad \bar{M}: \begin{cases} D(\bar{M}) = \{\phi \in H^0 : \sigma^i\partial_i\phi + h\phi \in H^0\}, \\ \bar{M}\phi(x) = \sigma^i(x)\partial_i\phi(x) + h(x)\phi(x), \quad \text{for } \phi \in D(\bar{M}), \qquad x \in R^d. \end{cases}$$

Moreover, $\bar{M}^* = M^{***} = M^*$.

LEMMA A.1. *There exists a constant* N, *which is given in Corollary* 3.1, *such that*

$$(A.4)\qquad |(\bar{M}\phi, \phi)_0| \leqq N\|\phi\|_0^2, \quad \text{for } \phi \in D(\bar{M});$$

$$(A.5)\qquad |(\bar{M}^*\phi, \phi)_0| \leqq N\|\phi\|_0^2, \quad \text{for } \phi \in D(\bar{M}^*).$$

Proof. Since $\bar{M}$ is the closed extension of M, for $\phi \in D(\bar{M})$, there exist $\{\phi_n\} \subset H^1$, such that $\phi_n \xrightarrow{H^0} \phi$, $M\phi_n \xrightarrow{H^0} \bar{M}\phi$. Hence (A.4) follows from Corollary 3.1. On the other hand, $\bar{M}^*$ is the closed extension of $K\phi := -\partial_i(\sigma^i\phi) + h\phi$ with $D(K) := H^1$, so (A.5) is proved by a similar argument. □

Proof of Proposition 3.2. Let $\lambda \in R^1$ such that $|\lambda| > N$. For $\phi \in D(\bar{M})$, set $\psi := (\lambda I - \bar{M})\phi$, then

$$(A.6)\qquad |(\psi, \phi)_0| \geqq |\lambda|\,\|\phi\|_0^2 - |(\bar{M}\phi, \phi)| \geqq (|\lambda| - N)\|\phi\|_0^2,$$

which yields $\lambda I - \bar{M}$ is one-to-one, and Range $(\lambda I - \bar{M})$ is a closed set in H^0. However, by virtue of (A.5), $\lambda I - \bar{M}^*$ is also one-to-one; thus Range $(\lambda I - \bar{M}) = \text{Ker}\,(\lambda I - \bar{M}^*)^\perp = H^0$. Then, by (A.6), λ is in the resolvent set of $\bar{M}$, and

$$(A.7)\qquad \|(\lambda I - \bar{M})^{-1}\|_{L(H^0 \to H^0)} \leqq (|\lambda| - N)^{-1}.$$

By [2, Cor. 17, p. 628], $\bar{M}$ generates a strongly continuous group $\{e^{\bar{M}t}: -\infty < t < +\infty\}$ on H^0 and (3.5) holds.

On the space H^1, consider the following densely defined operator M_1:

$$(A.8)\qquad M_1: \begin{cases} D(M_1) := H^2, \\ M_1\phi(x) := \sigma^i(x)\partial_i\phi(x) + h(x)\phi(x), \qquad \phi \in H^2, \quad x \in R^d. \end{cases}$$

There is also a closed extension $\bar{M}_1$ of M_1. By virtue of Corollary 3.1,

$$|(\bar{M}_1\phi, \phi)_1| \leqq N\|\phi\|_1^2, \quad \text{for } \phi \in D(\bar{M}_1).$$

So by a similar argument as above, $\bar{M}_1$ generates a strongly continuous group $\{e^{\bar{M}_1 t}: -\infty < t < +\infty\}$ on H^1, and

$$(A.9)\qquad \|e^{\bar{M}_1 t}\|_{L(H^1 \to H^1)} \leqq e^{N|t|}, \quad \text{for } t \in (-\infty, +\infty).$$

For $\phi \in H^2 \subset D(\bar{M}_1)$, since obviously $\bar{M} \equiv \bar{M}_1$ on $D(\bar{M}_1)$,

$$(A.10)\qquad d(e^{\bar{M}_1 t}\phi)/dt = \bar{M}_1 e^{\bar{M}_1 t}\phi = \bar{M}e^{\bar{M}_1 t}\phi,$$

where "d/dt" is in H^1-topology. Consequently, (A.10) holds on H^0, which means $e^{\bar{M}_1 t}\phi = e^{\bar{M}t}\phi$ for any $\phi \in H^2$. However, H^2 is dense in H^1, so $e^{\bar{M}t}|_{H^1} \equiv e^{\bar{M}_1 t}$. Now (3.6) follows from (A.9). This proves (i) of Proposition 3.2.

Note $\bar{M}^*$ is also a closed extension of a first-order differential operator K, so (ii) of Proposition 3.2 is proved in a completely analogous way to (i). Moreover, since H^0 is reflexive, $e^{\bar{M}^* t} \equiv (e^{\bar{M}t})^*$.

Finally, (iii) of Proposition 3.2 is clear. □

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REFERENCES

[1] A. Bensoussan and M. Nisio, *Non linear semi-group arising in the control of diffusions with partial observation*, Stochastics, 30 (1990), pp. 1–46.

[2] N. Dunford and J. T. Schwartz, *Linear Operators, Part I*, Interscience, New York, 1958.

[3] G. Da Prato, M. Iannelli, and L. Tubaro, *Some results on linear stochastic differential equations in Hilbert spaces*, Stochastics, 6 (1982), pp. 105–116.

[4] N. El Karoui, D. Huù Nguyen, and M. Jeablanc-Picqué, *Existence of an optimal Markovian filter for the control under partial observations*, SIAM J. Control Optim., 26 (1988), pp. 1025–1061.

[5] X. Fernique, *Intégrabilité des vecteurs gaussiens*, C.R. Acad. Sci. Paris Sér. I Math., 270 (1970), pp. 1698–1699.

[6] W. H. Fleming and M. Nisio, *On stochastic relaxed control for partially observed diffusions*, Nagoya Math. J., 93 (1984), pp. 71–108.

[7] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusions Processes*, 2nd ed., Kodansha/North-Holland, Tokyo, 1989.

[8] N. V. Krylov and B. L. Rozovskii, *On the Cauchy problem for linear stochastic partial differential equations*, Ivz. Akad. Nauk SSSR Ser. Mat., 41 (1977), pp. 1329–1347; Math. USSR-Izv., 11 (1977), pp. 1267–1284.

[9] ———, *On characteristics of the degenerate parabolic Itô equations of the second order*, Proc. Petrovskii Sem., 8 (1982), pp. 153–168. (In Russian.)

[10] ———, *Stochastic partial differential equations and diffusion processes*, Uspekhi Mat. Nauk, 37 (1982), pp. 75–95; Russian Math. Surveys, 37 (1982), pp. 81–105.

[11] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaire*, Dunod, Paris, 1969.

[12] N. Nagase, *On the existence of optimal control for controlled stochastic partial differential equations*, Nagoya Math. J., 115 (1989), pp. 73–85.

[13] N. NAGASE AND M. NISIO, *Optimal controls for stochastic partial differential equations*, SIAM J. Control Optim., 28 (1990), pp. 186–213.
[14] E. PARDOUX, *Stochastic partial differential equations and filtering of diffusion processes*, Stochastics, 3 (1979), pp. 127–167.
[15] B. L. ROZOVSKII, *Nonnegative L^1-solutions of second order stochastic parabolic equations with random coefficients*, Steklov Seminar, Stat. Control of Stoch. Proc., 1984, Transl. Math. Monographs, American Mathematical Society, Providence, RI, 1985, pp. 410–427.