A new risk-sensitive maximum principle[*]

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Abstract

In this paper a new maximum principle for the risk-sensitive control problem is established. One important feature of this result is that it applies to systems in which the diffusion term may depend on the control. Such control dependence gives rise to interesting phenomena not observed in the usual setting where control independence of the diffusion term is assumed. In particular, there is an additional second order adjoint equation and additional terms in the maximum condition that involve this second order process as well as the risk-sensitive parameter. Moreover, contrary to a conventional maximum principle, the first order adjoint equation involved in our maximum principle is a *nonlinear* equation. An advantage of considering this new type of adjoint equation is that the risk-sensitive maximum principle derived is similar in form to its risk-neutral counterpart. The approach is based on the logarithmic transformation and the relationship between the adjoint variables and the value function. As an example, a linear-quadratic risk-sensitive problem is solved using the maximum principle derived.

Index Terms– risk-sensitive control; maximum principle; backward stochastic differential equations; adjoint equations; logarithmic transformation.

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1 Introduction

Risk-sensitive control has attracted much research attention in recent years. In this paper, we derive a maximum principle, namely, necessary conditions for optimality, for optimally controlled diffusion processes with an exponential-of-an-integral risk-sensitive cost functional. A key feature of our result is that it applies to systems in which the control input may appear in the diffusion term, and hence, affects the variance of the noise.

Although dynamic programming has been the tool predominantly used to study the risk-sensitive control, several papers have been devoted to the maximum principle. For systems with control *independent* diffusion terms, there are the papers by Whittle [17, 18], and Charalambous and Hibey [4]. In [17, 18], ideas from the theory of large deviations form the basis of a heuristic, rather than rigorous, derivation of a maximum principle for both full observation and partial observation problems. In [4], a measure-valued decomposition (to transform the Zakai equation) and weak control variations are used to obtain a minimum principle for the partial observation problem.

Since the publication of the deterministic maximum principle by Pontryagin *et. al.* [15], much work has been done on various generalizations to stochastic systems; see [6, 9] for example. However, in these and other papers, it is assumed that the diffusion term is control independent, and the resulting stochastic maximum principles are quite similar to the deterministic result. A stochastic maximum principle for systems with a control dependent diffusion term was first obtained by Peng [14] by considering the *second order* term in the Taylor expansion for the spike variation. His result brought to light certain key differences between deterministic and stochastic optimal control problems not seen in earlier results. For systems with control dependent diffusions, there is an *additional* (second order) adjoint equation, and the maximum condition involves both the first and second order adjoint processes. (We note that the risk-sensitive maximum principles [4, 5, 17, 18] only involve one adjoint process). One good illustration of these ideas is the stochastic linear-quadratic (LQ) problem. It is shown in [3] that if the diffusion depends on the control, a stochastic LQ problem can be well posed even if the control weighting matrix is negative definite. This contrasts sharply with the deterministic case or the stochastic case with control-independent diffusion terms where, in general, a positive semi-definite control weight is required for well posedness. We also mention that LQ problems with indefinite control weighting matrices and control dependent diffusions arise frequently in finance applications; for example mean-variance portfolio selection [10] and insurance problems [16]. For a unified account on these results, the reader should consult [19].

To the best of our knowledge, the risk-sensitive maximum principle established in this paper is entirely new, and is fundamentally different from the existing results. Apart from the aforementioned feature of control-dependent diffusion, the first order adjoint equation

2

is a *nonlinear* backward stochastic differential equation. This is contrary to any conventional maximum principle, deterministic and stochastic alike, where the (first order) adjoint equation is a linear equation. On the other hand, the maximum condition in our maximum principle is similar in form to that in a risk-neutral maximum principle, with some coefficient matrix appropriately modified incorporating the risk-sensitive parameter. This is again not seen in the literature.

In addition to the risk-sensitive maximum principle, one of the main contributions of this paper is the method of proof, which relies on novel transformations of the adjoint variables, and is considerably shorter and less technical than a typical 'first principles' derivation. The derivation consists of two steps outlined as follows. The first is a simple reformulation of the risk-sensitive problem as a (standard) risk-neutral problem by augmenting the state with an additional variable (that represents the evolution of the cost over time). An intermediate maximum principle is then obtained by a standard application of Peng's (risk-neutral) result to the augmented problem. This intermediate result is unsatisfactory however as it is significantly more complex than its risk-neutral counterpart and involves additional adjoint variables (due to the state space enlargement) that seem unnecessary. Consequently, the next step is to simplify. A key element of the simplification is a transformation of the intermediate first and second order adjoint variables that is inspired by the logarithmic transform used in the dynamic programming approach to the risk-sensitive problem [7] together with the fundamental relationship between the maximum principle and dynamic programming [20]. This transformation reduces the dimension of the (intermediate) system of adjoint equations and results in the *new* first and second order adjoint system associated with the risk-sensitive problem. After the new maximum principle is derived, we further prove that the maximum condition is *sufficient* under some additional convexity/concavity conditions. It should be noted that our approach relies on the value function being sufficiently smooth, a common assumption in dynamic programming but one not usually required for the maximum principle. Save for this however, our assumptions are generally weaker than those typically required for optimality results on risk-sensitive control; see remarks in Section 2.

Finally, when the diffusion term is independent of the control, the maximum principle is not very different from the deterministic one: The second order adjoint equation does not play any role, and the maximum condition contains no additional terms. However, in the case when the diffusion is control dependent, the maximum condition has additional terms involving second order adjoint variable and the risk-sensitive parameter; that is, the optimal control depends explicitly on these quantities.

The paper is organized as follows. In Section 2, we state the problem and assumptions.

In Section 3, we give statements of the new risk-sensitive maximum principle as well as its sufficiency for optimality. For the sake of readability, the proofs of the results are spread over several subsections in Section 4. In Section 5, we present a risk-sensitive LQ problem as an example of applying the general results established. We end with some concluding remarks in Section 6.

2 Problem statement

Let $s \in [0, T]$ and $W(\cdot)$ be a standard Brownian motion on $[s, T]$ defined on a given filtered probability space $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t^s\}_{t \geq s})$. Throughout this paper, $L_{\mathcal{F}}^2(s, T; \mathbb{R}^n)$ will denote the set of $\mathbb{R}^n$ valued, $\{\mathcal{F}_t^s\}_{t \geq s}$-adapted, square integrable processes on $[s, T]$, and $L_{\mathcal{F}}^\infty(s, T; \mathbb{R}^n)$ the set of $\mathbb{R}^n$-valued, $\{\mathcal{F}_t^s\}_{t \geq s}$-adapted, essentially bounded processes on $[s, T]$. For convenience, we shall assume that $W(\cdot)$ is one-dimensional. (The subsequent analysis and results can easily be extended to the case of higher-dimensional Brownian motions). For any $(s, x) \in [0, T] \times \mathbb{R}^n$, consider the following Ito stochastic differential equation (SDE):

$$\begin{cases} dx(t) &= b(t, x(t), u(t))\, dt + \sigma(t, x(t), u(t)) dW(t), \\ x(s) &= x. \end{cases} \tag{1}$$

We shall work in the weak formulation [19, p. 248]. For any $s \in [0, T]$, the class of *admissible controls* $\mathcal{U}[s, T]$ is the set of all 5-tuples $(\Omega, \mathcal{F}, P, W(\cdot), u(\cdot))$ satisfying the following conditions:

1. $(\Omega, \mathcal{F}, P)$ is a complete probability space.

2. $W(\cdot)$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$ over $[s, T]$ (with $W(s) = 0$, P-a.s.), and $\mathcal{F}_t^s$ is $\sigma\{W(r) : s \leq r \leq t\}$ augmented by all the P-null sets in $\mathcal{F}$.

3. $u : [s, T] \times \Omega \to U$ is an $\{\mathcal{F}_t^s\}_{t \geq s}$ adapted process on $(\Omega, \mathcal{F}, P)$.

4. Under $u(\cdot)$, for any $y \in \mathbb{R}^n$, the equation (1) admits a unique (in the sense of probability law) solution $x(\cdot)$ on $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t^s\}_{t \geq 0})$.

We write $(\Omega, \mathcal{F}, P, W(\cdot), u(\cdot)) \in \mathcal{U}[s, T]$. However, if there is no ambiguity, we may simply write $u(\cdot) \in \mathcal{U}[s, T]$ instead of the entire 5-tuple. If $x(\cdot)$ is the unique solution of (1) associated with the input $u(\cdot) \in \mathcal{U}[s, T]$, we refer to $(x(\cdot), u(\cdot))$ as an *admissible pair*.

The cost $J^\theta(s, x; u(\cdot))$ associated with the initial condition $(s, x) \in [0, T] \times \mathbb{R}^n$ and control $u(\cdot) \in \mathcal{U}[s, T]$ is given by

$$J^\theta(s, x; u(\cdot)) = E e^{\theta\left[g(x(T)) + \int_s^T f(t, x(t), u(t))\, dt\right]}, \tag{2}$$

4

where $\theta > 0$, the risk-sensitive parameter, is a given fixed constant. The risk-sensitive control problem associated with (1)-(2) is defined as follows:

$$
\begin{cases}
\text{Minimize} & J^\theta(s,\,x;\,u(\cdot)) \\[2mm]
\text{subject to:} & \begin{cases} u(\cdot) \in \mathcal{U}[s,\,T] \\ (x(\cdot), u(\cdot)) \text{ satisfies (1).} \end{cases}
\end{cases} \tag{3}
$$

The value function $v^\theta : [0,\,T] \times \mathbb{R}^n \to \mathbb{R}$ associated with (3) is defined by:

$$
v^\theta(s,\,x) := \inf_{u(\cdot) \in \mathcal{U}[s,\,T]} J^\theta(s,\,x;\,u(\cdot)). \tag{4}
$$

Due to the exponential function involved, it is clear that $v^\theta(s,\,x) \geq 0$. Hence, we say that (3) is *well posed* if $v^\theta(s,\,x) > 0$, or equivalently, if $\ln v^\theta(s,\,x) > -\infty$.

We introduce the following assumptions.

Assumptions:

(B1) U is a separable metric space and $T > 0$.

(B2) The maps $b : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $\sigma : [0,T] \times U \to \mathbb{R}^n$, $f : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are measurable, and there exists a constant $L > 0$ and a modulus of continuity $\bar{\omega} : [0,\infty) \to [0,\infty)$ such that for $\varphi(t,x,u) = b(t,x,u)$, $\sigma(t,x)$, $f(t,x,u)$, $g(x)$,

$$
\begin{cases}
|\varphi(t,x,u) - \varphi(t,y,v)| \leq L|x-y| + \bar{\omega}(d(u,v)), \quad \forall t \in [0,T],\ x,y \in \mathbb{R}^n,\ u,v \in U, \\[2mm]
|\varphi(t,0,u)| \leq L, \quad \forall t \in [0,T],\ u \in U.
\end{cases}
$$

Also, f and g are uniformly bounded.

(B3) b, f, g are C^2 in x, and there exists a modulus of continuity $\bar{\omega} : [0,\infty) \to [0,\infty)$ such that for $\varphi(t,x,u) = b(t,x,u)$, $f(t,x,u)$, $g(x)$,

$$
\begin{cases}
|\varphi_x(t,x,u) - \varphi_x(t,y,v)| \leq L|x-y| + \bar{\omega}(d(u,v)), \\[2mm]
|\varphi_{xx}(t,x,u) - \varphi_{xx}(t,y,v)| \leq \bar{\omega}(|x-y| + d(u,v)), \\[2mm]
\forall t \in [0,T],\ x,y, \in \mathbb{R}^n,\ u,v \in U.
\end{cases}
$$

(B4) $v^\theta \in C^{1,3}([0,\,T] \times \mathbb{R}^n)$.

For the sufficiency of the maximum principle, we require the following additional assumption.

(B5) U is a convex subset of $\mathbb{R}^k$. The maps b, σ, and f are locally Lipschitz in u, and their derivatives in x are continuous in $(x,\,u)$.

As mentioned in the Introduction, our results depend on the value function $v^\theta(s,\,x)$ being sufficiently smooth; see **(B4)**. This is due to our use, in Section 4, of results on the relationship between the maximum principle and dynamic programming; see (26). A sufficient condition

for (**B4**) to hold is the so-called non-degeneracy assumption; that is, there exists a constant $\delta > 0$ such that $\sigma\sigma' \geq \delta I$; see [8]. On the other hand, apart from the assumption that f and g are uniformly bounded (which is standard in the risk-sensitive literature), our remaining assumptions are typical for a maximum principle result and weaker than those usually invoked in risk-sensitive control; see [1, 4, 7] for typical assumptions for the risk-sensitive problem and [19, p. 114] for the maximum principle.

3   Statement of risk-sensitive maximum principle

In this section, we present a maximum principle for the risk-sensitive control problem (3) as well as sufficient conditions for optimality. For the purpose of readability, proofs of these results will be given in the next section.

Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be an admissible pair for the system (1). We introduce the first order adjoint variable $(\bar{p}(\cdot), \bar{q}(\cdot)) \in L^2_{\mathcal{F}}(s, T; \mathbb{R}^n) \times L^2_{\mathcal{F}}(s, T; \mathbb{R}^n)$ and the second order adjoint variable $(\bar{P}(\cdot), \bar{Q}(\cdot)) \in L^2_{\mathcal{F}}(s, T; \mathbb{R}^{n \times n}) \times L^2_{\mathcal{F}}(s, T; \mathbb{R}^{n \times n})$ associated with the admissible pair $(\bar{x}(\cdot), \bar{u}(\cdot))$, which are the solutions of the following first order and second order adjoint equations respectively:

$$
\begin{cases}
d\bar{p}(t) = -\Big[\bar{b}_x(t)'\bar{p}(t) - \bar{f}_x(t)' - \theta\,\bar{p}(t)'\bar{\sigma}(t)\bar{\sigma}_x(t)'\bar{p}(t) \\
\qquad\qquad -\theta\,\bar{p}(t)'\bar{\sigma}(t)\bar{q}(t) + \bar{\sigma}_x(t)'\bar{q}(t)\Big]dt + \bar{q}(t)dW(t), \\
\bar{p}(T) = -g_x(\bar{x}(T)),
\end{cases}
\tag{5}
$$

$$
\begin{cases}
d\bar{P}(t) = -\Big\{\bar{b}_x(t)'\bar{P}(t) + \bar{P}(t)\bar{b}_x(t) + \bar{\sigma}_x(t)'[\bar{P}(t) - \theta\,\bar{p}(t)\bar{p}(t)']\bar{\sigma}_x(t) \\
\qquad\qquad +\bar{\sigma}_x(t)'[\bar{Q}(t) - \theta\,\bar{p}(t)\bar{q}(t)' - \theta\,\bar{p}(t)'\bar{\sigma}(t)\bar{P}(t)] \\
\qquad\qquad +[\bar{Q}(t) - \theta\,\bar{q}(t)\bar{p}(t)' - \theta\,\bar{p}(t)'\bar{\sigma}(t)\bar{P}(t)]\bar{\sigma}_x(t) \\
\qquad\qquad -\theta\,\bar{p}(t)'\bar{\sigma}(t)\bar{Q}(t) - \theta\,\bar{q}(t)\bar{q}(t)' + \bar{H}^\theta_{xx}(t, \bar{x}(t), \bar{u}(t), \bar{p}(t), \bar{q}(t))\Big\}dt \\
\qquad\qquad +\bar{Q}(t)dW(t), \\
\bar{P}(T) = -g_{xx}(\bar{x}(T)),
\end{cases}
\tag{6}
$$

where $\bar{b}_x(t) := b_x(t, \bar{x}(t), \bar{u}(t))$ (with similar interpretations for $\bar{f}_x(t)$, $\bar{\sigma}_x(t)$, etc.), and $\bar{H}^\theta : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, called the Hamiltonian associated with the pair $(\bar{x}(\cdot), \bar{u}(\cdot))$, is given by

$$
\bar{H}^\theta(t, x, u, p, q) = \langle p, b(t, x, u) \rangle - f(t, x, u) + \sigma(t, x, u)'\left[q - \theta\,pp'\,\sigma(t, \bar{x}(t), \bar{u}(t))\right].
\tag{7}
$$

Note that (5) and (6) are backward stochastic differential equations (BSDEs). In particular, the terminal conditions in (5) and (6) are $\mathcal{F}^s_T$-measurable random variables, and the solution

pairs $(\bar{p}(\cdot), \bar{q}(\cdot))$ and $(\bar{P}(\cdot), \bar{Q}(\cdot))$ must be $\{\mathcal{F}_t^s\}_{t \geq s}$-adapted. (For systematic discussions on BSDEs, the reader may refer to [13, 12, 19].) In addition, unlike the risk-neutral case where the adjoint equations are linear (see [14, 19]), (5) is a *nonlinear* equation. Moreover, while it does not appear that the usual conditions (Lipschitz continuity and linear growth) for existence and uniqueness of the solutions to nonlinear BSDEs ([12, 13, 19]) are satisfied by this equation, it will be shown however that our assumptions are sufficient to guarantee the existence of unique solutions $(\bar{p}(\cdot), \bar{q}(\cdot)) \in L_{\mathcal{F}}^2(s, T; \mathbb{R}^n) \times L_{\mathcal{F}}^2(s, T; \mathbb{R}^n)$ and $(\bar{P}(\cdot), \bar{Q}(\cdot)) \in L_{\mathcal{F}}^2(s, T; \mathbb{R}^{n \times n}) \times L_{\mathcal{F}}^2(s, T; \mathbb{R}^{n \times n})$ of (5) and (6), respectively.

Let the $\mathcal{H}$-function $\bar{\mathcal{H}}^\theta : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, associated with the pair $(\bar{x}(\cdot), \bar{u}(\cdot))$, be defined as

$$
\begin{aligned}
\bar{\mathcal{H}}^\theta(t, x, u) := & \langle \bar{p}(t), b(t, x, u) \rangle - f(t, x, u) \\
& + \frac{1}{2} \sigma(t, x, u)' [\bar{P}(t) - \theta \bar{p}(t) \bar{p}(t)'] \sigma(t, x, u) \\
& + \sigma(t, x, u)' [\bar{q}(t) - \bar{P}(t) \sigma(t, \bar{x}(t), \bar{u}(t))].
\end{aligned}
\tag{8}
$$

The risk-sensitive maximum principle can be stated as follows:

Theorem 3.1 (Risk-sensitive maximum principle) *Suppose that* **(B1)-(B4)** *hold. Let* $(\bar{x}(\cdot), \bar{u}(\cdot))$ *be an optimal pair for the risk-sensitive optimal control problem (3). Then there are unique solutions* $(\bar{p}(\cdot), \bar{q}(\cdot)) \in L_{\mathcal{F}}^2(s, T; \mathbb{R}^n) \times L_{\mathcal{F}}^2(s, T; \mathbb{R}^n)$ *and* $(\bar{P}(\cdot), \bar{Q}(\cdot)) \in L_{\mathcal{F}}^2(s, T; \mathbb{R}^{n \times n}) \times L_{\mathcal{F}}^2(s, T; \mathbb{R}^{n \times n})$ *of the first order and second order adjoint equations (5) and (6), respectively, such that*

$$
\begin{aligned}
& \bar{H}^\theta(t, \bar{x}(t), \bar{u}(t), \bar{p}(t), \bar{q}(t)) - \bar{H}^\theta(t, \bar{x}(t), u, \bar{p}(t), \bar{q}(t)) \\
& \quad - \frac{1}{2} [\sigma(t, \bar{x}(t), \bar{u}(t)) - \sigma(t, \bar{x}(t), u)]' [\bar{P}(t) - \theta \bar{p}(t) \bar{p}(t)'] [\sigma(t, \bar{x}(t), \bar{u}(t)) - \sigma(t, \bar{x}(t), u)] \\
& \geq 0, \quad \text{a.e. } t \in [s, T], \text{ P} - \text{a.s.},
\end{aligned}
\tag{9}
$$

or equivalently

$$
\bar{\mathcal{H}}^\theta(t, \bar{x}(t), \bar{u}(t)) = \max_{u \in U} \bar{\mathcal{H}}^\theta(t, \bar{x}(t), u), \quad \text{a.e. } t \in [s, T], \text{ P} - \text{a.s..}
\tag{10}
$$

Sufficient conditions for optimality of the pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ are as follows:

Theorem 3.2 (Sufficient conditions for optimality) *Suppose that* **(B1)-(B5)** *hold. Let* $(\bar{x}(\cdot), \bar{u}(\cdot))$ *be an admissible pair, and* $(\bar{p}(\cdot), \bar{q}(\cdot))$, $(\bar{P}(\cdot), \bar{Q}(\cdot))$ *the associated first and second order adjoint variables respectively. Suppose that* $g(\cdot)$ *is convex,* $\bar{H}^\theta(t, \cdot, \cdot, \bar{p}(t), \bar{q}(t))$ *is concave for all* $t \in [0, T]$ *almost surely, and*

$$
\bar{\mathcal{H}}^\theta(t, \bar{x}(t), \bar{u}(t)) = \max_{u \in U} \bar{\mathcal{H}}^\theta(t, \bar{x}(t), u), \quad \text{a.e. } t \in [s, T], \text{ P} - \text{a.s..}
\tag{11}
$$

Then $(\bar{x}(\cdot), \bar{u}(\cdot))$ *is an optimal pair for the problem (3).*

7

Proofs of the preceding two theorems are deferred to Section 4.

Several remarks are in order. First, observe that the form of the variational inequality (9) is very similar to its risk-neutral counterpart (see [19, p. 118, Theorem 3.2]) with $\bar{P}(t)$ in the quadratic form there replaced by $\bar{P}(t) - \theta\bar{p}(t)\bar{p}(t)'$. If one substitutes $\theta = 0$ in (5)-(8), then Theorem 3.1 is nothing else than the risk-neutral stochastic maximum principle that appears in [14, 19], and Theorem 3.2 the sufficient condition [19, 22]. More important however is to observe the changes to the usual stochastic maximum principle due to risk-sensitivity (i.e. $\theta > 0$), and in particular, the changes due to the control appearing in the diffusion term. As mentioned risk-sensitivity results in additional terms appearing in the adjoint equations (5)-(6) as well as in the Hamiltonian (7) and the $\mathcal{H}$-function (8). However, if σ is independent of u, then the additional risk-sensitive term that appears in the $\mathcal{H}$-function does not have any influence on the maximization (10). In fact, if σ is independent of u, the terms in (8) which involve the second order process $\bar{P}(t)$ may also be ignored, and we can replace $\bar{\mathcal{H}}^\theta$ in (10) by

$$\mathcal{H}(t, x, u) = \langle \bar{p}(t), b(t, x, u) \rangle - f(t, x, u). \tag{12}$$

This is simply the Hamiltonian function that appears in the deterministic maximum principle, evaluated along the adjoint process $\bar{p}(t)$. In summary, if the diffusion term is control independent, then the only effect of risk-sensitivity is on the solution $(\bar{p}(\cdot), \bar{q}(\cdot))$ of the first order adjoint equation (5). In particular, the control $\bar{u}(t)$ which maximizes the $\mathcal{H}$-function does not depend explicitly on θ and is exactly the same (as a function of $\bar{p}(t)$, $\bar{x}(t)$ and t) as the control that maximizes the Hamiltonian in the deterministic problem. (Of course, $\bar{p}(\cdot)$ will depend implicitly on the value of θ via (5)). On the other hand, if σ is control dependent, then the maximum principle involves the second order adjoint process $\bar{P}(\cdot)$, as well as a terms depending on the risk-sensitive parameter θ. Both $\bar{P}(\cdot)$ and θ appear in $\bar{\mathcal{H}}^\theta(t, x, u)$. The optimal control $\bar{u}(t)$, which maximizes (10), depends *explicitly* on $\bar{P}(t)$ and θ as well as $\bar{p}(t)$, $\bar{x}(t)$ and t.

4   Proofs of Theorems 3.1 and 3.2

This section is devoted to proofs of the two main theorems of the paper, Theorems 3.1 and 3.2. The proofs will be spread over several subsections.

4.1   Applying risk-neutral maximum principle

The risk-sensitive control problem (3) is not of a form in which the existing (risk-neutral) stochastic maximum principle found in [14, 19] can be applied. In this subsection, the result from [14, 19] is applied to an optimal control problem that is equivalent to (3). In the subsequent subsections, transformations are then introduced which enable us to represent the risk-sensitive maximum principle in the more familiar form as stated in Theorem 3.1.

Consider the following optimal control problem:

$$
\begin{cases}
\text{Minimize} \quad J^\theta(s,\, x,\, y;\, u(\cdot)) = E e^{\theta\, [g(x(T)) + y(T)]}, \\[2mm]
\text{subject to:} \\[2mm]
\quad dx(t) = b(t, x(t), u(t))\, dt + \sigma(t, x(t), u(t)) dW(t), \\[2mm]
\quad dy(t) = f(t, x(t), u(t)) dt, \\[2mm]
\quad x(s) = x, \quad y(s) = y, \\[2mm]
\quad u(\cdot) \in \mathcal{U}[s,\, T].
\end{cases}
\tag{13}
$$

Clearly, (3) corresponds to the case when $y = 0$ in (13). The value function $v^\theta : [0,\, T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ associated with (13) is

$$
v^\theta(s,\, x,\, y) := \inf_{u(\cdot) \in \mathcal{U}[s, T]} J^\theta(s,\, x,\, y;\, u(\cdot)).
\tag{14}
$$

Note that

$$
v^\theta(s,\, x,\, y) = e^{\theta\, y} v^\theta(s,\, x),
\tag{15}
$$

where $v^\theta(s,\, x)$ is defined by (4). In particular, **(B4)** implies that $v^\theta \in C^{1,\, 3,\, \infty}([0,\, T] \times \mathbb{R}^n \times \mathbb{R})$.

Assuming **(B1)**-**(B3)**, we can apply the result from [14] or [19, Theorem 3.2, p. 118] to (13): Suppose that $((\bar{x}(\cdot), \bar{y}(\cdot)), \bar{u}(\cdot))$ is an optimal triple for the optimal control problem (13). Denoting $\bar{b}_x(t) := b_x(t, \bar{x}(t), \bar{u}(t))$ (with similar interpretations for $\bar{f}_x(t)$, $\bar{\sigma}_x(t)$, etc.), the first order and second order adjoint equations are:

$$
\begin{cases}
dp(t) = -\left\{ \begin{bmatrix} \bar{b}_x(t) & 0 \\ \bar{f}_x(t) & 0 \end{bmatrix}' p(t) + \begin{bmatrix} \bar{\sigma}_x(t) & 0 \\ 0 & 0 \end{bmatrix}' q(t) \right\} dt + q(t)\, dW(t), \\[4mm]
p(T) = -\theta e^{\theta[g(\bar{x}(T)) + \bar{y}(T)]} \begin{bmatrix} g_x(\bar{x}(T)) \\ 1 \end{bmatrix},
\end{cases}
\tag{16}
$$

9

$$
\begin{cases}
dP(t) = -\left\{ \begin{bmatrix} \bar{b}_x(t) & 0 \\ \bar{f}_x(t) & 0 \end{bmatrix}' P(t) + P(t) \begin{bmatrix} \bar{b}_x(t) & 0 \\ \bar{f}_x(t) & 0 \end{bmatrix} \right. \\[2ex]
\qquad + \begin{bmatrix} \bar{\sigma}_x(t) & 0 \\ 0 & 0 \end{bmatrix}' P(t) \begin{bmatrix} \bar{\sigma}_x(t) & 0 \\ 0 & 0 \end{bmatrix} \\[2ex]
\qquad + \begin{bmatrix} \bar{\sigma}_x(t) & 0 \\ 0 & 0 \end{bmatrix}' Q(t) + Q(t) \begin{bmatrix} \bar{\sigma}_x(t) & 0 \\ 0 & 0 \end{bmatrix} \\[2ex]
\qquad \left. + \begin{bmatrix} H^\theta_{xx}(t,\bar{x}(t),\bar{u}(t),p(t),q(t)) & 0 \\ 0 & 0 \end{bmatrix} \right\} dt + Q(t)\,dW(t), \\[3ex]
P(T) = -\theta\, e^{\theta\,[g(\bar{x}(T))+\bar{y}(T)]} \begin{bmatrix} \theta\, g_x(\bar{x}(T))g_x(\bar{x}(T))' + g_{xx}(\bar{x}(T)) & \theta\, g_x(\bar{x}(T)) \\ \theta\, g_x(\bar{x}(T))' & \theta \end{bmatrix},
\end{cases} \tag{17}
$$

where the Hamiltonian $H^\theta : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ is given by

$$
H^\theta(t,x,u,p,q) = \left\langle p, \begin{pmatrix} b(t,x,u) \\ f(t,x,u) \end{pmatrix} \right\rangle + \left\langle q, \begin{pmatrix} \sigma(t,x,u) \\ 0 \end{pmatrix} \right\rangle. \tag{18}
$$

The adjoint equations (16)-(17) are *linear* BSDEs. Under **(B1)**-**(B3)**, for every admissible triple $((\bar{x}(\cdot), \bar{y}(\cdot)), \bar{u}(\cdot))$, there are unique solutions $(p(\cdot), q(\cdot)) \in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n+1}) \times L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n+1})$ and $(P(\cdot), Q(\cdot)) \in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{(n+1)\times(n+1)}) \times L^2_{\mathcal{F}}(s,T;\mathbb{R}^{(n+1)\times(n+1)})$ of (16)-(17), respectively; see [19, Theorem 2.2, p. 349].

The $\mathcal{H}$-function for problem (13) associated with $((\bar{x}(\cdot), \bar{y}(\cdot)), \bar{u}(\cdot))$ is defined by:

$$
\begin{aligned}
\mathcal{H}^\theta&(t,x,u) \\
&= H^\theta(t,x,u,p(t),q(t)) - \frac{1}{2} \begin{bmatrix} \sigma(t,\bar{x}(t),\bar{u}(t)) \\ 0 \end{bmatrix}' P(t) \begin{bmatrix} \sigma(t,\bar{x}(t),\bar{u}(t)) \\ 0 \end{bmatrix} \\
&\quad + \frac{1}{2} \begin{bmatrix} \sigma(t,x,u) - \sigma(t,\bar{x}(t),\bar{u}(t)) \\ 0 \end{bmatrix}' P(t) \begin{bmatrix} \sigma(t,x,u) - \sigma(t,\bar{x}(t),\bar{u}(t)) \\ 0 \end{bmatrix}.
\end{aligned} \tag{19}
$$

The maximum principle for (13) can be stated as follows:

Proposition 4.1 ([14, 19]) *Let* **(B1)**-**(B3)** *hold. Let* $((\bar{x}(\cdot), \bar{y}(\cdot)), \bar{u}(\cdot))$ *be an optimal triple for the optimal control problem (13). Then there are unique solutions* $(p(\cdot), q(\cdot)) \in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n+1}) \times L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n+1})$ *and* $(P(\cdot), Q(\cdot)) \in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{(n+1)\times(n+1)}) \times L^2_{\mathcal{F}}(s,T;\mathbb{R}^{(n+1)\times(n+1)})$ *of (16)-(17), respectively, such that*

$$
\begin{aligned}
H^\theta&(t,\bar{x}(t),\bar{u}(t),p(t),q(t)) - H^\theta(t,\bar{x}(t),u,p(t),q(t)) \\
&- \frac{1}{2} \begin{bmatrix} \sigma(t,x,u) - \sigma(t,\bar{x}(t),\bar{u}(t)) \\ 0 \end{bmatrix}' P(t) \begin{bmatrix} \sigma(t,x,u) - \sigma(t,\bar{x}(t),\bar{u}(t)) \\ 0 \end{bmatrix} \\
&\geq 0, \ \forall u \in U, \ \text{a.e.} \ t \in [0,T], \ \text{P} - \text{a.s.},
\end{aligned} \tag{20}
$$

or equivalently

$$\mathcal{H}^{\theta}(t, \bar{x}(t), \bar{u}(t)) = \max_{u \in U} \mathcal{H}^{\theta}(t, \bar{x}(t), u), \text{ a.e. } t \in [s, T], \text{ P} - \text{a.s..} \tag{21}$$

Sufficient conditions for the optimality of $((\bar{x}(\cdot), \bar{y}(\cdot)), \bar{u}(\cdot))$ are as follows:

Proposition 4.2 ([22]) *Let* **(B1)-(B3)** *hold. Let* $((\bar{x}(\cdot), \bar{y}(\cdot)), \bar{u}(\cdot))$ *be an admissible triple and* $(p(\cdot), q(\cdot), P(\cdot), Q(\cdot))$ *satisfy (16) and (17). Suppose that* $g(\cdot)$ *is convex,* $H^{\theta}(t, \cdot, \cdot, p(t), q(t), P(t), Q(t))$ *is concave for all* $t \in [0, T]$, $P - a.s.$, *and*

$$\mathcal{H}^{\theta}(t, \bar{x}(t), \bar{u}(t)) = \max_{u \in U} \mathcal{H}^{\theta}(t, \bar{x}(t), u), \text{ a.e. } t \in [s, T], \text{ P} - \text{a.s..} \tag{22}$$

Then $((\bar{x}(\cdot), \bar{y}(\cdot)), \bar{u}(\cdot))$ *is an optimal triple for (13).*

Note that Propositions 4.1 and 4.2 do *not* depend on assumption **(B4)**.

4.2 Transformation of first order adjoint

Although Proposition 4.1 can be regarded as *a* maximum principle for the underlying risk-sensitive control problem, it is not a desirable one since the adjoint variables there involve additional and unnecessary components (for example, the dimension of $p(\cdot)$ is one plus that of the state variable) and the adjoint equations appear to be complicated. To cope with this problem, we need to transform the adjoint variables $(p(\cdot), q(\cdot))$ and $(P(\cdot), Q(\cdot))$ in certain ways. While we could have presented such transformations immediately without any explanation, for the benefit of the readers we choose to unfold the process of finding those transformations which is inspired by the logarithmic transformation and the relationship between the maximum principle and dynamic programming [19, 20]. This will be carried out in this and next subsections.

Let $((\bar{x}(\cdot), \bar{y}(\cdot)), \bar{u}(\cdot))$ be an optimal triple for (13), and $(p(\cdot), q(\cdot)) \equiv \left(\begin{bmatrix} p_1(\cdot) \\ p_2(\cdot) \end{bmatrix}, \begin{bmatrix} q_1(\cdot) \\ q_2(\cdot) \end{bmatrix} \right) \in$ $L^2_{\mathcal{F}}(s, T; \mathbb{R}^{n+1}) \times L^2_{\mathcal{F}}(s, T; \mathbb{R}^{n+1})$ be the associated first order adjoint variables satisfying equation (16), where $p_1(\cdot), q_1(\cdot) \in L^2_{\mathcal{F}}(s, T; \mathbb{R}^n)$ and $p_2(\cdot), q_2(\cdot) \in L^2_{\mathcal{F}}(s, T; \mathbb{R})$. Recall that $v^{\theta}(s, x, y)$ is the corresponding value function. Under assumption **(B4)**, $v^{\theta}(s, x, y) \in C^{1, 3, \infty}([0, T] \times \mathbb{R}^n \times \mathbb{R})$; see (15). By the relationship between the maximum principle and dynamic programming, we have (see [19, Theorem 4.1, p. 250]):

$$p(t) = -v^{\theta}_{(x,y)}(t, \bar{x}(t), \bar{y}(t)), \tag{23}$$

where $v^{\theta}_{(x,y)}$ signifies the gradient of v^{θ} in (x, y). On the other hand, the following logarithmic transformation is typically used to derive a partial differential equation for the value function associated with the risk-sensitive problem (13)

$$V^{\theta}(t, x, y) = \frac{1}{\theta} \ln v^{\theta}(t, x, y); \tag{24}$$

11

see [7] for instance. Transformation (24) and relation (23) suggest that we take the following transformation of the first order adjoint variable (by taking gradient on the right hand side of (24)):

$$\tilde{p}(t) \;=\; \frac{1}{\theta}\frac{p(t)}{v(t)} \tag{25}$$

where $v(t) := v^{\theta}(t, \bar{x}(t), \bar{y}(t)) > 0$.

Next, we are going to derive the equation for $\tilde{p}(\cdot) \equiv \begin{bmatrix} \bar{p}(\cdot) \\ \tilde{p}_2(\cdot) \end{bmatrix}$ where $\bar{p}(\cdot)$ is $\mathbb{R}^n$-valued. First notice that v^{θ} is the value function of the optimal control problem (13) which has no running cost; hence, it follows from [19, p. 251-252, (4.18) and (4.19)] that

$$\begin{cases} dv(t) = -p_1(t)'\,\sigma\,(t, \bar{x}(t), \bar{u}(t))\,dW(t) \\[2mm] v(T) = e^{\theta\,[g(\bar{x}(T))+\bar{y}(T)]}. \end{cases} \tag{26}$$

On the other hand, rearranging (25), we obtain:

$$p(t) = \theta\,v(t)\tilde{p}(t).$$

Applying Ito's formula, and assuming that $\tilde{p}(t)$ satisfies an equation of the following form:

$$d\tilde{p}(t) = \alpha(t)\,dt + \tilde{q}(t)dW(t), \tag{27}$$

we obtain

$$\begin{aligned} dp(t) &= d\{\theta\,v(t)\tilde{p}(t)\} \\ &= \theta\,v(t)\,d\tilde{p}(t) - \theta\,p_1(t)'\bar{\sigma}(t)\tilde{p}(t)dW(t) - \theta\,p_1(t)'\bar{\sigma}(t)\tilde{q}(t)dt. \end{aligned} \tag{28}$$

Rearranging (28) for $d\tilde{p}(t)$, and noting (from (25)) that $p_1(t) = \theta v(t)\bar{p}(t)$, we obtain the following expression:

$$d\tilde{p}(t) = \frac{1}{\theta\,v(t)}dp(t) + \theta\,\bar{p}(t)'\bar{\sigma}(t)\tilde{p}(t)dW(t) + \theta\,\bar{p}(t)'\bar{\sigma}(t)\tilde{q}(t)dt. \tag{29}$$

Substituting the expression (16) for $dp(t)$ into (29) we find that the diffusion term of $d\tilde{p}(t)$ is

$$\tilde{q}(t) \equiv \begin{bmatrix} \bar{q}(t) \\ \tilde{q}_2(t) \end{bmatrix} \;=\; \frac{1}{\theta}\frac{q(t)}{v(t)} + \theta\,\bar{p}(t)'\,\sigma(t, \bar{x}(t), \bar{u}(t))\,\tilde{p}(t), \tag{30}$$

where $\bar{q}(\cdot)$ is $\mathbb{R}^n$-valued. Substituting (30) back to (29) and using (16), it follows that the transformed first order adjoint variable $\tilde{p}(\cdot)$ satisfies the following equation, where the terminal

condition for $\tilde{p}(T)$ is easily determined from (25) and the terminal condition $p(T)$:

$$
\begin{cases}
d\tilde{p}(t) & = & -\left\{ \begin{bmatrix} \bar{b}_x(t) & 0 \\ \bar{f}_x(t) & 0 \end{bmatrix}' \tilde{p}(t) - \theta\,\bar{p}(t)'\bar{\sigma}(t) \begin{bmatrix} \bar{\sigma}_x(t) & 0 \\ 0 & 0 \end{bmatrix}' \tilde{p}(t) \right. \\[3mm]
& & \left. -\theta\,\bar{p}(t)'\bar{\sigma}(t)\tilde{q}(t) + \begin{bmatrix} \bar{\sigma}_x(t) & 0 \\ 0 & 0 \end{bmatrix}' \tilde{q}(t) \right\} dt + \tilde{q}(t)dW(t), \\[3mm]
\tilde{p}(T) & = & -\begin{bmatrix} g_x(\bar{x}(T)) \\ 1 \end{bmatrix}.
\end{cases}
\tag{31}
$$

Expanding (31), it can be easily seen that

$$
\tilde{p}_2(t) = -1, \qquad \tilde{q}_2(t) = 0, \quad \forall t \in [0, T], \tag{32}
$$

and $(\bar{p}(\cdot), \bar{q}(\cdot)) \in L^2_{\mathcal{F}}(s, T; \mathbb{R}^n) \times L^2_{\mathcal{F}}(s, T; \mathbb{R}^n)$ is a solution of (5). This explains how equation (5) is derived. Also, since our derivation can be reversed, it follows from the uniqueness property of (16) that this solution is unique.

4.3   Transformation of second order adjoint

Let $(P(\cdot), Q(\cdot)) \in L^2_{\mathcal{F}}(s, T; \mathbb{R}^{(n+1)\times(n+1)}) \times L^2_{\mathcal{F}}(s, T; \mathbb{R}^{(n+1)\times(n+1)})$ be the second order adjoint variables satisfying (17). Again, inspired by the relation that $P(\cdot)$ corresponds to the second order derivative of v^θ (along the optimal triple) as well as (25), we propose the following transformation of the second order adjoint variable:

$$
\tilde{P}(t) \quad = \quad \frac{1}{\theta}\frac{P(t)}{v(t)} + \theta\,\tilde{p}(t)\,\tilde{p}(t)'. \tag{33}
$$

Then

$$
d\tilde{P}(t) = d\left\{\frac{P(t)}{\theta\,v(t)}\right\} + d\left\{\theta\,\tilde{p}(t)\tilde{p}(t)'\right\}. \tag{34}
$$

Putting

$$
\Gamma(t) = \frac{P(t)}{\theta\,v(t)}, \tag{35}
$$

and assuming that

$$
d\Gamma(t) = X(t)\,dt + Y(t)\,dW(t), \tag{36}
$$

for some processes $X(\cdot), Y(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{(n+1)\times(n+1)})$, it follows from Ito's formula and (35) that

$$
d\left\{\frac{P(t)}{\theta\,v(t)}\right\} \equiv d\Gamma(t) = \frac{1}{\theta\,v(t)}dP(t) + \theta\,\bar{p}(t)'\bar{\sigma}(t)\left\{\frac{P(t)}{\theta\,v(t)}\right\}dW(t) + \theta\,\bar{p}(t)'\bar{\sigma}(t)Y(t)dt. \tag{37}
$$

Substituting the expression for $dP(t)$ (equation (17)) into (37), and noting (36), gives

$$Y(t) = \frac{Q(t)}{\theta\, v(t)} + \theta\, \bar{p}(t)'\bar{\sigma}(t)\tilde{P}(t) - \theta^2\, \bar{p}(t)'\bar{\sigma}(t)\tilde{p}(t)\tilde{p}(t)'. \tag{38}$$

Substituting (37) into (34) and using Ito's formula we obtain (after some manipulation):

$$
\begin{aligned}
&d\tilde{P}(t) \\
&= \quad -\left\{ \begin{bmatrix} \bar{b}_x(t) & 0 \\ \bar{f}_x(t) & 0 \end{bmatrix}' \tilde{P}(t) + \tilde{P}(t) \begin{bmatrix} \bar{b}_x(t) & 0 \\ \bar{f}_x(t) & 0 \end{bmatrix} + \begin{bmatrix} \bar{\sigma}_x(t) & 0 \\ 0 & 0 \end{bmatrix}' \left(\frac{P(t)}{\theta v(t)} \right) \begin{bmatrix} \bar{\sigma}_x(t) & 0 \\ 0 & 0 \end{bmatrix} \right. \\
&\quad + \begin{bmatrix} \bar{\sigma}_x(t) & 0 \\ 0 & 0 \end{bmatrix}' \left(\frac{Q(t)}{\theta v(t)} + \theta\, \tilde{q}(t)\tilde{p}(t)' - \theta^2\, \bar{p}(t)'\bar{\sigma}(t)\tilde{p}(t)\tilde{p}(t)' \right) \\
&\quad + \left(\frac{Q(t)}{\theta v(t)} + \theta\, \tilde{p}(t)\tilde{q}(t)' - \theta^2 \bar{p}(t)'\bar{\sigma}(t)\tilde{p}(t)\tilde{p}(t)' \right) \begin{bmatrix} \bar{\sigma}_x(t) & 0 \\ 0 & 0 \end{bmatrix} \\
&\quad + \frac{1}{\theta v(t)} \begin{bmatrix} H^\theta_{xx}(t,\bar{x}(t),\bar{u}(t),p(t),q(t)) & 0 \\ 0 & 0 \end{bmatrix} \\
&\quad \left. - \theta\, \bar{p}(t)'\bar{\sigma}(t)\left\{ Y(t) + \theta\, \tilde{p}(t)\tilde{q}(t)' + \theta\, \tilde{q}(t)\tilde{p}(t)' \right\} - \theta\, \tilde{q}(t)\tilde{q}(t)' \right\} dt + \tilde{Q}(t)dW(t), \tag{39}
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{Q}(t) \quad = \quad & \frac{1}{\theta}\frac{Q(t)}{v(t)} + \theta\, \tilde{p}(t)\,\tilde{q}(t)' + \theta\, \tilde{q}(t)\,\tilde{p}(t)' \\
& + \theta\, \bar{p}(t)'\,\sigma(t,\bar{x}(t),\bar{u}(t))\left(\tilde{P}(t) - \theta\, \tilde{p}(t)\,\tilde{p}(t)' \right). \tag{40}
\end{aligned}
$$

Recalling the definition of $\bar{H}^\theta$ (i.e. (7)), we have:

$$\frac{1}{\theta v(t)} \begin{bmatrix} H^\theta_{xx}(t,\bar{x}(t),\bar{u}(t),p(t),q(t)) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{H}^\theta_{xx}(t,\bar{x}(t),\bar{u}(t),\bar{p}(t),\bar{q}(t)) & 0 \\ 0 & 0 \end{bmatrix}. \tag{41}$$

Noting that

$$Y(t) = \tilde{Q}(t) - \theta\, \tilde{p}(t)\tilde{q}(t)' - \theta\, \tilde{q}(t)\tilde{p}(t)' \tag{42}$$

together with the transformations (30), (33), (40) as well as the relationship (41), equation

(39) becomes

$$
\begin{cases}
d\tilde{P}(t) = -\left\{ \begin{bmatrix} \bar{b}_x(t) & 0 \\ \bar{f}_x(t) & 0 \end{bmatrix}' \tilde{P}(t) + \tilde{P}(t) \begin{bmatrix} \bar{b}_x(t) & 0 \\ \bar{f}_x(t) & 0 \end{bmatrix} \right. \\[3mm]
\qquad\qquad + \begin{bmatrix} \bar{\sigma}_x(t) & 0 \\ 0 & 0 \end{bmatrix}' \left(\tilde{P}(t) - \theta\, \tilde{p}(t)\tilde{p}(t)' \right) \begin{bmatrix} \bar{\sigma}_x(t) & 0 \\ 0 & 0 \end{bmatrix} \\[3mm]
\qquad\qquad + \begin{bmatrix} \bar{\sigma}_x(t) & 0 \\ 0 & 0 \end{bmatrix}' \left(\tilde{Q}(t) - \theta\, \tilde{p}(t)\tilde{q}(t)' - \theta\, \bar{p}(t)'\bar{\sigma}(t)\tilde{P}(t) \right) \\[3mm]
\qquad\qquad + \left(\tilde{Q}(t) - \theta\, \tilde{q}(t)\tilde{p}(t)' - \theta\, \bar{p}(t)'\bar{\sigma}(t)\tilde{P}(t) \right)' \begin{bmatrix} \bar{\sigma}_x(t) & 0 \\ 0 & 0 \end{bmatrix}' \\[3mm]
\qquad\qquad + \begin{bmatrix} \bar{H}^\theta_{xx}(t,\bar{x}(t),\bar{u}(t),\bar{p}(t),\bar{q}(t)) & 0 \\ 0 & 0 \end{bmatrix} \\[3mm]
\qquad\qquad \left. - \theta\, \bar{p}(t)'\bar{\sigma}(t)\tilde{Q}(t) - \theta\, \tilde{q}(t)\tilde{q}(t)' \right\} dt + \tilde{Q}(t)\, dW(t), \\[3mm]
\tilde{P}(T) = -\begin{bmatrix} g_{xx}(\bar{x}(T)) & 0 \\ 0 & 0 \end{bmatrix}.
\end{cases}
$$

Therefore, it follows that

$$
\tilde{P}(t) = \begin{bmatrix} \bar{P}(t) & 0 \\ 0 & 0 \end{bmatrix}, \qquad\qquad \tilde{Q}(t) = \begin{bmatrix} \bar{Q}(t) & 0 \\ 0 & 0 \end{bmatrix},
$$

where $(\bar{P}(\cdot), \bar{Q}(\cdot)) \in L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n\times n}) \times L^2_{\mathcal{F}}(s,T;\mathbb{R}^{n\times n})$ is the solution of (6). As in the first order case, this solution is unique.

4.4 Maximum condition

In this subsection, we complete the proofs for Theorem 3.1 and Theorem 3.2 by translating the maximum condition in Propositions 4.1 and 4.2 into the one stated in (9). To start, consider the variational inequality (20). It can be shown, in view of (25), (30) and (32), that

$$
H^\theta(t,x,u,p(t),q(t)) = [\theta\, v(t)]\, \bar{H}^\theta(t,x,u,\bar{p}(t),\bar{q}(t)), \tag{43}
$$

where $\bar{H}^\theta(t,x,u,p,q)$ is given by (7), and

$$
\begin{aligned}
\frac{1}{2} &\begin{bmatrix} \sigma(t,\bar{x}(t),\bar{u}(t)) - \sigma(t,\bar{x}(t),u) \\ 0 \end{bmatrix}' P(t) \begin{bmatrix} \sigma(t,\bar{x}(t),\bar{u}(t)) - \sigma(t,\bar{x}(t),u) \\ 0 \end{bmatrix} \\
&= \left[\frac{\theta\, v(t)}{2}\right] \left(\sigma(t,\bar{x}(t),\bar{u}(t)) - \sigma(t,\bar{x}(t),u) \right)' \left(\bar{P}(t) - \theta\, \bar{p}(t)\bar{p}(t)' \right) \\
&\quad \times \left(\sigma(t,\bar{x}(t),\bar{u}(t)) - \sigma(t,\bar{x}(t),u) \right).
\end{aligned}
$$

Since $v(t) > 0$, it follows that the maximum condition (20) is equivalent to

$$\bar{H}^\theta(t, \bar{x}(t), \bar{u}(t), \bar{p}(t), \bar{q}(t)) - \bar{H}^\theta(t, \bar{x}(t), u, \bar{p}(t), \bar{q}(t))$$
$$-\frac{1}{2}\left(\sigma(t, \bar{x}(t), \bar{u}(t)) - \sigma(t, \bar{x}(t), u)\right)'\left(\bar{P}(t) - \theta\,\bar{p}(t)\bar{p}(t)'\right)\left(\sigma(t, \bar{x}(t), \bar{u}(t)) - \sigma(t, \bar{x}(t), u)\right)$$
$$\geq \quad 0,$$

which gives us (9). The equivalent condition (10) is seen by direct manipulation. This completes proofs for both Theorem 3.1 and Theorem 3.2.

5 An example: Linear-quadratic case

Consider the following linear-quadratic (LQ) risk-sensitive problem:

$$\begin{cases} \text{Minimize } J^\theta(x_0; u(\cdot)) = E e^{\theta[\frac{1}{2}\int_0^T (x(t)'M(t)x(t) + u(t)'N(t)u(t))\,dt + \frac{1}{2}x(T)'Hx(T)]}, \\ \text{subject to:} \\ \quad dx(t) = [A(t)x(t) + B(t)u(t) + f(t)]dt \\ \qquad\qquad + [C(t)x(t) + D(t)u(t) + \sigma(t)]dW(t), \\ \quad x(0) = x_0. \end{cases} \tag{44}$$

We assume that A, $C \in L_{\mathcal{F}}^\infty(0, T; \mathbb{R}^{n \times n})$, B, $D \in L_{\mathcal{F}}^\infty(0, T; \mathbb{R}^{n \times m})$, $M \in L_{\mathcal{F}}^\infty(0, T; \mathbb{S}^{n \times n})$, $N \in L_{\mathcal{F}}^\infty(0, T; \mathbb{S}^{m \times m})$, f, $\sigma \in L_{\mathcal{F}}^2(0, T; \mathbb{R}^n)$, $H \in \mathbb{S}^{n \times n}$, $x_0 \in \mathbb{R}^n$, and $\theta > 0$ are given and fixed. Here $\mathbb{S}^{n \times n}$ denotes the set of all $n \times n$ symmetric matrices. Initially, we impose no additional positive semidefiniteness assumptions on M, N or H.

In [3], the authors consider a (risk-neutral) stochastic linear-quadratic regulator (LQR) problem with cost:

$$J_{LQ}(x_0; u(\cdot)) := E\left\{\frac{1}{2}\int_0^T \left(x(t)'M(t)x(t) + u(t)'N(t)u(t)\right)dt + \frac{1}{2}x(T)'Hx(T)\right\} \tag{45}$$

and dynamics as in (44). An easily verifiable necessary and sufficient condition for solvability of the associated Riccati equation is given. One consequence of their result is that the LQR problem with $D \neq 0$ may be well posed (i.e., (45) has a finite infimum), even if N is indefinite or negative definite. Unfortunately, it appears that the optimal control for (44) can not be expressed in terms of a Riccati type equation when $C \neq 0$ or $D \neq 0$. However, it is still possible to see the effect of having $D \neq 0$ on the well posedness of (44). By Jensen's inequality:

$$J_{LQ}(x_0; u(\cdot)) \leq \frac{1}{\theta}\ln J^\theta(x_0; u(\cdot)), \quad \forall u(\cdot) \text{ admissible, } \forall x_0. \tag{46}$$

Therefore, well posedness of the LQR problem with dynamics as in (44) and cost (45) is sufficient for well posedness of the risk-sensitive problem (44). In particular, it follows from [3] that if $D \neq 0$, then (44) may be well posed even if N in indefinite. However, it remains an

open question whether well posedness of the LQR problem (45) is necessary for well posedness of (44).

For the remainder of the paper, we assume that $M(t) \geq 0$ and $N(t) > 0$ for a.e. $t \in [0, T]$ and $H \geq 0$. **Unfortunately, it seems that the optimal control can not be expressed in terms of Riccati type equations unless $C = 0$ and $D = 0$, which we also assume. The existence of an elegant solution of (44) when $C \neq 0$ or $D \neq 0$ remains an open issue. Also, for convenience, we assume $f = 0$ though this is not crucial. Under these assumptions, the** problem (44) can be solved using completion of squares [2] or dynamic programming [1, 7]. We now use the risk-sensitive maximum principle established in the previous sections. (Note that the minimum principle derived in [5] is used to solve the partial observation problem. This involves solving a pair of stochastic partial differential equations. On the other hand, the problem (44) involves solving BSDEs). **One interesting bi-product of this analysis is that the second-order adjoint, under these assumptions, reduces to an ordinary differential equation of Lyapunov type.** It should be noted that the optimal control for the **(risk-neutral)** stochastic LQR problem with dynamics (44) and cost (45) has been derived using the completion of square argument, dynamic programming or the maximum principle; see [19, Chapter 6].

We now determine an admissible pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ which satisfies the necessary conditions in Theorem 3.1. First of all, associated with an admissible pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ the adjoint equations (5) and (6) reduce to

$$\begin{cases} d\bar{p}(t) &= -\Big[A(t)'\,\bar{p}(t) - M(t)\bar{x}(t) - \theta\,\bar{p}(t)'\,\sigma(t)\,\bar{q}(t)\Big]\,dt + \bar{q}(t)\,dW(t) \\ \bar{p}(T) &= -H\bar{x}(T), \end{cases} \tag{47}$$

$$\begin{cases} d\bar{P}(t) &= -\Big[\bar{P}(t)A(t) + A(t)'\bar{P}(t) - M(t) - \theta\Big(\bar{p}(t)'\sigma(t)\bar{Q}(t) + \bar{q}(t)\bar{q}(t)'\Big)\Big]\,dt \\ &\quad + \bar{Q}(t)\,dW(t), \\ \bar{P}(T) &= -H. \end{cases} \tag{48}$$

Let $(\bar{p}(\cdot), \bar{q}(\cdot))$ and $(\bar{P}(\cdot), \bar{Q}(\cdot))$ be the solutions of the preceding two equations respectively.

The associated $\mathcal{H}$-function is

$$\begin{aligned} \bar{\mathcal{H}}^\theta(t, x, u) &= \bar{p}(t)\,'(A(t)x + B(t)u) - \frac{1}{2}(x'M(t)x + u'N(t)u) \\ &\quad + \frac{1}{2}\sigma(t)'\bar{P}(t)\sigma(t) + \sigma(t)'[\bar{q}(t) - \bar{P}(t)\sigma(t)]. \end{aligned} \tag{49}$$

Maximizing, as in (11), $\bar{\mathcal{H}}^\theta(t, \bar{x}(t), u)$ over $u \in \mathbb{R}^m$, we obtain:

$$\bar{u}(t) = N(t)^{-1}B(t)'\bar{p}(t). \tag{50}$$

Substituting (50) into the SDE in (44) gives

$$\begin{cases} d\bar{x}(t) = [A(t)\bar{x}(t) + B(t)N(t)^{-1}B(t)'\bar{p}(t)]\,dt + \sigma(t)dW(t), \\ \\ \bar{x}(0) = x_0. \end{cases} \qquad (51)$$

Therefore, an admissible pair that satisfies the necessary conditions can be obtained by solving the system of forward-backward stochastic differential equations (FBSDEs) (47) and (51). We conjecture a solution of the form:

$$\bar{p}(t) = -P(t)\bar{x}(t) \qquad (52)$$

for some deterministic $\mathbb{R}^{n \times n}$-valued differentiable function $P(t)$ which we now determine. Applying Ito's formula to (52) gives:

$$d\bar{p}(t) = (-\dot{P}(t) - P(t)A(t) + P(t)B(t)N(t)^{-1}B(t)'P(t))\,\bar{x}(t)\,dt - P(t)\sigma(t)\,dW(t). \qquad (53)$$

On the other hand, after substituting (52) into (47), we arrive at:

$$d\bar{p}(t) = -(-A(t)'P(t) - M(t) + \theta\bar{q}(t)\sigma(t)'P(t))\,\bar{x}(t)\,dt + \bar{q}(t)\,dW(t). \qquad (54)$$

Equating the coefficients of (53) and (54) gives

$$(\bar{p}(t),\,\bar{q}(t)) = (-P(t)\bar{x}(t),\,-P(t)\sigma(t)) \qquad (55)$$

where $P(t)$ is a solution of the differential equation

$$\begin{cases} \dot{P}(t) + P(t)A(t) + A(t)'P(t) - P(t)\Big(B(t)N(t)^{-1}B(t)' - \theta\,\sigma(t)\sigma(t)'\Big)P(t) + M(t) = 0, \\ \\ P(T) = H. \end{cases} \qquad (56)$$

It is easy to show that

$$(\bar{P}(t),\,\bar{Q}(t)) = (-\Gamma(t),\,0), \qquad (57)$$

where $\Gamma(\cdot)$ is the solution of the Lyapunov equation:

$$\begin{cases} \dot{\Gamma}(t) + \Gamma(t)A(t) + A(t)'\Gamma(t) + M(t) + \theta P(t)\sigma(t)\sigma(t)'P(t) = 0, \\ \\ \Gamma(T) = H, \end{cases} \qquad (58)$$

is a solution of the second order adjoint equation (48). Finally, with $(\bar{p}(\cdot),\,\bar{q}(\cdot))$ given by (55), it follows from (44), (50) that $(\bar{x}(\cdot),\,\bar{u}(\cdot))$ satisfies the necessary conditions, where $\bar{x}(\cdot)$ is the unique solution of

$$\begin{cases} d\bar{x}(t) = (A(t) - B(t)N(t)^{-1}B(t)'P(t))\,\bar{x}(t)\,dt + \sigma(t)dW(t), \\ \\ \bar{x}(0) = x_0, \end{cases} \qquad (59)$$

and

$$\bar{u}(t) = -N(t)^{-1}B(t)'P(t)\bar{x}(t). \tag{60}$$

Finally, we verify that in the present case the necessary conditions of optimality (Theorem 3.1) are also sufficient. In view of the sufficient conditions given by Theorem 3.2 it suffices to show the convexity of $g(x) = \frac{1}{2}x'Hx$ and the joint concavity of the Hamiltonian $\bar{H}^\theta(t, x, u, \bar{p}(t), \bar{q}(t))$ in (x, u). Since $H \geq 0$, the convexity of g is evident. On the other hand,

$$\bar{H}^\theta(t, x, u, \bar{p}(t), \bar{q}(t)) = \langle \bar{p}(t), A(t)x + B(t)u \rangle - \frac{1}{2}x'M(t)x - \frac{1}{2}u'N(t)u$$
$$+ \sigma(t)'[\bar{q}(t) - \theta\,\bar{p}(t)\,\bar{p}(t)'\,\sigma(t)],$$

which is concave in (x, u) due to $M(t) \geq 0$ and $N(t) > 0$.

To summarize, we have the following result.

Theorem 5.1 *Assume that $C = 0$, $D = 0$, $f = 0$, $M(t) \geq 0$ and $N(t) > 0$ for a.e. $t \in [0, T]$ and $H \geq 0$. If the equation (56) admits a solution $P(\cdot)$, then the state feedback control (60) is optimal for the risk-sensitive LQ problem (44).*

To conclude this section, we remark that the equation (56) resembles the celebrated Riccati equation. If

$$B(t)N(t)^{-1}B(t)' - \theta\,\sigma(t)\sigma(t)' > 0,$$

then it is indeed a Riccati equation which admits a unique solution by the classical Riccati theory.

6 Conclusion

In this paper we have derived a new maximum principle for the risk-sensitive control problem. One interesting feature of our result is that it involves a nonlinear adjoint equation which has not been seen in existing relevant results. Also, our maximum principle applies to systems in which the diffusion term may be control dependent. By allowing such control dependence, we obtain results quite different from related work on this problem already in the literature. In particular, the maximum principle involves an additional second order adjoint process and the maximum condition contains additional terms associated with the second order process as well as the risk-sensitive parameter. Importantly, the second order process and the risk-sensitive parameter do not appear in the maximum condition when the diffusion term is control independent.

Our derivation consists of two steps: Applying Peng's result [14] to a standard form, risk-neutral stochastic optimal control problem (that is equivalent to the risk-sensitive problem)

and transforming the resulting adjoint variables and maximum condition to obtain the risk-sensitive maximum principle as stated in Theorem 3.1. This approach allows us to obtain a completely rigorous result, while avoiding many of the technicalities associated with a 'first principles' derivation.

We wish to emphasize the nonlinear transformations (25) and (33) which are used together with results on the relationship between dynamic programming and the maximum principle to obtain our results. These transformations are analogous to the logarithmic transformation used in the dynamic programming approach to risk-sensitive control. One drawback of our approach is the requirement that the value function is smooth. The need for this assumption is brought about by our use of results on the relationship between the maximum principle and dynamic programming. Typically, this assumption is not required for a maximum principle. Our remaining assumptions however are standard (in fact weaker) than those commonly invoked when studying the risk-sensitive problem.

One interesting issue that we have not addressed concerns the relationship between the risk-sensitive control problem and stochastic differential games (which is studied in [7, 11], for example), and in particular, the derivation of a 'min-max principle' for differential games through its relationship with the risk-sensitive problem and the maximum principle that is derived in this paper. This is an important issue which we shall leave for future research.

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