A diffusion model for optimal dividend distribution for a company with constraints on risk control

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Abstract

We investigate a model of a corporation which faces constant liability payments and which can choose a production/business policy from an available set of control policies with different expected profits and risks. The objective is to maximize the expected present value of the total dividend distributions. The main purpose of this paper is to deal with the impact of constraints on business activities such as inability to completely eliminate risk (even at the expense of reducing the potential profit to zero) or when such a risk cannot exceed a certain level. We analyze the case in which there is no restriction on the dividend pay-out rates as well as the case when such a restriction does exist. By delicate analysis on the corresponding Hamilton-Jacobi-Bellman equation we compute explicitly the optimal return function and determine the optimal policy.

1 Introduction

Recently there has been an upsurge of interest in diffusion models for optimal dividend optimization/risk control techniques (see Jeanblanc Piqué and Shiryaev [10], Asmussen and Taksar [2], Radner and Shepp [13], Boyle et al. [3], Højgaard and Taksar [7], [8], [9], Paulsen and Grjessing [12], and Taksar and Zhou [15]). In those models the liquid assets of the company are modeled by a Brownian motion with constant drift and diffusion coefficients. The drift term corresponds to the expected (potential) profit per unit time, while the diffusion term is interpreted as risk. The larger the diffusion coefficient the greater the business risk the company takes on. If the company wants to decrease the risk from its business activities, it also faces a decrease in its potential profit. In other words, different business activities in this model correspond to changing *simultaneously* the drift and the diffusion

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coefficients of the underlying process. This sets a scene for an optimal stochastic control model where the controls affect not only the drift, but also the diffusion part of the dynamic of the system.
Another important consideration in this paper is dividend distribution. Dividends are paid from the liquid reserve of the company and distributed to the shareholders. In the control model the dividend distribution plan is represented by an increasing functional, $C(t)$, whose meaning is the cumulative amount of dividends paid out up to time t. Moreover, the control model is the so-called singular control if the dividend pay-out rate is unbounded, while it is a regular control model if the dividend pay-out rate is bounded. The risk control/dividend distribution policy determines uniquely the dynamics of the liquid reserve. The company is bankrupt when its liquid assets vanish. The objective is to find the policy which maximizes the expected cumulative discount dividend pay-outs up to the time of the bankruptcy.
Insurance is one of the natural areas where those models become widely applied. The risk control in insurance takes on a natural form of reinsurance. Specifically, if at any fixed time both the drift and diffusion coefficients of the controlled stochastic process are multiples of one and the same control parameter $a, 0 \leq a \leq 1$, then this would be the limiting case of the so-called *proportional reinsurance*, which is employed by a *cedent* in order to reduce the insurance risks. Other types of reinsurance schemes result in different types of drift/diffusion control models (see, e.g., [1], [14]).
In this paper we consider a company whose business activities are modeled by a control process $a_t,\ \ t \geq 0$, which takes on values in the interval $[\alpha, \beta]$, $0 < \alpha < \beta < +\infty$ with risk and potential profit at any time t proportional to a_t. The restriction $\alpha > 0$ reflects the fact that there are institutional or statutory reasons (e.g., the company is public) that its business activities cannot be reduced to zero, unless the company faces bankruptcy. In addition in our model the company has a constant rate of liability payments, such as mortgage payments on its property or amortization of bonds. In the case of an insurance company, when the control parameter a_t lies within $[0, 1]$ this problem was considered by Taksar and Zhou [15]. In this regard, the model treated in [15] can be viewed as a limiting case of $\alpha \to 0+$ and $\beta = 1$. In addition, we investigate the case in which there are restrictions on the dividend pay out rate and analyze how those restrictions affect the optimal policy.
The paper is structured as follows. In the next section we give a rigorous mathematical formulation of the problem and obtain some preliminary results. In Section 3 we analyze the case of unbounded dividend rate without liability, which is interesting in its own right. In Section 3 we extend the results to the case of a constant liability payments. In Section 5 we investigate the case of bounded dividend rate with nonzero liability. Section 6 is devoted to the construction of optimal policies based on the results of the preceding sections. The last section is devoted to economic interpretation of the obtained results.

2 Mathematical model

We start with a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and a one-dimensional standard Brownian motion W_t (with $W_0 = 0$) on it, adapted to the filtration $\mathcal{F}_t$. We denote by R_t^π the reserve of the company at time t under a control policy $\pi = (a_t^\pi, C_t^\pi; t \geq 0)$ (to be specified below). The dynamic of the reserve process R_t^π is described by

$$dR_t^\pi = (a_t^\pi \mu - \delta)dt + a_t^\pi \sigma dW_t - C_t^\pi, \tag{2.1}$$

with initial condition

$$R_0^\pi = x, \tag{2.2}$$

where μ is the expected profit per unit time (profit rate) and σ is the volatility rate of the reserve process (in the absence of any risk control), δ represents the amount of money the company has to

pay per unit time (the debt rate) irrespective of what business activities it chooses, and x is the initial reserve.
The control in this model is described by a pair of $\mathcal{F}_t$-adapted processes $\pi = (a_t^\pi, C_t^\pi; t \geq 0)$. A control $\pi = (a_t^\pi, C_t^\pi; t \geq 0)$ is admissible if $\alpha \leq a_t^\pi \leq \beta$, $\forall t \geq 0$, and C_t^π is a non-decreasing, right continuous process, where $0 < \alpha < \beta < +\infty$ are given scalars. We denote the set of all admissible controls by $\mathcal{A}$. The control component a_t^π represents one of the possible business activities available for the company at time t, and the component C_t^π corresponds to the total amount of dividends paid out by the company up to time t.
Given a control policy π, the time of bankruptcy is defined as

$$\tau^\pi = \inf\{t \geq 0 : R_t^\pi = 0\}. \tag{2.3}$$

The *performance functional* associated with each control π is

$$J_x(\pi) = E\left(\int_0^{\tau^\pi} e^{-\gamma t} dC_t^\pi\right), \tag{2.4}$$

where $\gamma > 0$ is an a priori given discount factor (used in calculating the present value of the future dividends), and the subscript x denotes the initial state x. The objective is to find

$$v(x) = \sup_{\pi \in \mathcal{A}} J_x(\pi) \tag{2.5}$$

and the optimal policy π^* such that

$$J_x(\pi^*) = v(x). \tag{2.6}$$

The exogenous parameters of the problem are $\mu, \sigma, \delta, \alpha, \beta$ and γ. The aim of this paper is to obtain the optimal return function v and the optimal policy *explicitly* in terms of these parameters.
A few remarks on the control component a_t^π are in order. The way this quantity enters into the dynamics (2.1) clearly shows that it reduces or increases the risk simultaneously reducing or increasing the expected profit rate at the same scale. In other words, the diffusion coefficient of the dynamic system (2.1) depends on the control component a_t^π. In [15], the problem is formulated in the context of an insurance company where $1 - a_t^\pi$ signifies the reinsurance fraction and the constraint $0 \leq a_t^\pi \leq 1$ is imposed which is a limiting case of $\alpha \to 0+$ and $\beta = 1$ (note that while in our analysis below we require $\alpha > 0$, the solution we obtain does have a limit when $\alpha \to 0+$ and this limit coincides with the solution in [15]. In this sense the model in [15] is indeed a special case of the model presented here). It is certainly meaningful to relax this constraint to one with any arbitrary upper and lower bounds. Thus for the insurance company case $\alpha > 1$ would mean that the company can take an extra insurance business from other companies (that is act as a reinsurer for other cedents). Moreover, our formulation can model risk control problems for companies other than insurance ones. On the other hand, the two general bounds α and β add a new, nontrivial feature to this model, as will be evident in the sequel.
The main tools for solving the problem are the dynamic programming and Hamilton-Jacobi-Bellman (HJB) equation (see Fleming and Rishel [5], Fleming and Soner [6], and Yong and Zhou [16], as well as relevant discussions in [2], [8] and [15]). First of all, the optimal return function v has the following basic properties.

Proposition 2.1 *The function v as defined by (2.5) is a concave, non-decreasing function subject to*

$$v(0) = 0. \tag{2.7}$$

The proof of this proposition is standard and can be found in [2], [8] and [15].
Next, the dynamic programming principle yields that if the optimal return function v is a C^2-function, then v is a classical solution to the following HJB equation

$$\begin{array}{l}\max\left(\max_{\alpha\le a\le\beta}\left(\frac{1}{2}\sigma^2a^2V''(x)+(a\mu-\delta)V'(x)-\gamma V(x)\right),\ 1-V'(x)\right)=0,\ \ x>0,\\ V(0)=0.\end{array}\tag{2.8}$$

Note that we do not know a priori whether the HJB equation has any solution other than the optimal return function. However, the following verification theorem, which says that *any* concave solution V to the HJB equation (2.8) whose derivative is finite at 0 majorizes the performance functional for any policy π, is sufficient for us to identify optimal policies.

Theorem 2.2 *Let V be a concave, twice continuously differentiable solution of (2.8), such that $V'(0)<+\infty$. Then for any policy $\pi=(a_t^\pi, C_t^\pi; t\ge 0)$,*

$$V(x)\ge J_x(\pi).\tag{2.9}$$

Proof. Let R_t^π be the reserve process given by (2.1) and (2.2). Denote the operator

$$L^a=\frac{1}{2}\sigma^2a^2\frac{d^2}{dx^2}+(a\mu-\delta)\frac{d}{dx}-\gamma.$$

Then applying a generalized Ito's formula (see Dellacherie and Meyer [4, Theorem VIII.27]) to the process $e^{-\gamma t}V(R_t^\pi)$, we get (below $C_\cdot^{\pi c}$ stands for a continuous part of the increasing functional $C_\cdot^\pi$)

$$\begin{aligned}e^{-\gamma t\wedge\tau}V(R_{t\wedge\tau}^\pi)&=V(x)+\int_0^{t\wedge\tau}e^{-\gamma s}\sigma a_s^\pi V'(R_s^\pi)dW_s\\&\quad+\int_0^{t\wedge\tau}e^{-\gamma s}L^{a_s^\pi}V(R_s^\pi)ds-\int_0^{t\wedge\tau}e^{-\gamma s}V'(R_s^\pi)dC_s^\pi\\&\quad+\sum_{s\le t\wedge\tau}e^{-\gamma s}[V(R_s^\pi)-V(R_{s-}^\pi)-V'(R_{s-}^\pi)(R_s^\pi-R_{s-}^\pi)]\\&=V(x)+\int_0^{t\wedge\tau}e^{-\gamma s}\sigma a_s^\pi V'(R_s^\pi)dW_s+\int_0^{t\wedge\tau}e^{-\gamma s}L^{a_s^\pi}V(R_s^\pi)ds\\&\quad-\int_0^{t\wedge\tau}e^{-\gamma s}V'(R_s^\pi)dC_s^{\pi c}+\sum_{s\le t\wedge\tau}e^{-\gamma s}[V(R_s^\pi)-V(R_{s-}^\pi)],\end{aligned}\tag{2.10}$$

where we used the equality $R_s^\pi-R_{s-}^\pi=-(C_s^\pi-C_{s-}^\pi)$. Since V is non-decreasing, concave with finite derivative at the origin, $V'(x)$ is bounded and the stochastic integral in (2.10) is a square integrable martingale whose expectation vanishes. In view of the HJB equation (2.8) the quantity $L^{a_s^\pi}V(R_s^\pi)$ is always non-positive. Taking expectations of both sides of (2.10), we get

$$\begin{aligned}E(e^{-\gamma(t\wedge\tau)}V(R_{t\wedge\tau}^\pi))&\le V(x)-E\int_0^{t\wedge\tau}e^{-\gamma s}V'(R_s^\pi)dC_s^{\pi c}\\&\quad+E\sum_{s\le t\wedge\tau}e^{-\gamma s}[V(R_s^\pi)-V(R_{s-}^\pi)].\end{aligned}\tag{2.11}$$

Since $V'(x)\ge 1$, the mean-value theorem implies $V(R_s^\pi)-V(R_{s-}^\pi)\le R_s^\pi-R_{s-}^\pi=-(C_s^\pi-C_{s-}^\pi)$. Combining this with (2.11), we get

$$E(e^{-\gamma(t\wedge\tau)}V(R_{t\wedge\tau}^\pi))+E\int_0^{t\wedge\tau}e^{-\gamma s}V'(R_s^\pi)dC_s^\pi\le V(x).\tag{2.12}$$

Note that in view of boundedness of V',

$$e^{-\gamma(t\wedge\tau)}V(R^{\pi}_{t\wedge\tau}) \le e^{-\gamma t}K(1+R^{\pi}_{t\wedge\tau}) \le e^{-\gamma t}K(1+|R^{\pi}_t|)$$

for some constant K. Since R^{π}_t is a diffusion process with uniformly bounded drift and diffusion coefficient, standard arguments yield $E|R^{\pi}_t| \le x + K_1 t$ for some constant K_1. Therefore

$$Ee^{-\gamma(t\wedge\tau)}V(R^{\pi}_{t\wedge\tau}) \to 0 \tag{2.13}$$

as $t \to \infty$. Thus taking limit in (2.12) as $t \to \infty$ we arrive at

$$V(x) \ge E\int_0^{\tau} e^{-\gamma s}V'(R^{\pi}_s)dC^{\pi}_s = J_{\pi}(x).$$

□

The idea of solving the original optimization problem is to first find a concave, smooth function to the HJB equation (2.8), and then construct a control policy (via solving a Skorohod problem; for details see Section 6) whose performance functional can be shown to coincide with the founded solution to (2.8). Then, the above verification theorem establishes the optimality of the constructed control policy. As a by-product, there is no other concave solution to (2.8) than the optimal return function.

3 Case of unbounded dividend rate without liability

In this section we study the case where the dividend rate is unbounded and there is no debt liability, namely, $\delta = 0$. While being part of a more general case, it is interesting in its own right and will provide some valuable insights into the general problem.
In this case, the HJB equation reads

$$\begin{aligned}&\max\left(\max_{\alpha\le a\le\beta}\left(\tfrac{1}{2}\sigma^2a^2V''(x)+a\mu V'(x)-\gamma V(x)\right),\ 1-V'(x)\right)=0, \quad x>0,\\ &V(0)=0.\end{aligned} \tag{3.1}$$

As mentioned, the key is to find a concave, smooth function V satisfying (3.1). While we could have presented such a solution immediately without any explanation (one would need only to check if it does satisfy (3.1), which is a relatively easy task), we believe that it is better to unfold the entire process of finding the solution for the benefit of the readers. Therefore, what we are going to present below is indeed the original process of tracking down the solution. Suppose such a solution, V, to the HJB equation (3.1) is found. Then due to concavity, V' is a non-increasing function. Let $x_1 = \inf\{x \ge 0 \ :\ \ V'(x) \le 1\}$. Suppose $x_1 > 0$. Since V is concave we have $V'(x) > 1$ for all $x < x$. In view of (3.1), V satisfies

$$\max_{\alpha\le a\le\beta}\left(\frac{1}{2}\sigma^2a^2V''(x)+a\mu V'(x)-\gamma V(x)\right)=0, \quad \forall x<x_1. \tag{3.2}$$

Let

$$a(x) \equiv -\frac{\mu V'(x)}{\sigma^2 V''(x)} > 0, \quad x < x_1, \tag{3.3}$$

be the maximizer of $\frac{1}{2}\sigma^2a^2V''(x)+a\mu V'(x)-\gamma V(x)$ over all $a \ge 0$. Since $V''(x) \le 0$, the expression $\frac{1}{2}\sigma^2a^2V''(x)+a\mu V'(x)-\gamma V(x)$ as a function of a increases on $[0,\ a(x)]$ and decreases on $[a(x),\ \infty)$. Assume that $a(x)$ is a non-decreasing function of x (which will be shown a posteriori). Then there exist x_α and x_β, $0 \le x_\alpha < x_\beta \le +\infty$, such that $a(x) \le \alpha$, $\forall x \le x_\alpha$ and $a(x) \ge \beta$, $\forall x \ge x_\beta$; and $a(x)$ is increasing from α to β on $[x_\alpha,\ x_\beta]$. Our next step would be to show that $x_\alpha > 0$. To this end it is sufficient to analyze $a(0)$.

Proposition 3.1 *The following holds*

$$a(0) = \frac{1}{2}\alpha. \tag{3.4}$$

Proof. Put

$$\phi(x,\ a) = \frac{1}{2}\sigma^2 a^2 V''(x) + a\mu V'(x) - \gamma V(x), \quad x \geq 0,\ a \geq 0. \tag{3.5}$$

It follows from (3.2) that $\max_{\alpha \leq a \leq \beta} \phi(0,\ a) = 0$. Let $\tilde{a} \in [\alpha,\ \beta]$ such that $\phi(0,\ \tilde{a}) = \max\limits_{\alpha \leq a \leq \beta} \phi(0,\ a) = 0$. Since $V(0) = 0$ we conclude

$$\tilde{a} = \frac{-2\mu V'(0)}{\sigma^2 V''(0)} \equiv 2a(0).$$

On the other hand, $\phi(0, a) = a[\frac{1}{2}\sigma^2 a V''(0) + \mu V'(0)]$ where $\frac{1}{2}\sigma^2 V''(0) \leq 0$. Hence the maximum of $\phi(0, \cdot)$ is attained at the lower end of the interval $[\alpha, \beta]$, namely, $\tilde{a} = \alpha$. This proves (3.4). □

Continuity of $a(x)$ and the inequality $a(0) < \alpha$ shows that for all x in a right neighborhood of 0, the maximum over $a \in [\alpha, \beta]$ in (3.2) is attained at α. Substituting $a = \alpha$ into (3.2) and solving the resulting second-order linear ordinary differential equation (ODE), we get

$$V(x) = k_1(\alpha,\ \beta)\left(e^{r_+(\alpha)x} - e^{r_-(\alpha)x}\right), \tag{3.6}$$

where $k_1(\alpha,\ \beta)$ is a free constant to be determined, and

$$r_+(z) \equiv \frac{-\mu + [\mu^2 + 2\sigma^2\gamma]^{1/2}}{z\sigma^2} > 0, \quad r_-(z) \equiv \frac{-\mu - [\mu^2 + 2\sigma^2\gamma]^{1/2}}{z\sigma^2} < 0,\ \forall z > 0. \tag{3.7}$$

From (3.6) and (3.3) it follows

$$a'(x) = \frac{-\mu r_-(\alpha) r_+(\alpha) k_1^2(\alpha,\ \beta) e^{(r_+(\alpha)+r_-(\alpha))x}\,(r_+(\alpha) - r_-(\alpha))^2}{(\sigma V''(x))^2} > 0,$$

for all x in the right neighborhood of 0. Therefore in the right neighborhood of 0 the function $a(x)$ increases. Let x_α be such that $a(x_\alpha) = \alpha$. From (3.3) and (3.6), we obtain

$$x_\alpha = \frac{1}{r_+(\alpha) - r_-(\alpha)} \ln\left(-\frac{r_-(\alpha)}{r_+(\alpha)}\right) > 0. \tag{3.8}$$

Proposition 3.2 *For each $x \geq x_\alpha$,*

$$a(x) \geq \alpha. \tag{3.9}$$

Proof. Suppose there exists $x_0 > x_\alpha$ such that $a(x_0) < \alpha$. Then there exists $\varepsilon > 0$ such that $a(x) < \alpha$ for each x with $|x - x_0| < \varepsilon$. Let $x' = \sup\{x < x_0 :\ \ a(x) = \alpha\}$. Then $x_\alpha \leq x' < x_0 < x_0 + \varepsilon$ and $a(x') = \alpha$. Since $a(x) \leq \alpha$ for all $x \in [x',\ x_0 + \varepsilon)$, the function V satisfies (3.2) with the maximum there attained at $a = \alpha$. Therefore

$$V(x) = K_1 e^{r_+(\alpha)(x-x')} + K_2 e^{r_-(\alpha)(x-x')}, \ \ \forall x \in [x',\ x_0 + \varepsilon). \tag{3.10}$$

From (3.10) and (3.3), the equation $a(x') = \alpha$ can be rewritten as

$$K_1 r_+(\alpha) = -K_2 r_-(\alpha) \frac{\mu + \alpha\sigma^2 r_-(\alpha)}{\mu + \alpha\sigma^2 r_+(\alpha)},$$

which establishes a relation between the constants K_1 and K_2. Using this relation, we calculate

$$a(x) = \frac{-\mu V'(x)}{\sigma^2 V''(x)} = \frac{-\mu \left(e^{(r_+(\alpha)-r_-(\alpha))(x-x')} - \frac{\mu+\alpha\sigma^2 r_+(\alpha)}{\mu+\alpha\sigma^2 r_-(\alpha)} \right)}{\sigma^2 \left(r_+(\alpha) e^{(r_+(\alpha)-r_-(\alpha))(x-x')} - r_-(\alpha) \frac{\mu+\alpha\sigma^2 r_+(\alpha)}{\mu+\alpha\sigma^2 r_-(\alpha)} \right)}, \ \forall x \in [x', \ x_0+\varepsilon). \quad (3.11)$$

However, we have $a(x) < \alpha$ for $x > x'$, which after a simple algebraic transformation of (3.11) is equivalent to $e^{(r_+(\alpha)-r_-(\alpha))(x-x')} < 1$. This leads to a contradiction. Therefore (3.9) holds. □

In view of $a(x_\alpha) = \alpha < \beta$ and (3.9), we have

$$\alpha \le a(x) < \beta$$

in the right neighborhood of x_α. Therefore

$$\phi(x, \ a(x)) = \max_{\alpha \le a \le \beta} \phi(x, \ a) = 0. \quad (3.12)$$

From (3.3), we have $V''(x) = -\dfrac{\mu V'(x)}{\sigma^2 a(x)}$. Substituting this expression for V'' into (3.12), we get

$$\mu a(x) V'(x)/2 = \gamma V(x). \quad (3.13)$$

Differentiating this equation and again using $V''(x) = -\frac{\mu V'(x)}{\sigma^2 a(x)}$, we arrive at

$$a'(x) = \frac{\mu^2 + 2\sigma^2\gamma}{\mu\sigma^2}.$$

Integrating this equation results in (recall that $a(x_\alpha) = \alpha$)

$$a(x) = \frac{\mu^2 + 2\sigma^2\gamma}{\mu\sigma^2}(x - x_\alpha) + \alpha. \quad (3.14)$$

Let

$$x_\beta = \frac{\mu\sigma^2}{\mu^2 + 2\sigma^2\gamma}(\beta - \alpha) + x_\alpha, \quad (3.15)$$

which is obtained by setting $a(x_\beta) = \beta$. Then $\alpha \le a(x) < \beta$, $\forall x \in [x_\alpha, \ x_\beta)$. It follows from (3.13) that (noting $a(x_\alpha) = \alpha$)

$$\frac{V(x_\alpha)}{V'(x_\alpha)} = \frac{\mu\alpha}{2\gamma} \equiv \frac{y_\alpha}{(1-\Gamma)}, \quad (3.16)$$

where

$$0 < \Gamma \equiv \frac{\mu^2}{\mu^2 + 2\sigma^2\gamma} < 1, \qquad y_\alpha \equiv \frac{\mu\sigma^2\alpha}{\mu^2 + 2\sigma^2\gamma}. \quad (3.17)$$

Substituting the expression (3.14) for $a(x)$ into (3.3), and then solving the resulting equation for $V(x)$ on $[x_\alpha, \ x_\beta)$ while taking into account (3.16), we get

$$V(x) = \frac{\mu\alpha}{2\gamma} V'(x_\alpha) \left(\frac{x - x_\alpha + y_\alpha}{y_\alpha} \right)^{1-\Gamma}, \quad x_\alpha \le x < x_\beta, \quad (3.18)$$

where the free constant $V'(x_\alpha)$ can be determined by

$$V'(x_\alpha) = k_1(\alpha, \beta)(r_+(\alpha) e^{r_+(\alpha) x_\alpha} - r_-(\alpha) e^{r_-(\alpha) x_\alpha}), \quad (3.19)$$

in view of (3.6) and a smooth fit at $x = x_\alpha$. Straightforward computations show that the function V defined by (3.6) and (3.18) is continuous with continuous first and second derivatives at x_α.
So far, we have obtained the forms of V on two intervals, $[0, x_\alpha)$ and $[x_\alpha, x_\beta)$, by (3.6) and (3.18), respectively. Now we proceed to the interval beyond x_β. To this end, we first have the following

Proposition 3.3 *For all $x \geq x_\beta$*

$$a(x) \geq \beta.$$

Proof. By virtue of Proposition 3.2, $a(x) \geq \alpha$ for all $x \geq x_\beta$. Suppose there exists $x' > x_\beta$ such that $a(x') < \beta$. Then there exists $\varepsilon > 0$ such that $a(x) < \beta$ for all $x < x' + \varepsilon$. Let $\bar{x} = \sup\{x < x' \ : \ a(x) = \beta\}$. Then $x_\beta \leq \bar{x} < x'$ and $a(\bar{x}) = \beta$. In addition $\alpha \leq a(x) < \beta$ for all $\bar{x} < x \leq x'$. Repeating the arguments of Proposition 3.2, we get $a(x) = \frac{\mu^2+2\sigma^2\gamma}{\mu\sigma^2}(x - \bar{x}) + \beta > \beta$ for each $x > \bar{x}$, which is a contradiction. □

The above proposition implies that the maximum in (3.2) is obtained at $a = \beta$ for $x \geq x_\beta$. The resulting equation of (3.2) then becomes a second-order linear ODE, whose solution is of the form

$$V(x) = k_1(\beta)e^{r_+(\beta)(x-x_1)} + k_2(\beta)e^{r_-(\beta)(x-x_1)}, \quad x_\beta \leq x < x_1, \tag{3.20}$$

where $x_1 > x_\beta$, as defined earlier, is also the first point such that $V''(x_1) = 0$ (see Proposition 3.4 below).

Proposition 3.4 *Let $x_1 > x_\beta$ be the first point where V'' vanishes. Then $V'(x_1) = 1$.*

Proof. Suppose that $V'(x_1) > 1$. Then there exists $\varepsilon > 0$ such that $V'(x) > 1$ for all $x_1 \leq x \leq x_1 + \varepsilon$. Therefore on the interval $[x_1, x_1 + \varepsilon]$, the function V satisfies (3.2). In view of Proposition 3.3, $a(x) \geq \beta$, and hence on the interval $[x_1, x_1 + \varepsilon]$, V is of the form given by (3.20). Since $V''(x_1) = 0$, we conclude $k_1(\beta)r_+^2(\beta) = -k_2(\beta)r_-^2(\beta) > 0$ (the positivity of $k_1(\beta)$ follows from $V' > 0$). Thus $V''(x) = k_1(\beta)r_+^2(\beta)\left(e^{(r_+(\beta)-r_-(\beta))(x-x_1)} - 1\right)$. This expression is positive for each $x > x_1$. This contradicts the concavity of V. □

The following corollary is straightforward in view of the above proposition and the inequality $V'(x) \geq 1$.

Corollary 3.5 *Under the assumption of Proposition 3.4,*

$$V'(x) = 1, \quad \forall x \geq x_1.$$

Those results enable us to present the function V in the following form:

$$V(x) = \begin{cases} k_1(\alpha,\ \beta)(e^{r_+(\alpha)x} - e^{r_-(\alpha)x}), & 0 \leq x < x_\alpha, \\ \frac{\mu\alpha}{2\gamma}V'(x_\alpha)\left(\frac{x-x_\alpha+y_\alpha}{y_\alpha}\right)^{1-\Gamma}, & x_\alpha \leq x < x_\beta, \\ k_1(\beta)e^{r_+(\beta)(x-x_1)} + k_2(\beta)e^{r_-(\beta)(x-x_1)}, & x_\beta \leq x < x_1, \\ k_1(\beta) + k_2(\beta) + x - x_1, & x_1 \leq x, \end{cases} \tag{3.21}$$

where $r_+(\alpha)$, $r_-(\alpha)$, $r_+(\beta)$, $r_-(\beta)$, x_α and x_β, and Γ and y_α are given by (3.7), (3.8), (3.15), and (3.17) respectively.

The next step is to determine the remaining constants in (3.21). To do so we use the principle of smooth fit at the points x_β and x_1. Namely, we have to choose the unknown constants $k_1(\beta), k_2(\beta), k_1(\alpha, \beta)$ and x_1 in such a way that the function V, its first and second derivatives are continuous at these points. To this end, first by virtue of Proposition 3.4,

$$V'(x_1) = 1, \quad V''(x_1) = 0.$$

For V of the form (3.20) this translates into

$$k_1(\beta)r_+(\beta) + k_2(\beta)r_-(\beta) = 1, \ k_1(\beta)r_+^2(\beta) + k_2(\beta)r_-^2(\beta) = 0.$$

As a result,

$$k_1(\beta) = \frac{-r_-(\beta)}{r_+(\beta)(r_+(\beta) - r_-(\beta))}, \quad k_2(\beta) = \frac{r_+(\beta)}{r_-(\beta)(r_+(\beta) - r_-(\beta))}. \tag{3.22}$$

Next, let $\Delta = x_\beta - x_1$, then we can calculate V' and V'' at x_β as

$$\begin{aligned} V'(x_\beta) &= k_1(\beta)r_+(\beta)e^{r_+(\beta)\Delta} + k_2(\beta)r_-(\beta)e^{r_-(\beta)\Delta}, \\ V''(x_\beta) &= k_1(\beta)r_+^2(\beta)e^{r_+(\beta)\Delta} + k_2(\beta)r_-^2(\beta)e^{r_-(\beta)\Delta}. \end{aligned} \tag{3.23}$$

Recall that $V''(x_\beta) = \frac{-\mu V'(x_\beta)}{\sigma^2 a(x_\beta)} = \frac{-\mu V'(x_\beta)}{\sigma^2 \beta}$, which results in

$$x_\beta - x_1 \equiv \Delta = \frac{\beta\sigma^2}{[\mu^2 + 2\sigma^2\gamma]^{1/2}} \log\left(\frac{-\mu + [\mu^2 + 2\sigma^2\gamma]^{1/2}}{\mu + [\mu^2 + 2\sigma^2\gamma]^{1/2}}\right) < 0. \tag{3.24}$$

On the other hand, a smooth fit, in terms of $V'(x_\beta)$, for (3.18) and (3.23) while taking (3.19) into consideration yields

$$\frac{\alpha\mu}{2\gamma y_\alpha} k_1(\alpha,\ \beta)\left(r_+(\alpha)e^{r_+(\alpha)x_\alpha} - r_-(\alpha)e^{r_-(\alpha)x_\alpha}\right)\left(\frac{\beta}{\alpha}\right)^{-\Gamma} = k_1(\beta)r_+(\beta)e^{r_+(\beta)\Delta} + k_2(\beta)r_-(\beta)e^{r_-(\beta)\Delta}. \tag{3.25}$$

Formulas (3.24) and (3.25) determine x_1 and $k_1(\alpha, \beta)$. Note that from (3.22) and (3.24) it follows

$$\frac{V(x_\beta)}{V'(x_\beta)} = \frac{k_1(\beta)e^{r_+(\beta)\Delta} + k_2(\beta)e^{r_-(\beta)\Delta}}{k_1(\beta)r_+(\beta)e^{r_+(\beta)\Delta} + k_2(\beta)r_-(\beta)e^{r_-(\beta)\Delta}} = \frac{e^{(r_+(\beta) - r_-(\beta))\Delta} - \frac{r_-(\beta)}{r_+(\beta)}}{r_+(\beta)e^{(r_+(\beta) - r_-(\beta))\Delta} - r_-(\beta)} = \frac{\mu\beta}{2\gamma}.$$

Therefore $\lim_{x\to x_\beta,\ x<x_\beta} V(x) = \frac{\mu\alpha}{2\gamma}V'(x_\alpha)\left(\frac{\beta}{\alpha}\right)^{1-\Gamma} = \frac{\mu\beta}{2\gamma}V'(x_\beta) = V(x_\beta)$, which proves the continuity of V at x_β.

These calculations enable us to formulate the main result of this section.

Theorem 3.6 *Let $r_+(\alpha)$, $r_-(\alpha)$, $r_+(\beta)$ and $r_-(\beta)$, Γ and y_α, x_α, x_β, x_1, $k_1(\beta)$ and $k_2(\beta)$, and $k_1(\alpha,\ \beta)$ be given by (3.7), (3.17), (3.8), (3.15), (3.24), (3.22), and (3.25) respectively. Then $V(x)$ given by (3.21) is a concave, twice continuously differentiable solution of the HJB equation (3.1).*

Proof. From the way we constructed V, it must be a twice continuously differentiable solution to the HJB equation (3.1). What remains to show is the concavity. From (3.21), we deduce that

$$V'''(x) = k_1(\alpha,\ \beta)\left(r_+^3(\alpha)e^{r_+(\alpha)x} - r_-^3(\alpha)e^{r_-(\alpha)x}\right) > 0, \quad \forall 0 \le x < x_\alpha,$$

due to $r_-(\alpha) < 0 < k_1(\alpha,\ \beta)$. Hence on this interval V'' is increasing and

$$V''(x) < V''(x_\alpha) = k_1(\alpha,\ \beta)\left(r_+^2(\alpha)e^{r_+(\alpha)x_\alpha} - r_-^2(\alpha)e^{r_-(\alpha)x_\alpha}\right) < 0,$$

due to $\frac{r_-(\alpha)}{r_+(\alpha)} = e^{(r_+(\alpha) - r_-(\alpha))x_\alpha}$ and $|r_-(\alpha)| > r_+(\alpha)$.

For $x_\alpha \le x < x_\beta$, $V''(x) = \frac{-\mu V'(x)}{\sigma^2 a(x)} < 0$. For $x_\beta \le x < x_1$,

$$V'''(x) = k_1(\beta)r_+^3(\beta)e^{r_+(\beta)(x-x_1)} + k_2(\beta)r_-^3(\beta)e^{r_-(\beta)(x-x_1)} > 0,$$

since $k_2(\beta)$ and $r_-(\beta)$ are of the same signs. Thus $V''(x) < V''(x_1) = 0$, $\forall x_\beta \le x < x_1$. Finally, $V''(x) = 0$, $\forall x \ge x_1$. This establishes the concavity of V. □

4 Case of unbounded dividend rate with nonzero liability

This section deals with the general model (2.1) where $\delta > 0$. In this case the HJB equation is given by (2.8). Again we are looking for a smooth concave function that solves this equation. As before, suppose that such a solution V exists and consider $x_1 = \inf\{x \geq 0 \ : \ V'(x) \leq 1\}$, then it is obvious that $x_1 = 0$ if and only if $V(x) = x, \ \ \forall x \geq 0$. Our first step is to characterize the existence of such a trivial solution to (2.8).

Theorem 4.1 $V(x) = x, \ \ \forall x \geq 0$ *iff*

$$\beta\mu \leq \delta. \tag{4.1}$$

Proof. Suppose $V(x) = x$ for each $x \geq 0$. Then in view of (2.8)

$$\max_{\alpha \leq a \leq \beta} \left(\frac{1}{2}\sigma^2 a^2 V''(0) + (a\mu - \delta)V'(0) - \gamma V(0) \right) \equiv \beta\mu - \delta \leq 0.$$

Conversely, if $\beta\mu \leq \delta$, then due to concavity

$$\max_{\alpha \leq a \leq \beta} \left(\frac{1}{2}\sigma^2 a^2 V''(x) + (a\mu - \delta)V'(x) - \gamma V(x) \right) \leq -\gamma V(x) < 0, \quad \forall x > 0.$$

Thus, (2.8) is satisfied only if $V'(x) = 1, \ \ \forall x > 0$. ☐

Remark 4.2 Proposition 4.1 is a mathematical formulation of the intuition that if a company has a liability rate not smaller than the maximal expected profit rate, then it is optimal to declare bankruptcy immediately, distributing the whole reserve as the dividend.

In the rest of this section we assume $\beta\mu > \delta$. In view of (2.8)

$$0 = \max_{\alpha \leq a \leq \beta} \left(\frac{1}{2}\sigma^2 a^2 V''(x) + (a\mu - \delta)V'(x) - \gamma V(x) \right), \quad \forall x < x_1. \tag{4.2}$$

For each $x \geq 0$ and $a \geq 0$ define

$$\phi(x, \ a) = \frac{1}{2}\sigma^2 a^2 V''(x) + (a\mu - \delta)V'(x) - \gamma V(x). \tag{4.3}$$

The maximizer of the function $\phi(x, \ a)$ over $a \geq 0$ is given by

$$a(x) = -\frac{\mu V'(x)}{\sigma^2 V''(x)} > 0, \quad x \geq 0. \tag{4.4}$$

Following the same scheme as in the no-liability case, we will prove that there exist $x_\alpha \leq x_\beta < x_1$ such that $a(x) \leq \alpha, \ \ \forall x \leq x_\alpha$, and $a(x) \geq \beta, \ \ \forall x \geq x_\beta$, and the function $a(x)$ increases from α to β on the interval $[x_\alpha, \ x_\beta]$.

As before we start with analyzing $a(0)$.

Proposition 4.3 *Let* $\beta\mu > \delta$. *Then*
(i) $\frac{2\delta}{\mu} < \alpha$ *if and only if* $a(0) < \alpha$. *In this case* $a(0) = \frac{\mu\alpha^2}{2(\mu\alpha - \delta)}$.
(ii) $\alpha \leq \frac{2\delta}{\mu} < \beta$ *if and only if* $\alpha \leq a(0) < \beta$. *In this case* $a(0) = 2\frac{\delta}{\mu}$.
(iii) $\beta \leq \frac{2\delta}{\mu}$ *if and only if* $a(0) \geq \beta$. *In this case* $a(0) = \frac{\mu\beta^2}{2(\mu\beta - \delta)}$.

Proof. Let $\tilde{a} \in [\alpha,\ \beta]$ be such that

$$0 = \max_{\alpha \le a \le \beta} \left(\frac{1}{2}\sigma^2 a^2 V''(0) + (a\mu - \delta)V'(0) \right) = \frac{1}{2}\sigma^2 \tilde{a}^2 V''(0) + (\tilde{a}\mu - \delta)V'(0). \tag{4.5}$$

Comparing (4.5) with (4.4) we obtain

$$\tilde{a}^2 - 2a(0)\tilde{a} + \frac{2\delta}{\mu}a(0) = 0. \tag{4.6}$$

From (4.6), it follows $a(0) \ge \frac{2\delta}{\mu}$. Moreover, by definition, $a(0) \in [\alpha,\ \beta]$ is equivalent to $\tilde{a} = a(0)$, which is further equivalent to $a(0) = \frac{2\delta}{\mu} \in [\alpha,\ \beta]$. Thus we conclude:
(i) If $a(0) < \alpha$, then $\frac{2\delta}{\mu} \le a(0) < \alpha$. Conversely, suppose $\frac{2\delta}{\mu} < \alpha$. If $a(0) \in [\alpha,\ \beta]$, then by the above $a(0) = \frac{2\delta}{\mu} < \alpha$ which is a contradiction. Thus either $a(0) < \alpha$ or $a(0) > \beta$. Suppose $a(0) > \beta$, then $\tilde{a} = \beta$ and by (4.6), $a(0) = \frac{\mu\beta^2}{2(\mu\beta-\delta)} < \beta$ (due to $\frac{2\delta}{\mu} < \alpha < \beta$). This is again a contradiction. Hence we have $a(0) < \alpha$. Then $\tilde{a} = \alpha$ and in view of (4.6), we get $a(0) = \frac{\mu\alpha^2}{2(\alpha\mu-\delta)}$.
(ii) Suppose $\alpha \le \frac{2\delta}{\mu} < \beta$. Then due to (i) we have $a(0) \ge \alpha$. Now we proceed to prove that $a(0) \le \frac{2\delta}{\mu} < \beta$. Suppose $a(0) > \frac{2\delta}{\mu}$. Then $a(0) > \beta \equiv \tilde{a}$. On the other hand, in view of (4.6) we have $a(0) = \frac{\mu\beta^2}{2(\beta\mu-\delta)}$, thus $\frac{\mu\beta^2}{2(\beta\mu-\delta)} \ge \beta$, which is equivalent to $2\frac{\delta}{\mu} \ge \beta$. This however is a contradiction and therefore $a(0) = \frac{2\delta}{\mu} \in [\alpha,\ \beta)$. Conversely, if $a(0) \in [\alpha,\ \beta)$, then $a(0) = \frac{2\delta}{\mu} \in [\alpha,\ \beta)$.
(iii) Suppose $\beta \le \frac{2\delta}{\mu}$. Then $a(0) \ge \frac{2\delta}{\mu} \ge \beta$, leading to $\tilde{a} = \beta$ and $a(0) = \frac{\mu\beta^2}{2(\beta\mu-\delta)} \ge \beta$. Conversely, if $a(0) \ge \beta$, then $\tilde{a} = \beta$ and $a(0) = \frac{\mu\beta^2}{2(\mu\beta-\delta)} \ge \beta$, which is equivalent to $\frac{2\delta}{\mu} \ge \beta$. □

As it will be seen in the sequel, the structure of the solution to our original optimization problem depends on the three cases as specified by (i), (ii) and (iii) above. Accordingly in the rest of the section we will analyze these three cases.

4.1 Case of $\frac{2\delta}{\mu} < \alpha$

We begin our analysis with an observation that in this case, in view of Proposition 4.3-(i), $a(x) < \alpha$ for all x in the right neighborhood of 0. Substituting $a = \alpha$ in (4.2) and solving the resulting second-order linear ODE, we obtain

$$V(x) = k_1(\alpha,\ \beta)(e^{r_+(\alpha)x} - e^{r_-(\alpha)x}), \tag{4.7}$$

where $k_1(\alpha,\ \beta)$ is a free constant to be determined, and

$$\begin{aligned} r_+(z) &= \frac{-(z\mu-\delta) + [(z\mu-\delta)^2 + 2\sigma^2 z^2 \gamma]^{1/2}}{\sigma^2 z^2}, \\ r_-(z) &= \frac{-(z\mu-\delta) - [(z\mu-\delta)^2 + 2\sigma^2 z^2 \gamma]^{1/2}}{\sigma^2 z^2},\ z > 0. \end{aligned} \tag{4.8}$$

Due to (4.4) and (4.7),

$$a'(x) = \frac{-\mu}{\sigma^2}\frac{(V''(x))^2 - V'(x)V^{(3)}(x)}{(V''(x))^2} = \frac{-\mu r_+(\alpha) r_-(\alpha) e^{(r_+(\alpha)+r_-(\alpha))x}\,(r_+(\alpha) - r_-(\alpha))^2}{\sigma^2 (V''(x))^2} > 0$$

for each x in the right neighborhood of 0. Therefore $a(x)$ increases and reaches α at the point x_α given by

$$x_\alpha = \frac{1}{r_+(\alpha) - r_-(\alpha)} \log\left(\frac{r_-(\alpha)\,(\mu + \alpha\sigma^2 r_-(\alpha))}{r_+(\alpha)\,(\mu + \alpha\sigma^2 r_+(\alpha))} \right) > 0. \tag{4.9}$$

Proposition 4.4 *For each $x \geq x_\alpha$,*

$$a(x) \geq \alpha.$$

Proof. Suppose there exists $x' > x_\alpha$ such that $a(x) < \alpha$ and let $\bar{x} = \sup\{x < x' \ : \ a(x) = \alpha\}$. Then $x_\alpha \leq \bar{x} < x'$, $a(\bar{x}) = \alpha$, and $a(x) < \alpha$ for $\bar{x} < x \leq x'$. Substituting $a = \alpha$ into (4.2) and solving the resulting second-order linear ODE, we get $V(x) = k_1 e^{r_+(\alpha)(x-\bar{x})} + k_2 e^{r_-(\alpha)(x-\bar{x})}$. Therefore

$$a(x) = \frac{-\mu V'(x)}{\sigma^2 V''(x)} = -\frac{\mu}{\sigma^2} \frac{k_1 r_+(\alpha) e^{(r_+(\alpha)-r_-(\alpha))(x-\bar{x})} + k_2 r_-(\alpha)}{k_1 r_+^2(\alpha) e^{(r_+(\alpha)-r_-(\alpha))(x-\bar{x})} + k_2 r_-^2(\alpha)}, \quad \bar{x} < x \leq x'.$$

Since $a(\bar{x}) = \alpha$, we have

$$k_1 r_+(\alpha) \left(1 + \frac{\alpha\sigma^2 r_+(\alpha)}{\mu}\right) = -k_2 r_+(\alpha) \left(1 + \frac{\alpha\sigma^2 r_-(\alpha)}{\mu}\right).$$

Thus, for $\bar{x} < x \leq x'$ the inequality $a(x) < \alpha$ is equivalent to $e^{(r_+(\alpha)-r_-(\alpha))(x-\bar{x})} < 1$, which is a contradiction. ◻

By virtue of Proposition 4.4, $\alpha \leq a(x) < \beta$ in the right neighborhood of x_α. In this case we have

$$\phi(x,\ a(x)) = \max_{\alpha \leq a \leq \beta} \phi(x,\ a) = 0. \tag{4.10}$$

Substituting

$$V''(x) = \frac{-\mu V'(x)}{\sigma^2 a(x)} \tag{4.11}$$

into (4.10), differentiating the resulting equation and substituting $V''(x) = \frac{-\mu V'(x)}{\sigma^2 a(x)}$ once more, we arrive at $\frac{\mu a'(x)}{2} + \frac{\mu\delta}{\sigma^2 a(x)} = \frac{\mu^2 + 2\gamma\sigma^2}{2\sigma^2}$. As a result

$$a'(x) = \frac{\mu^2 + 2\gamma\sigma^2}{\mu\sigma^2}(1 - \frac{c}{a(x)}) \tag{4.12}$$

with

$$c \equiv 2\delta\mu/(\mu^2 + 2\gamma\sigma^2). \tag{4.13}$$

Integrating equation (4.12) we get $G(a(x)) \equiv \frac{\mu^2+2\gamma\sigma^2}{\mu\sigma^2}(x - x_\alpha) + G(\alpha)$, where

$$G(u) = u + c\log(u - c). \tag{4.14}$$

Therefore

$$a(x) = G^{-1}\left(\frac{\mu^2 + 2\gamma\sigma^2}{\mu\sigma^2}(x - x_\alpha) + G(\alpha)\right). \tag{4.15}$$

Thus $a(x)$ is increasing and $a(x_\beta) = \beta$ for

$$x_\beta \equiv \frac{\mu\sigma^2}{\mu^2 + 2\gamma\sigma^2}[G(\beta) - G(\alpha)] + x_\alpha = \frac{\mu\sigma^2}{\mu^2 + 2\gamma\sigma^2}(\beta - \alpha) + \frac{\mu\sigma^2 c}{\mu^2 + 2\gamma\sigma^2}\log(\frac{\beta - c}{\alpha - c}). \tag{4.16}$$

Solving equation (4.11) we obtain

$$V(x) = V(x_\alpha) + V'(x_\alpha)\int_{x_\alpha}^{x} \exp\left(-\frac{\mu}{\sigma^2}\int_{x_\alpha}^{y} \frac{du}{a(u)}\right) dy, \quad x_\alpha \leq x < \ x_\beta, \tag{4.17}$$

where $V(x_\alpha)$ and $V'(x_\alpha)$ are free constants. Choosing $V(x_\alpha)$ and $V'(x_\alpha)$ as the value and the derivative respectively of the right hand side of (4.7) at x_α, we can ensure that the function V given by (4.7) and (4.17) is continuous with its first and second derivatives at the point x_α no matter what the choice of $k(\alpha, \beta)$ is. (Note that due to the HJB equation continuity of V and its first derivative at x_α automatically implies continuity of the second derivative as well.)
Next we are to simplify (4.17). First, changing variables $a(u) = \theta$ we get

$$\int_{x_\alpha}^{x} \exp\left(-\mu/\sigma^2 \int_{x_\alpha}^{y} \frac{du}{a(u)} dy\right) = \frac{\mu\sigma^2}{\mu^2 + 2\gamma\sigma^2} \int_{\alpha}^{a(x)} (1 + \frac{c}{\theta - c}) \left(\frac{\theta - c}{\alpha - c}\right)^{-\Gamma} d\theta, \; x_\alpha \leq x < x_\beta.$$

On the other hand, relations (4.9) and (4.7) imply

$$V(x_\alpha) = \frac{\alpha\mu - 2\delta}{2\gamma} V'(x_\alpha).$$

Simple algebraic transformations yield

$$\left(\frac{\mu\sigma^2}{\mu^2 + 2\gamma\sigma^2}\right)\left(\frac{c}{\Gamma} - \frac{z - c}{1 - \Gamma}\right) = \frac{z\mu - 2\delta}{2\gamma}, \; \forall z > 0, \tag{4.18}$$

where c is given by (4.13) and

$$\Gamma = \frac{\mu^2}{\mu^2 + 2\gamma\sigma^2}. \tag{4.19}$$

Therefore

$$V(x) = V'(x_\alpha) \frac{\mu a(x) - 2\delta}{2\gamma} \left(\frac{a(x) - c}{\alpha - c}\right)^{-\Gamma}, \quad x_\alpha \leq x < x_\beta. \tag{4.20}$$

Now, we proceed to the next piece of V on the interval beyond x_β.

Proposition 4.5 *For each $x \geq x_\beta$,*

$$a(x) \geq \beta.$$

Proof. Suppose that there exists $x' > x_\beta$ such that $a(x) < \beta$. Since $x' \geq x_\alpha$, we have $\beta > a(x) \geq \alpha$. Denote $\bar{x} = \sup\{x < x' \; : \; a(x) = \beta\}$. Then $x_\beta \leq \bar{x} < x'$, $a(\bar{x}) = \beta$, and $\alpha \leq a(x) < \beta$ for $\bar{x} < x \leq x'$. Thus $a(x)$ satisfies (4.12) for $\bar{x} < x \leq x'$ and

$$a(x) = G^{-1}\left(\frac{\mu^2 + 2\gamma\sigma^2}{\mu\sigma^2}(x - \bar{x}) + G(\beta)\right) > \beta.$$

This is a contradiction. □

In view of the above proposition,

$$\phi(x, \; \beta) = \max_{\alpha \leq a \leq \beta} \phi(x, \; a) = 0, \quad x_\beta \leq x < x_1, \tag{4.21}$$

where x_1, which is defined earlier, is also the first point such that $V''(x_1) = 0$. This results in

$$V(x) = k_1(\beta) e^{r_+(\beta)(x - x_1)} + k_2(\beta) e^{r_-(\beta)(x - x_1)}, \qquad x_\beta \leq x < x_1, \tag{4.22}$$

where $k_1(\beta)$ and $k_2(\beta)$ are two free constants also to be determined.

Proposition 4.6 *For $x \geq x_1$,*

$$V'(x) = 1.$$

Proof. Suppose that there exists $x' \geq x_1$ such that $V''(x') < 0$. Since $x' \geq x_\beta$ inequality $a(x) \geq \beta$ holds. Let $\bar{x} = \sup\{x < x' \; : \; V''(x) = 0\}$. For any $x \in (\bar{x},\; x']$, we have $V(x) = K_1' e^{r_+(\beta)(x-\bar{x})} + K_2' e^{r_-(\beta)(x-\bar{x})}$. Equality $V''(\bar{x}) = 0$ results in $0 < K_1' r_+^2(\beta) = -K_2' r_-^2(\beta)$. Consequently $V''(x) = K_1' r_+^2(\beta) e^{r_+(\beta)(x-\bar{x})} + K_2' r_-^2(\beta) e^{r_-(\beta)(x-\bar{x})} = K_1' r_+^2(\beta) \left(e^{r_+(\beta)(x-\bar{x})} - e^{r_-(\beta)(x-\bar{x})}\right) > 0$. This contradicts the concavity of the function V. □

From Proposition 4.6 it follows

$$V(x) = x - x_1 + k_1(\beta) + k_2(\beta), \quad x \geq x_1.$$

To compute the free constants $k_1(\beta)$ and $k_2(\beta)$, we use the relationship

$$V'(x_1) = 1, \quad V''(x_1) = 0.$$

From (4.22) it follows

$$k_1(\beta) r_+(\beta) + k_2(\beta) r_-(\beta) = 1, \quad k_1(\beta) r_+^2(\beta) + k_2(\beta) r_-^2(\beta) = 0.$$

As a result

$$k_1(\beta) = \frac{-r_-(\beta)}{r_+(\beta)(r_+(\beta) - r_-(\beta))} > 0, \quad k_2(\beta) = \frac{r_+(\beta)}{r_-(\beta)(r_+(\beta) - r_-(\beta))} < 0. \tag{4.23}$$

To determine the remaining unknown constants we apply the principle of smooth fit at the point x_β. Let $\Delta = x_\beta - x_1$. By (4.22) we have

$$\begin{aligned} V'(x_\beta) &= k_1(\beta) r_+(\beta) e^{r_+(\beta)\Delta} + k_2(\beta) r_-(\beta) e^{r_-(\beta)\Delta}, \\ V''(x_\beta) &= k_1(\beta) r_+^2(\beta) e^{r_+(\beta)\Delta} + k_2(\beta) r_-^2(\beta) e^{r_-(\beta)\Delta}. \end{aligned} \tag{4.24}$$

However, the relation (4.11) (recall that $a(x_\beta) = \beta$) yield $V''(x_\beta) = \frac{-\mu V'(x_\beta)}{\sigma^2 \beta}$. This leads to

$$x_\beta - x_1 = \Delta = \frac{1}{r_+(\beta) - r_-(\beta)} \log \left(\frac{\frac{1}{r_-(\beta)} + \frac{\sigma^2 \beta}{\mu}}{\frac{1}{r_+(\beta)} + \frac{\sigma^2 \beta}{\mu}} \right) < 0, \tag{4.25}$$

which determines x_1 and, in turn, determines $V'(x_\beta)$ via (4.24). To proceed, simple but tedious algebraic transformations show that from (4.20) and (4.17) it follows

$$V'(x_\alpha) = V'(x_\beta) \left(\frac{\beta - c}{\alpha - c}\right)^\Gamma. \tag{4.26}$$

As in the previous section this implies the continuity of V at x_β. Finally, the continuity of V at x_α gives rise to

$$k_1(\alpha,\; \beta) = \frac{V'(x_\beta) \left(\frac{\beta - c}{\alpha - c}\right)^\Gamma}{r_+(\alpha) e^{r_+(\alpha) x_\alpha} - r_-(\alpha) e^{r_-(\alpha) x_\alpha}}. \tag{4.27}$$

This enables us to establish the main result of this section.

Theorem 4.7 *Suppose* $\frac{2\delta}{\mu} < \alpha$. *Let* $k_1(\alpha,\; \beta)$, $r_+(\alpha)$, $r_-(\alpha)$, $r_+(\beta)$, $r_-(\beta)$, x_α, x_β, x_1 $k_1(\beta)$, $k_2(\beta)$, $a(x)$, c, Γ, *and* $V'(x_\alpha)$ *be given by (4.27), (4.8), (4.9), (4.16), (4.25), (4.23), (4.15), (4.13), (4.19), and (4.26) respectively. Then*

$$V(x) = \begin{cases} k_1(\alpha,\; \beta) \left(e^{r_+(\alpha)x} - e^{r_-(\alpha)x}\right), & 0 \leq x < x_\alpha, \\ V'(x_\alpha) \frac{\mu a(x) - 2\delta}{2\gamma} \left(\frac{a(x) - c}{\alpha - c}\right)^{-\Gamma}, & x_\alpha \leq x < x_\beta, \\ k_1(\beta) e^{r_+(\beta)(x - x_1)} + k_2(\beta) e^{r_-(\beta)(x - x_1)}, & x_\beta \leq x < x_1, \\ k_1(\beta) + k_2(\beta) + x - x_1, & x \geq x_1 \end{cases} \tag{4.28}$$

is a concave, twice continuously differentiable solution of the HJB equation (2.8).

Proof. As before we need only to show the concavity. To do this consider V'''. From (4.28), we get

$$\begin{array}{ll} V'''(x) = k_1(\alpha,\ \beta)\left(r_+^3(\alpha)e^{r_+(\alpha)x} - r_-^3(\alpha)e^{r_-(\alpha)x}\right) > 0, & 0 \le x < x_\alpha, \\ V'''(x) = \frac{-\mu V''(x)}{\sigma^2 a(x)} + \frac{\mu a'(x)V'(x)}{\sigma^2(a(x))^2} > 0, & x_\alpha \le x < x_\beta, \\ V'''(x) = k_1 r_+^3(\beta)e^{r_+(\beta)(x-x_1)} + k_2(\beta)r_-^3(\beta)e^{r_-(\beta)(x-x_1)} > 0, & x_\beta \le x < x_1. \end{array}$$

Thus $V''(x) < V''(x_1) = 0$ for each $x < x_1$. On the other hand, $V''(x) = 0$ for each $x \ge x_1$. These lead to the concavity of V. □

4.2 Case of $\alpha \le \frac{2\delta}{\mu} < \beta$

Applying Propositions 4.3 and 4.4, we see that in this case $a(0) = \dfrac{2\delta}{\mu} \ge \alpha$ and $a(x) \ge \alpha$ for all $x \ge 0$. Then in the right neighborhood of 0, $\alpha \le a(x) < \beta$. It follows that for ϕ given by (4.3), equation (4.10) holds. Proceeding as in Section 4.1, we see that $a(x)$ satisfies (4.12). Therefore

$$a(x) = G^{-1}\left(\frac{\mu^2 + 2\gamma\sigma^2}{\mu\sigma^2}x + G(2\delta/\mu)\right) \in [2\delta/\mu,\ \ \infty), \tag{4.29}$$

where G is given by (4.14). As a result $a(x)$ increases and $a(x_\beta) = \beta$ where

$$x_\beta = \frac{\mu\sigma^2}{\mu^2 + 2\gamma\sigma^2}[G(\beta) - G(2\delta/\mu)] = \frac{\mu\sigma^2}{\mu^2 + 2\gamma\sigma^2}(\beta - 2\delta/\mu) + \frac{2\delta\mu c}{\mu^2 + 2\gamma\sigma^2}\log(\frac{\beta - c}{2\delta/\mu - c}). \tag{4.30}$$

Integrating (4.4), we get

$$V(x) = V'(0)\frac{\mu a(x) - 2\delta}{2\gamma}\left(\frac{a(x) - c}{2\delta/\mu - c}\right)^{-\Gamma}, \quad 0 \le x < x_\beta. \tag{4.31}$$

By virtue of Proposition 4.5 we have $a(x) \ge \beta$ for $x \ge x_\beta$. Let x_1 be such that $V''(x_1) = 0$. Then for $x_\beta \le x < x_1$,

$$V(x) = k_1(\beta)e^{r_+(\beta)(x-x_1)} + k_2(\beta)e^{r_-(\beta)(x-x_1)}. \tag{4.32}$$

Using the principle of smooth fit for V in (4.32) at x_1, we see that $k_1(\beta)$ and $k_2(\beta)$ are given by (4.23). Put $\Delta = x_\beta - x_1$. Applying the principle of smooth fit at x_β for V' and V'', we deduce that Δ is given by (4.25). Therefore

$$x_1 = x_\beta + \Delta = x_\beta + \frac{1}{r_+(\beta) - r_-(\beta)}\log\left(\frac{\frac{1}{r_-(\beta)} + \frac{\sigma^2\beta}{\mu}}{\frac{1}{r_+(\beta)} + \frac{\sigma^2\beta}{\mu}}\right). \tag{4.33}$$

Theorem 4.8 *Suppose $\alpha \le \frac{2\delta}{\mu} < \beta$. Let $a(x)$, c, Γ, x_β, x_1, $r_+(\beta)$, $r_-(\beta)$, $k_1(\beta)$, and $k_2(\beta)$ be given by (4.29), (4.13), (4.19), (4.30), (4.33), (4.8), and (4.23) respectively. Let $V'(0)$ be determined from (4.26), in which x_α and α are replaced by 0 and $2\delta/\mu$ respectively. Then*

$$V(x) = \begin{cases} V'(0)\frac{\mu a(x)-2\delta}{2\gamma}\left(\frac{a(x)-c}{2\delta/\mu-c}\right)^{-\Gamma}, & 0 \le x < x_\beta, \\ k_1(\beta)e^{r_+(\beta)(x-x_1)} + k_2(\beta)e^{r_-(\beta)(x-x_1)}, & x_\beta \le x < x_1, \\ k_1(\beta) + k_2(\beta) + x - x_1, & x \ge x_1 \end{cases} \tag{4.34}$$

is a concave, twice continuously differentiable solution of the HJB equation (2.8).

Proof. The proof of this theorem is similar to that of Theorem 4.7. (One needs only to repeat the proof of Theorem 4.7, substituting x_α, x_β and x_1 by 0, x_β and x_1 respectively.) □

Remarks 4.9 Denote by $V_{\alpha,\ \beta}(x)$ the concave solution to the HJB equation (2.8) corresponding to the parameters (α, β). Then the results of this section show that

$$V_{\alpha,\ \beta}(x) = V_{\frac{2\delta}{\mu},\ \beta}(x), \quad x \geq 0,$$

for each $\alpha \leq \frac{2\delta}{\mu}$. It is interesting to notice that as $\alpha \to 0+$ and $\beta = 1$, Theorem 4.8 gives us a different expression for V than the one in [15]. However, one can show that those are two different expressions of the same function.

4.3 Case of $\beta \leq \frac{2\delta}{\mu}$

Since in this case, $a(0) \geq \beta$, we can apply Proposition 4.5 to conclude $a(x) \geq \beta, \quad \forall x \geq 0$. Substituting $a = \beta$ in (4.2), we get

$$V(x) = k_1(\beta)e^{r_+(\beta)(x-x_1)} + k_2(\beta)e^{r_-(\beta)(x-x_1)}, \quad 0 \leq x < x_1. \tag{4.35}$$

Using the principle of smooth fit at x_1, we get that $k_1(\beta)$ and $k_2(\beta)$ are given by (4.23). Using the initial condition $V(0) = 0$, we obtain

$$\Delta = -x_1 = \frac{1}{r_+(\beta) - r_-(\beta)} \log\left(\frac{r_+^2(\beta)}{r_-^2(\beta)}\right). \tag{4.36}$$

Note that the expression on the right hand side of (4.36) is negative if and only if $|r_+(\beta)| < |r_-(\beta)|$. The later is true iff

$$\frac{\delta}{\mu} < \beta; \tag{4.37}$$

see (4.8).

Theorem 4.10 *Suppose $\frac{\delta}{\mu} < \beta \leq \frac{2\delta}{\mu}$. Let $k_1(\beta)$, $k_2(\beta)$, $r_+(\beta)$, $r_-(\beta)$, and x_1 be given by (4.23), (4.8), and (4.36) respectively. Then*

$$V(x) = \begin{cases} k_1(\beta)e^{r_+(\beta)(x-x_1)} + k_2(\beta)e^{r_-(\beta)(x-x_1)}, & 0 \leq x < x_1, \\ k_1(\beta) + k_2(\beta) + x - x_1, & x \geq x_1 \end{cases} \tag{4.38}$$

is a concave, twice continuously differentiable solution of the HJB equation (2.8).

Proof. The proof of this theorem follows the lines of the proof of Theorem 4.8, in which one replaces x_β and x_1 by 0 and x_1 respectively. □

Remark 4.11 The case when (4.37) fails is a trivial case; see Theorem 4.1.

Remark 4.12 From the expressions for $V_{\alpha,\ \beta}$ obtained in this subsection and the previous subsection, one can see that

$$\lim_{\beta\to\infty} V_{\alpha,\ \beta}(x) = \infty, \quad \lim_{\beta\to 0} V_{\alpha,\ \beta}(x) = 0, \quad \forall x > 0.$$

5 Case of bounded dividend rate with nonzero liability

When the pay-out rate of dividends is bounded, the functional C_t^π in (2.1) is absolutely continuous with respect to t and the dynamics of the reserve process can be represented as

$$dR_t^\pi = (a_t^\pi \mu - \delta)dt + a_t^\pi \sigma dW_t - c_t^\pi dt, \quad R_0^\pi = x, \tag{5.1}$$

where $a^\pi = (a_t^\pi; t \geq 0)$ and $c^\pi = (c_t^\pi; t \geq 0)$ are two $\mathcal{F}_t$-adapted processes such that $\alpha \leq a_t^\pi \leq \beta$ and $0 \leq c_t^\pi \leq M$, $\forall t \geq 0$, P-a.s, with $M > 0$ being a given bound on dividend rate. Thus, instead of the control process $\pi = (a^\pi, C^\pi; t \geq 0)$ we can consider $\pi = (a^\pi, c^\pi; t \geq 0)$, which we will call hereafter an admissible control. So we are now in the realm of the regular control.
The optimal return function v can be defined as

$$v(x) = \sup_{\pi \in \mathcal{A}} E_x \left(\int_0^{\tau^\pi} e^{-\gamma t} c_t^\pi dt \right), \tag{5.2}$$

where $\mathcal{A}$ stands for the set of all admissible controls.

Proposition 5.1 *The optimal return function v is a concave, non-decreasing function subject to $v(0) = 0$ and*

$$0 \leq v(x) \leq \frac{M}{\gamma}, \quad \forall x > 0. \tag{5.3}$$

Proof. The concavity and the monotonicity as well as the boundary condition, $v(0) = 0$, are standard and can be found in [2], [8] and [15]. To show (5.3), consider

$$0 \leq E\left(\int_0^{\tau^\pi} e^{-\gamma t} c_t^\pi dt \right) \leq M \int_0^\infty e^{-\gamma t} dt = \frac{M}{\gamma}.$$

□

If the optimal return function v is twice continuously differentiable, then it must be a solution to the following HJB equation

$$\begin{aligned} 0 = \;& \max_{\alpha \leq a \leq \beta, 0 \leq c \leq M} \left(\tfrac{1}{2}\sigma^2 a^2 V''(x) + (a\mu - \delta - c)V'(x) - \gamma V(x) + c \right) \\ \equiv \;& \max_{\alpha \leq a \leq \beta} \left(\tfrac{1}{2}\sigma^2 a^2 V''(x) + (a\mu - \delta)V'(x) - \gamma V(x) + M\left(1 - V'(x)\right)^+ \right), \\ 0 = \;& V(0), \end{aligned} \tag{5.4}$$

where $x^+ = \max(x, \ 0)$. This equation is rather standard and its derivation can be found in [6], [5], [16]; see also [8] and [9].

As before, we are looking for a concave, smooth solution to (5.4). Assume that such a solution, V, has been found. Let

$$x_1 = \inf\{x \geq 0 \ : \ V'(x) \leq 1\}. \tag{5.5}$$

Then for $0 \leq x < x_1$, (5.4) becomes

$$0 = \max_{\alpha \leq a \leq \beta} \left(\frac{1}{2}\sigma^2 a^2 V''(x) + (a\mu - \delta)V'(x) - \gamma V(x) \right), \tag{5.6}$$

while for $x \geq x_1$, (5.4) can be rewritten as

$$0 = \max_{\alpha \leq a \leq \beta} \left(\frac{1}{2}\sigma^2 a^2 V''(x) + (a\mu - \delta - M)V'(x) - \gamma V(x) + M \right). \tag{5.7}$$

We start with seeking a smooth solution to (5.7). Obviously if $V'(0) \leq 1$, then $x_1 = 0$ and (5.4) is equivalent to (5.7) for all $x \geq 0$.

Proposition 5.2 *If $\beta\mu \leq \delta$, then $V'(0) < 1$.*

Proof. It follows from (5.4) that there exits $\tilde{a} \in [\alpha,\ \beta]$ such that

$$0 = \frac{1}{2}\sigma^2\tilde{a}^2V''(0) + (\tilde{a}\mu - \delta)V'(0) + M\left(1 - V'(0)\right)^+. \tag{5.8}$$

If $\beta\mu < \delta$, then each of the first two terms on the right hand side of (5.8) is non-positive with the second being strictly negative. Therefore $M\left(1 - V'(0)\right)^+ > 0$, which implies $V'(0) < 1$. The same argument goes if $\beta\mu = \delta$ and $V''(0) < 0$. In this case either the first or the second term on the right hand side of (5.8) is strictly negative. If $\beta\mu = \delta$ and $V''(0) = 0$, then the maximizer of the right hand side of (5.4) is equal to β for all x in a right neighborhood of 0 (recall that $V'(0) > 0$). Substituting $a = \beta$ either into (5.6) or into (5.7) and solving the resulting linear ODE with constant coefficients we get a function V whose second derivative at 0 does not vanish, which is a contradiction. □

Remark 5.3 When the dividend rates are unrestricted, the condition $\beta\mu \leq \delta$ makes the problem trivial (see Theorem 4.1). This is not the case when the dividend rates are bounded. Even if $\beta\mu \leq \delta$ the second derivative of V at 0 is strictly negative which makes the problem nontrivial in contrast to a similar situation in the case of unrestricted dividends.

Now we analyze the solution to (5.7) under the condition $\beta\mu > \delta$. As we will see later the qualitative nature of this solution depends on whether $a(x_1) < \alpha$ or $\alpha \leq a(x_1) < \beta$ or $a(x_1) \geq \beta$, where $a(x)$ is defined by (3.3). Let

$$\tilde{r}_+(z) \equiv \frac{-(z\mu - \delta - M) + \sqrt{(z\mu - \delta - M)^2 + 2\gamma\sigma^2 z^2}}{z^2\sigma^2}, \tag{5.9}$$

$$\tilde{r}_-(z) \equiv \frac{-(z\mu - \delta - M) - \sqrt{(z\mu - \delta - M)^2 + 2\gamma\sigma^2 z^2}}{z^2\sigma^2}, \qquad z > 0. \tag{5.10}$$

First suppose that $a(x_1) \geq \beta$. Using the same arguments as in the proof of Proposition 4.5, we deduce that $a(x) \geq \beta$ for each $x \geq x_1$. Substituting $a = \beta$ into (5.7) and solving the resulting equation, we get

$$V(x) = \frac{M}{\gamma} + K_\beta e^{\tilde{r}_-(\beta)(x - x_1)}, \qquad \forall x \geq x_1.$$

Here K_β is a constant which takes on either the value $-\frac{M}{\gamma}$ or $\frac{1}{\tilde{r}_-(\beta)}$depending respectively on the fact whether x_1 in (5.5) is zero or not. Then straightforward calculations show that $a(x) = \dfrac{-\mu}{\sigma^2\tilde{r}_-(\beta)}$. Thus, the condition $a(x_1) \geq \beta$ is equivalent to

$$\tilde{c} \equiv \frac{2\mu(\delta + M)}{\mu^2 + 2\gamma\sigma^2} \geq \beta. \tag{5.11}$$

Next suppose $\alpha \leq a(x_1) < \beta$. The same arguments as in the proof of Proposition 4.4 show that $a(x) \geq \alpha$ for all $x \geq x_1$. As a result, $\alpha \leq a(x) < \beta$ in a right neighborhood of x_1. Substituting $V''(x) = -\frac{\mu V'(x)}{\sigma^2 a(x)}$ into (5.7), we get

$$0 = \frac{\mu a(x)}{2}V'(x) - (\delta + M)V'(x) - \gamma V(x) + M. \tag{5.12}$$

Differentiating (5.12) and again substituting $V''(x) = -\frac{\mu V'(x)}{\sigma^2 a(x)}$ into the resulting equation, we obtain

$$a(x)a'(x) = (a(x) - \tilde{c})\frac{\mu^2 + 2\sigma^2\gamma}{\mu\sigma^2}. \tag{5.13}$$

Suppose there exists $x' \geq x_1$ such that $a(x') < \tilde{c}$ (respectively $a(x') > \tilde{c}$), then from (5.13) we deduce that $a(x) < \tilde{c}$ (respectively $a(x) > \tilde{c}$), for each $x \geq x'$. Thus, by integrating (5.13) we obtain

$$a(x) - a(x') + \tilde{c}\,\log\left(\frac{a(x) - \tilde{c}}{a(x') - \tilde{c}}\right) = (x - x')\frac{\mu^2 + 2\sigma^2\gamma}{\mu\sigma^2}, \qquad \forall x \geq x'. \tag{5.14}$$

From (5.12) and (5.3) we see that $a(x) \leq \frac{2(\delta+M)}{\mu}$, $\forall x \geq x_1$. Therefore, the left hand side of (5.14) is bounded. This is a contradiction and we conclude that $a(x) = \tilde{c}$, for each $x \geq x_1$. In view of the above, the condition $\alpha \leq a(x_1) < \beta$ can be rewritten as $\alpha \leq \tilde{c} < \beta$. Now, substituting $a = \tilde{c}$ into (5.7) and solving the resulting equation (noting that $\tilde{r}_-(\tilde{c}) = -\frac{\sigma^2\tilde{c}}{\mu}$), we get

$$V(x) = \frac{M}{\gamma} + \tilde{K}e^{-\frac{\sigma^2\tilde{c}}{\mu}(x-x_1)}, \quad \forall x \geq x_1,$$

where $\tilde{K}$ is a constant which takes on either the value of $-\frac{M}{\gamma}$ or $-\frac{\mu}{\sigma^2\tilde{c}}$ depending respectively on whether $x_1 = 0$ or $x_1 > 0$.

Finally, suppose that $a(x_1) < \alpha$. Then it follows from the above that $\tilde{c} < \alpha$. Therefore $a(x) < \alpha$ for all x in a right neighborhood of x_1. Substituting $a = \alpha$ into (5.7) and solving the resulting linear differential equation, we get

$$V(x) = \frac{M}{\gamma} + K_1(\alpha)e^{\tilde{r}_+(\alpha)(x-x_1)} + K_2(\alpha)e^{\tilde{r}_-(\alpha)(x-x_1)}, \tag{5.15}$$

where $K_1(\alpha)$ and $K_2(\alpha)$ are free constants. If $K_1(\alpha) > 0$, then the right hand side of (5.15) is unbounded on $[x_1, \infty)$, which contradicts (5.3). If $K_1(\alpha) < 0$, then the right hand side of (5.15) becomes negative for x large enough, which again is a contradiction. Hence $K_1(\alpha) = 0$. On the other hand, we have $K_2(\alpha) < 0$ in view of $V''(0) < 0$. Therefore

$$V(x) = \frac{M}{\gamma} + K_\alpha e^{\tilde{r}_-(\alpha)(x-x_1)},$$

where K_α is a constant that takes on either the value $-\frac{M}{\gamma}$ or $\frac{1}{\tilde{r}_-(\alpha)}$ depending on whether x_1 is zero or not. Combining the above results, we can formulate the following theorem.

Theorem 5.4 *Let $\tilde{r}_-(\alpha)$, $\tilde{r}_-(\beta)$ and $\tilde{c}$ be the constants given by (5.10) and (5.11) respectively. Let x_1 be defined by (5.5). Then for $x_1 = 0$ (respectively for $x_1 > 0$) the following assertions hold.*
(i) If $\tilde{c} \geq \beta$, then

$$V(x) = \frac{M}{\gamma} + K_\beta e^{\tilde{r}_-(\beta)(x-x_1)}, \quad x \geq x_1 \tag{5.16}$$

is a concave, twice differentiable solution of the HJB equation (5.7) on $[x_1, \infty)$, where K_β is equal to $-\frac{M}{\gamma}$ (respectively to $\frac{1}{\tilde{r}_-(\beta)}$).
(ii) If $\alpha \leq \tilde{c} < \beta$, then

$$V(x) = \frac{M}{\gamma} + \tilde{K}e^{\frac{-\mu}{\sigma^2\tilde{c}}(x-x_1)}, \quad x \geq x_1 \tag{5.17}$$

is a concave, twice differentiable solution of the HJB equation (5.7) on $[x_1, \infty)$, where $\tilde{K}$ is equal to $-\frac{M}{\gamma}$ (respectively to $-\frac{\mu}{\sigma^2\tilde{c}}$).

(iii) If $\tilde{c} < \alpha$, then

$$V(x) = \frac{M}{\gamma} + K_\alpha e^{\tilde{r}_-(\alpha)(x-x_1)}, \quad x \geq x_1 \tag{5.18}$$

is a concave, twice differentiable solution of the HJB equation (5.7) on $[x_1, \infty)$, where K_α a constant equal to $-\frac{M}{\gamma}$ (respectively to $\frac{1}{\tilde{r}_-(\alpha)}$).

Corollary 5.5 *If $x_1 = 0$, then the solution to (5.4) subject to (5.3) is given by*

$$V(x) = \begin{cases} \frac{M}{\gamma}\left(1 - e^{\tilde{r}_-(\beta)x}\right), & \text{if } \tilde{c} \geq \beta, \\ \frac{M}{\gamma}\left(1 - e^{\frac{-\mu}{\sigma^2 \tilde{c}}x}\right), & \text{if } \alpha \leq \tilde{c} < \beta, \quad \forall x \geq 0. \\ \frac{M}{\gamma}\left(1 - e^{\tilde{r}_-(\alpha)x}\right), & \text{if } \tilde{c} < \alpha, \end{cases}$$

Corollary 5.5 shows that the qualitative nature of the solution depends on the relation between $\frac{2\delta}{\mu}$, α and β. Accordingly, we will consider three cases. However, in contrast to the situation with the unbounded dividend rates, each case here will consist of several subcases, each subcase being associated with a different range for the value of M. In the previous section, it was also shown that the qualitative nature of the solutions to (5.6) depends on $a(0)$; see Proposition 4.3. Thus in the spirit of this proposition, we will distinguish the cases as follows.

5.1 Case of $\frac{2\delta}{\mu} < \alpha$

Let $\tilde{c}$ be given by (5.11) and

$$M_z = \left(z - \frac{2\delta\mu}{\mu^2 + 2\sigma^2\gamma}\right)\frac{\mu^2 + 2\sigma^2\gamma}{2\mu}, \quad z > 0. \tag{5.19}$$

Then, $\tilde{c} \geq \beta$ (respectively $\tilde{c} = \beta$) is equivalent to $M \geq M_\beta$ (respectively $M = M_\beta$). This is the first subcase we will consider.

5.1.1 Case of $M > M_\beta$

Our assumptions imply that in this case, $a(x_1) > \beta$, which is equivalent to $x_1 > x_\beta > 0$. Combining the results of Section 4 and Theorem 5.4-(i) we can write a general form of the solution to (5.4) and (5.3):

$$V(x) = \begin{cases} K_1(\alpha,\ \beta)\left(e^{r_+(\alpha)x} - e^{r_-(\alpha)x}\right), & 0 \leq x < x_\alpha, \\ V'(x_\alpha)\frac{\mu a(x) - 2\delta}{2\gamma}\left(\frac{a(x)-c}{\alpha - c}\right)^{-\Gamma}, & x_\alpha \leq x < x_\beta, \\ K_1(\beta)e^{r_+(\beta)(x-x_1)} + K_2(\beta)e^{r_-(\beta)(x-x_1)}, & x_\beta \leq x < x_1, \\ \frac{M}{\gamma} + \frac{1}{\tilde{r}_-(\beta)}e^{\tilde{r}_-(\beta)(x-x_1)}, & x \geq x_1, \end{cases} \tag{5.20}$$

where $r_+(\alpha)$, $r_-(\alpha)$, $r_+(\beta)$ and $r_-(\beta)$, x_α and x_β are given by (4.8), (4.9) and (4.16) respectively, and $K_1(\beta)$, $K_2(\beta)$, $K_1(\alpha, \beta)$ and x_1 are unknown constants to be determined. Continuity of the first and the second derivatives at x_1 results in

$$V'(x_1) = 1, \quad V''(x_1) = \tilde{r}_-(\beta).$$

This gives us two equations

$$1 = K_1(\beta)r_+(\beta) + K_2(\beta)r_-(\beta), \ \tilde{r}_-(\beta) = K_1(\beta)r_+^2(\beta) + K_2(\beta)r_-^2(\beta),$$

whose solutions are

$$K_1(\beta) = \frac{\tilde{r}_-(\beta) - r_-(\beta)}{r_+(\beta)\left(r_+(\beta) - r_-(\beta)\right)}, \quad K_2(\beta) = \frac{r_+(\beta) - \tilde{r}_-(\beta)}{r_-(\beta)\left(r_+(\beta) - r_-(\beta)\right)}. \tag{5.21}$$

Put $\Delta = x_\beta - x_1$. As before, using the principle of smooth fit at x_β, we get

$$x_\beta - x_1 = \Delta = \frac{1}{\left(r_+(\beta) - r_-(\beta)\right)} \log\left(-\frac{\left(r_+(\beta) - \tilde{r}_-(\beta)\right)\left(\mu + \beta\sigma^2 r_-(\beta)\right)}{\left(\tilde{r}_-(\beta) - r_-(\beta)\right)\left(\mu + \beta\sigma^2 r_+(\beta)\right)}\right). \tag{5.22}$$

The expression on the right hand side of (5.22) is negative due to $\tilde{c} > \beta$. In view of (5.22) and (5.21) we can derive a simplified expression for $V'(x_\beta)$:

$$V'(x_\beta) = \beta\sigma^2 e^{r_-(\beta)\Delta} \frac{r_+(\beta) - \tilde{r}_-(\beta)}{\mu + \beta\sigma^2 r_+(\beta)}.$$

The principle of smooth fit for V' at x_α yields $V'(x_\alpha) = K_1(\alpha,\ \beta)\left(r_+(\alpha)e^{r_+(\alpha)x_\alpha} - r_-(\alpha)e^{r_-(\alpha)x_\alpha}\right)$. Combining this equality with (4.26), we get

$$K_1(\alpha,\ \beta) = \frac{V'(x_\beta)\left(\frac{\beta-c}{\alpha-c}\right)^\Gamma}{r_+(\alpha)e^{r_+(\alpha)x_\alpha} - r_-(\alpha)e^{r_-(\alpha)x_\alpha}}. \tag{5.23}$$

Theorem 5.6 *Let $a(x)$ be a function given by (4.15) and $r_+(\alpha)$, $r_-(\alpha)$, $r_+(\beta)$, $r_-(\beta)$, x_α, x_β, c, Γ, $\tilde{r}_-(\beta)$, $K_1(\beta)$, $K_2(\beta)$, $K_1(\alpha,\ \beta)$, and x_1 be given by (4.8), (4.9), (4.16), (4.13), (4.19), (5.10), (5.21), (5.23), and (5.22) respectively. If $\frac{2\delta}{\mu} < \alpha$ and $M > M_\beta$, then V given by (5.20) is a concave, twice differentiable solution of the HJB equation (5.4), subject to (5.3).*

Proof. The proof of this theorem results from Theorem 4.7 and Theorem 5.4-(i). □

5.1.2 Case of $M_\alpha < M \leq M_\beta$

Expression (5.19) shows that $\beta \geq \tilde{c} > \alpha$ if and only if $M_\beta \geq M > M_\alpha$. From (4.15) we see that the condition $\beta \geq \tilde{c} > \alpha$ is equivalent to $\beta \geq a(x_1) > \alpha$. In view of Theorem 5.4, $a(x) \leq a(x_1) \leq \beta$, for all $x \geq 0$. This also implies $V'(x_1) = 1$. As a result $\tilde{K} = -\frac{\sigma^2\tilde{c}}{\mu}$. Taking into account, (5.17) we can write the expression of V as follows

$$V(x) = \begin{cases} K_1(\alpha,\ \beta)\left(e^{r_+(\alpha)x} - e^{r_-(\alpha)x}\right), & 0 \leq x < x_\alpha, \\ V'(x_\alpha)\frac{\mu a(x) - 2\delta}{2\gamma}\left(\frac{a(x)-c}{\frac{2\delta}{\mu}-c}\right)^{-\Gamma}, & x_\alpha \leq x < x_1, \\ \frac{M}{\gamma} - \frac{\sigma^2\tilde{c}}{\mu}e^{-\frac{\mu}{\sigma^2\tilde{c}}(x-x_1)}, & x \geq x_1. \end{cases}$$

For $a(x)$ given by (4.15), the root of the equation $a(x_1) = \tilde{c}$ can be written as

$$x_1 = \frac{\mu\sigma^2}{\mu^2 + 2\sigma^2\gamma}\int_{\frac{2\delta}{\mu}}^{\tilde{c}} \frac{u\,du}{u-c} = \frac{\mu\sigma^2(\tilde{c} - \frac{2\delta}{\mu})}{\mu^2 + 2\sigma^2\gamma} + \frac{\mu\sigma^2}{\mu^2 + 2\sigma^2\gamma}\log\left(\frac{\tilde{c}-c}{\frac{2\delta}{\mu} - c}\right). \tag{5.24}$$

The principle of smooth fit for V' at x_1 leads to $V'(x_\alpha) = \left(\frac{\tilde{c}-c}{\frac{2\delta}{\mu}-c}\right)^\Gamma$. Consequently

$$K_1(\alpha,\ \beta) = \frac{\left(\frac{\tilde{c}-c}{\frac{2\delta}{\mu}-c}\right)^\Gamma}{r_+(\alpha)e^{r_+(\alpha)x_\alpha} - r_-(\alpha)e^{r_-(\alpha)x_\alpha}}. \tag{5.25}$$

Theorem 5.7 *Let $a(x)$ be a function given by (4.15) and $r_+(\alpha)$, $r_-(\alpha)$, $K_1(\alpha,\ \beta)$, x_α, x_1, c, Γ, and $\tilde{c}$ are given by (4.8), (5.25), (4.9), (5.24), (4.13), (4.19), and (5.11) respectively. If $\frac{2\delta}{\mu} < \alpha$ and $M_\alpha < M \le M_\beta$, then*

$$V(x) = \begin{cases} K_1(\alpha,\ \beta)\left(e^{r_+(\alpha)x} - e^{r_-(\alpha)x}\right), & 0 \le x < x_\alpha, \\ \frac{\mu a(x) - 2\delta}{2\gamma}\left(\frac{a(x)-c}{\tilde{c}-c}\right)^{-\Gamma}, & x_\alpha \le x < x_1, \\ \frac{M}{\gamma} - \frac{\sigma^2 \tilde{c}}{\mu} e^{-\frac{\mu}{\sigma^2 \tilde{c}}(x - x_1)}, & x \ge x_1 \end{cases} \tag{5.26}$$

is a concave, twice differentiable solution of the HJB equation (5.4) subject to (5.3).

Proof. The proof of this theorem follows from combining Theorem 4.7 and Theorem 5.4-(ii). □

Now suppose that $M \le M_\alpha$. Then $a(x) \le a(x_1) \le \alpha$ for each $x \ge 0$ (since $a(x)$ is increasing on $[0,\ x_1)$ and is constant for $x \ge x_1$; see Theorem 5.4). If $V'(0) > 1$, then $x_1 > 0$ and $V'(x_1) = 1$. As a result $\tilde{K}_\alpha = \frac{1}{\tilde{r}_-(\alpha)}$. In view of (5.18), the function V is given by

$$V(x) = \begin{cases} k_1(\alpha,\ \beta)\left(e^{r_+(\alpha)x} - e^{r_-(\alpha)x}\right), & 0 \le x < x_1, \\ \frac{M}{\gamma} + \frac{1}{\tilde{r}_-(\alpha)} e^{\tilde{r}_-(\alpha)(x - x_1)}, & x \ge x_1. \end{cases} \tag{5.27}$$

The smoothness of V requires

$$V'(x_1-) = 1, \ \ V''(x_1-) = \tilde{r}_-(\alpha),$$

which translates into

$$\begin{aligned} k_1(\alpha,\ \beta)\left(r_+(\alpha)e^{r_+(\alpha)x_1} - r_-(\alpha)e^{r_-(\alpha)x_1}\right) &= 1, \\ k_1(\alpha,\ \beta)\left(r_+^2(\alpha)e^{r_+(\alpha)x_1} - r_-^2(\alpha)e^{r_-(\alpha)x_1}\right) &= \tilde{r}_-(\alpha). \end{aligned} \tag{5.28}$$

Excluding $k_1(\alpha, \beta)$, we get an equation for x_1

$$e^{(r_+(\alpha) - r_-(\alpha))x_1} = \frac{r_-(\alpha)\left(r_-(\alpha) - \tilde{r}_-(\alpha)\right)}{r_+(\alpha)\left(r_+(\alpha) - \tilde{r}_-(\alpha)\right)}. \tag{5.29}$$

This equation has a positive solution if and only if

$$M > M_0(\alpha) \equiv \frac{\alpha^2 \sigma^2 \gamma}{2\left(\alpha\mu - \delta\right)}. \tag{5.30}$$

This proves the following

Proposition 5.8 *If $\frac{2\delta}{\mu} < \alpha$, then*

$$V'(0) > 1 \ \ \textit{iff} \ \ M > \frac{\alpha^2 \sigma^2 \gamma}{2(\alpha\mu - \delta)}.$$

Let M_z be given by (5.19) and

$$M_0(z) \equiv \frac{z^2 \sigma^2 \gamma}{2\left(z\mu - \delta\right)}, \ \ \forall z > \frac{\delta}{\mu}.$$

A simple analysis shows that $f(z) \equiv M_0(z) - M_z$ is a decreasing function of z and $f(\frac{2\delta}{\mu}) = 0$. Similarly, we claim that $M_0(z)$ is decreasing for $z \leq \frac{2\delta}{\mu}$ and increasing for $z \geq \frac{2\delta}{\mu}$. Thus, we derive the following inequalities

$$M_0(\frac{2\delta}{\mu}) < M_0(\alpha) < M_\alpha < M_\beta, \quad \text{if} \quad \frac{2\delta}{\mu} < \alpha. \tag{5.31}$$

$$M_\alpha \leq M_0(\alpha) \leq M_0(\frac{2\delta}{\mu}) < M_\beta, \quad \text{if} \quad \alpha \leq \frac{2\delta}{\mu} < \beta. \tag{5.32}$$

$$M_\alpha < M_\beta \leq M_0(\frac{2\delta}{\mu}) \leq M_0(\beta) < M_0(\alpha), \quad \text{if} \quad \beta \leq \frac{2\delta}{\mu}. \tag{5.33}$$

Since the qualitative behavior of the solution to (5.6) (respectively to (5.7)) depends on the value of $a(0)$ (respectively of $a(x_1)$), in accordance with (5.31) we will distinguish and study the remaining subcases in the following subsections.

5.1.3 Case of $M_0(\alpha) < M \leq M_\alpha$

This is the case when (5.29) has a positive solution x_1 given by

$$x_1 = \frac{1}{r_+(\alpha) - r_-(\alpha)} \log \left(\frac{r_-(\alpha)\,(r_-(\alpha) - \tilde{r}_-(\alpha))}{r_+(\alpha)\,(r_+(\alpha) - \tilde{r}_-(\alpha))} \right). \tag{5.34}$$

Theorem 5.9 *Let $r_+(\alpha)$, $r_-(\alpha)$, $\tilde{r}_-(\alpha)$ and x_1 are given by (4.8), (5.10) and (5.34) respectively and $k_1(\alpha,\ \beta)$ be determined by (5.28). If $\frac{2\delta}{\mu} < \alpha$ and $M_0(\alpha) < M \leq M_\alpha$, then V given by (5.27) is a concave, twice continuously differentiable solution of (5.4) subject to (5.3).*

Proof. The proof follows from combining Theorem 4.7 and Theorem 5.4-(iii). □

5.1.4 Case of $M \leq M_0(\alpha)$

By virtue of Proposition 5.8, this assumption results in $V'(0) \leq 1$. As a consequence, $x_1 = 0$. As shown in Theorem 5.4, this leads to $a(x) = a(0)$ for each $x \geq 0$. Since $M_\alpha > M_0(\alpha)$, we can apply Corollary 5.5 to deduce $a(0) < \alpha$.

Theorem 5.10 *Let $\tilde{r}_-(\alpha)$ be a constant given by (5.10). If $\frac{2\delta}{\mu} < \alpha$ and $M \leq M_0(\alpha)$, then*

$$V(x) = \frac{M}{\gamma}\left(1 - e^{\tilde{r}_-(\alpha)x}\right), \qquad x \geq 0, \tag{5.35}$$

is a concave, twice continuously differentiable solution of (5.4) subject to (5.3).

Proof. See Corollary 5.5. □

5.2 Case of $\alpha \leq \frac{2\delta}{\mu} < \beta$

In this subsection, we will investigate the second main case of $\alpha \leq a(0) < \beta$.

5.2.1 Case of $M > M_\beta$

As in Subsection 5.1.1, consider the case of $x_1 > x_\beta$. This case is characterized by $M > M_\beta$, which is also equivalent to $\tilde{c} > \beta$. As a result, we get $V'(x_1) = 1$. This leads to $\tilde{K}_\beta = \frac{1}{\tilde{r}_-(\beta)}$ (see Theorem 5.4-(i)). Then using (4.31), (4.32) and Theorem 5.4-(i), V can be represented in the following form

$$V(x) = \begin{cases} V'(0)\frac{\mu a(x)-2\delta}{2\gamma}\left(\frac{a(x)-c}{\frac{2\delta}{\mu}-c}\right)^{-\Gamma}, & 0 \leq x < x_\beta, \\ K_1(\beta)e^{r_+(\beta)(x-x_1)} + K_2(\beta)e^{r_-(\beta)(x-x_1)}, & x_\beta \leq x < x_1, \\ \frac{M}{\gamma} + \frac{1}{\tilde{r}_-(\beta)}e^{\tilde{r}_-(\beta)(x-x_1)}, & x \geq x_1, \end{cases} \tag{5.36}$$

where $a(x)$, $K_1(\beta)$, $K_2(\beta)$ and x_1 are given by (4.29), (5.21) and (5.22) respectively. Let $\Delta = x_\beta - x_1$ be given by (4.16). Continuity of V at x_β yields

$$V'(0) = \frac{\frac{2\gamma}{\mu\beta-2\delta}\left(\frac{\beta-c}{\frac{2\delta}{\mu}-c}\right)^{\Gamma}}{K_1(\beta)e^{r_+(\beta)\Delta} + K_2(\beta)e^{r_-(\beta)\Delta}}. \tag{5.37}$$

Theorem 5.11 *Let $V'(0)$, $a(x)$, c, Γ, $K_1(\beta)$, $K_2(\beta)$, x_1, and $\tilde{c}$ are given by (5.37), (4.29), (4.13), (4.19), (5.21), (5.22) and (5.11) respectively. If $\alpha \leq \frac{2\delta}{\mu} < \beta$ and $M \geq M_\beta$, then $V(x)$ given by (5.36) is a concave, twice continuously differentiable solution of (5.4) subject to (5.3).*

Proof. The proof results from combining Theorem 4.8 and Theorem 5.4-(ii). □

To classify the remaining cases, suppose $M \leq M_\beta$. Let $V'(0) > 1$. In this case x_1 defined by (5.5) is positive. Therefore, using (4.31) and Theorem 5.4, we can represent V as

$$V(x) = \begin{cases} V'(0)\frac{\mu a(x)-2\delta}{2\gamma}\left(\frac{a(x)-c}{\frac{2\delta}{\mu}-c}\right)^{-\Gamma}, & 0 \leq x < x_1, \\ V(x_1) + \frac{\sigma^2\tilde{c}}{\mu} - \frac{\sigma^2\tilde{c}}{\mu}e^{-\frac{\mu}{\sigma^2\tilde{c}}(x-x_1)}, & x \geq x_1, \end{cases} \tag{5.38}$$

where $a(x)$ and $\tilde{c}$ are given by (4.29) and (5.11) respectively. As a consequence, we get

$$a(x_1) = \tilde{c}. \tag{5.39}$$

Continuity of $V'(x)$ at $x = x_1$ (see (4.12) for the expression of the derivatives of $a(x)$) along with (5.39) results in

$$V'(0) = \left(\frac{\tilde{c} - c}{\frac{2\delta}{\mu} - c}\right)^{\Gamma}. \tag{5.40}$$

Substituting (5.40), (5.39) and (5.19) into (5.38), we obtain

$$V(x_1) = \frac{M}{\gamma} - \frac{\sigma^2\tilde{c}}{\mu}. \tag{5.41}$$

The unknown constant x_1 is the root of the equation (5.39). Recalling (4.29), we see that (5.39) admits a positive solution if and only if

$$M > M_0(\frac{2\delta}{\mu}) = \frac{2\delta^2\sigma^2\gamma}{\mu^2}. \tag{5.42}$$

Proposition 5.12 *Suppose $\alpha \leq \frac{2\delta}{\mu} < \beta$. Then*

$$V'(0) > 1 \quad \textit{iff} \quad M > M_0(\frac{2\delta}{\mu}).$$

In view of this proposition, we distinguish the remaining subcases as follows.

5.2.2 Case of $M_0(\frac{2\delta}{\mu}) < M \le M_\beta$

Substituting (4.29) into (5.39), we obtain

$$x_1 = \frac{\mu\sigma^2}{\mu^2+2\sigma^2\gamma}\int_{\frac{2\delta}{\mu}}^{\tilde{c}} \frac{udu}{u-c} = \frac{\mu\sigma^2(\tilde{c}-\frac{2\delta}{\mu})}{\mu^2+2\sigma^2\gamma} + \frac{\mu\sigma^2}{\mu^2+2\sigma^2\gamma}\log\left(\frac{\tilde{c}-c}{\frac{2\delta}{\mu}-c}\right). \tag{5.43}$$

Theorem 5.13 *Let $a(x)$, c, Γ, $\tilde{c}$, and x_1 be given by (4.29), (4.13), (4.19), (5.11), and (5.43) respectively. If $\alpha \le \frac{2\delta}{\mu} < \beta$ and $M_0(\frac{2\delta}{\mu}) < M \le M_\beta$, then*

$$V(x) = \begin{cases} \frac{\mu a(x)-2\delta}{2\gamma}\left(\frac{a(x)-c}{\tilde{c}-c}\right)^{-\Gamma}, & 0 \le x < x_1, \\ \frac{M}{\gamma} - \frac{\sigma^2\tilde{c}}{\mu}e^{-\frac{\mu}{\sigma^2\tilde{c}}(x-x_1)}, & x \ge x_1 \end{cases} \tag{5.44}$$

is a concave, twice continuously differentiable solution of (5.4) subject to (5.3).

Proof. The proof of this theorem is obtained by combining Theorem 4.8, Theorem 5.4-(ii) and formula (5.41). □

5.2.3 Case of $M \le M_0(\frac{2\delta}{\mu})$

Note that in this case, $V'(0) \le 1$ due to Proposition 5.12. From (5.32), it follows that $a(0) = \tilde{c} < \beta$.

Theorem 5.14 *Suppose $\alpha \le \frac{2\delta}{\mu} < b$ and $M \le M_0(\frac{2\delta}{\mu})$. Let $\tilde{c}$ and $\tilde{r}_-(\alpha)$ be given by (5.11) and (5.10) respectively.*
(i) If $\alpha < \frac{2\delta}{\mu}$ and $\alpha \le \tilde{c}$, then

$$V(x) = \frac{M}{\gamma}\left(1-e^{-\frac{\mu}{\sigma^2\tilde{c}}x}\right), \quad x \ge 0 \tag{5.45}$$

is a concave, twice continuously differentiable solution of (5.4) subject to (5.3).
(ii) If $\alpha = \frac{2\delta}{\mu}$ or $\tilde{c} < \alpha$, then

$$V(x) = \frac{M}{\gamma}\left(1-e^{\tilde{r}_-(\alpha)x}\right), x \ge 0 \tag{5.46}$$

is a concave, twice continuously differentiable solution of (5.4) subject to (5.3).

Proof. See Corollary 5.5. □

5.3 Case of $\frac{\delta}{\mu} < \beta \le \frac{2\delta}{\mu}$

We now investigate the final main case, $\frac{\delta}{\mu} < \beta \le \frac{2\delta}{\mu}$. Suppose $V'(0) > 1$. Then x_1 defined by (5.5) is positive. Therefore, using (4.35), (5.41) and Theorem 5.4-(i), V can be represented as follows

$$V(x) = \begin{cases} K\left(e^{r_+(\beta)x} - e^{r_-(\beta)x}\right), & 0 \le x < x_1, \\ \frac{M}{\gamma} + \frac{1}{\tilde{r}_-(\beta)}e^{\tilde{r}_-(\beta)(x-x_1)}, & x \ge x_1. \end{cases} \tag{5.47}$$

The principle of smooth fit for V at x_1 yields

$$V'(x_1-) = K\left(r_+(\beta)e^{r_+(\beta)x_1} - r_-(\beta)e^{r_-(\beta)x_1}\right) = 1, \tag{5.48}$$

$$V''(x_1-) = \tilde{r}_-(\beta).$$

Thus

$$e^{(r_+(\beta)-r_-(\beta))x_1} = \frac{r_-(\beta)\,(r_-(\beta)-\tilde{r}_-(\beta))}{r_+(\beta)\,(r_+(\beta)-\tilde{r}_-(\beta))}, \tag{5.49}$$

which admits a positive solution x_1 iff

$$M > M_0(\beta) = \frac{\sigma^2\beta^2\gamma}{2(\beta\mu-\delta)}. \tag{5.50}$$

Proposition 5.15 *If $\frac{\delta}{\mu} < \beta \leq \frac{2\delta}{\mu}$, then*

$$V'(0) > 1 \quad \textit{iff} \quad M > M_0(\beta).$$

Proof. The proof of this proposition follows from the calculations in this and in the previous subsections. □

In view of the above proposition we need only to treat two subcases, namely, $M > M_0(\beta)$ and $M \leq M_0(\beta)$, to complete our analysis.

5.3.1 Case of $M > M_0(\beta)$

In this case, (5.49) has a positive solution x_1 given by

$$x_1 = \frac{1}{r_+(\beta)-r_-(\beta)} \log\left(\frac{r_-(\beta)\,(r_-(\beta)-\tilde{r}_-(\beta))}{r_+(\beta)\,(r_+(\beta)-\tilde{r}_-(\beta))}\right). \tag{5.51}$$

Theorem 5.16 *Let x_1, $r_+(\beta)$, and $r_-(\beta)$ and $\tilde{r}_-(\beta)$ be given by (5.51), (4.8), and (5.10) respectively and let K be a constant determined from (5.48). If $\frac{\delta}{\mu} < \beta \leq \frac{2\delta}{\mu}$ and $M > M_0(\beta)$, then V given by (5.47) is a concave, twice continuously differentiable solution of (5.4) subject to (5.3).*

Proof. By differentiating the expression (5.47), we obtain

$$V^{(3)}(x) = K\left(r_+^3(\beta)e^{r_+(\beta)x} - r_-^3(\beta)e^{r_-(\beta)x}\right) > 0, \quad 0 \leq x < x_1.$$

As a result, $V''(x) < V''(x_1) = 0$ and $V'(x) > V'(x_1) = 1$. This proves that V is concave. □

5.3.2 Case of $M \leq M_0(\beta)$

In this case, in view of Proposition 5.15, $V'(0) \leq 1$. Therefore, x_1 defined by (5.5) equals zero.

Theorem 5.17 *Suppose that either $\beta \leq \frac{\delta}{\mu}$, or $\frac{\delta}{\mu} < \beta \leq \frac{2\delta}{\mu}$ and $M \leq M_0(\beta)$. Let $\tilde{r}_-(\beta)$, $\tilde{r}_-(\alpha)$ and $\tilde{c}$ be given by (5.10) and (5.11) respectively.*
(i) If $M \geq M_\beta$, then

$$V(x) = \frac{M}{\gamma}\left(1 - e^{\tilde{r}_-(\beta)x}\right), \qquad x \geq 0 \tag{5.52}$$

is a concave, twice continuously differentiable solution of (5.4)subject to (5.3).
(ii) If $M_\alpha \leq M < M_\beta$, then

$$V(x) = \frac{M}{\gamma}\left(1 - e^{-\frac{\mu}{\sigma^2\tilde{c}}x}\right), \qquad x \geq 0$$

is a concave, twice continuously differentiable solution of (5.4) subject to (5.3).
(iii) If $M < M_\alpha$ then

$$V(x) = \frac{M}{\gamma}\left(1 - e^{\tilde{r}_-(\alpha)x}\right), \qquad x \geq 0$$

is a concave, twice continuously differentiable solution of (5.4) subject to (5.3).

Proof. In the case of $\beta \le \frac{\delta}{\mu}$, the inequality $V'(0) \le 1$ holds due to Proposition 5.15. Then by applying Corollary 5.5, the desired result follows. On the other hand, if $\frac{\delta}{\mu} < \beta \le \frac{2\delta}{\mu}$ and $M \le M_0(\beta)$, then $V'(0) \le 1$ (see Proposition 5.2). Thus, in view of (5.33), Corollary 5.5 can be applied again to obtain the results. □

6 Optimal Policies

In this section we construct the optimal control policies based on the solutions to the HJB equations obtained in the previous sections. We first consider the case of unbounded dividend rates with nonzero liability. Recall that x_1 is the smallest number such that V'' vanishes. For each $x \le x_1$ define

$$a^*(x) \equiv \arg \max_{\alpha \le a \le \beta} \left(\frac{1}{2}\sigma^2 a^2 V''(x) + (a\mu - \delta)V'(x) - \gamma V(x) \right). \tag{6.1}$$

As evident from below the function $a^*(x)$ represents the optimal feedback control function for the control component $a_t^\pi, t \ge 0$. More precisely, the value $a^*(x)$ is the optimal risk that one should take when the value of the current reserve is x. From the analysis in Section 4, it follows that $a^*(x)$ can be represented as

$$a^*(x) = \begin{cases} \alpha, & 0 \le x \le x_\alpha, \\ a(x), & x_\alpha \le x \le x_\beta, \\ \beta, & x \ge x_\beta. \end{cases} \tag{6.2}$$

Note that the values of the critical points x_α, x_β as well as the function $a(x)$ depend on the three different cases studied in Section 4. Specifically, in the case of $\frac{2\delta}{\mu} < \alpha$ the values of x_α and x_β are specified by Theorem 4.7 while $a(x)$ is given by (4.15); in the case of $\alpha \le \frac{2\delta}{\mu} < \beta$, $x_\alpha = 0$ and x_β is given by Theorem 4.8 while $a(x)$ is determined by (4.29); and in the case of $\beta \le \frac{2\delta}{\mu}$, $x_\alpha = x_\beta = 0$.
To determine the other component of the optimal control, $C_t^\pi, t \ge 0$, which is the singular control in the terminology of control theory, we need to involve the so-called Skorohod problem. Let (R_t^*, C_t^*) be a solution to the following Skorohod problem on $t \ge 0$:

$$\begin{aligned} &R_t^* = x + \textstyle\int_0^t (a^*(R_s^*)\mu - \delta)ds + \int_0^t a^*(R_s^*)\sigma dW_s - C_t^*, \\ &R_t^* \le x_1, \\ &\textstyle\int_0^\infty 1_{\{R_s^* < x_1\}} dC_s^* = 0. \end{aligned} \tag{6.3}$$

Existence of a solution to such a Skorohod problem follows from Lions and Sznitman [11].

Theorem 6.1 *Let V be a concave, twice continuously differentiable solution of the HJB equation (2.8), and $(R_t^*, C_t^*; t \ge 0)$ be a solution to the Skorohod problem (6.3). Then for $\pi^* = (a^*(R_t^*), C_t^*; t \ge 0)$, we have*

$$J_x(\pi^*) = V(x), \ \ \forall x \ge 0. \tag{6.4}$$

Proof. For simplicity assume that the initial position $x \le x_1$. In this case both processes R_t^* and C_t^* as a solution to the Skorohod problem are continuous. In view of (6.1) and (5.7)

$$L^{a^*(R_s^*)} V(R_s^*) = 0, \tag{6.5}$$

where the operator L^a is defined in the proof of Theorem 2.2. Repeating the arguments of the proof of Theorem 2.2 and applying (6.5), we see that (we write τ instead of τ^π below, since there would be no confusion)

$$E(e^{-\gamma(t\wedge\tau)} V(R_{t\wedge\tau}^*)) = V(x) - E\int_0^{t\wedge\tau} e^{-\gamma s} V'(R_s^*) dC_s^*. \tag{6.6}$$

Since $V'(x_1) = 1$ and in view of (6.3)

$$1_{\{R_s^*=x_1\}} dC_s^* = dC_s^*, \tag{6.7}$$

we can replace $V'(R_s^*)$ in the integrand on the right hand side of (6.6) by $V'(R_s^*)1_{R_s^*=x_1} = V'(x_1)1_{R_s^*=x_1}$, to obtain

$$E(e^{-\gamma(t\wedge\tau)}V(R_{t\wedge\tau}^*)) = V(x) - E\int_0^{t\wedge\tau} e^{-\gamma s}V'(x_1)1_{\{R_s^*=x_1\}}dC_s^* = E\int_0^{t\wedge\tau} e^{-\gamma s}V'(x_1)dC_s^*, \tag{6.8}$$

where in the last equality we used (6.7) once more. Taking limit as $t \to \infty$, and applying (2.13), we obtain the desired result. □

Combining Theorems 2.2 and 6.1, we get the following result immediately.

Corollary 6.2 *The function V presented in the previous sections is the optimal return function and π^* is the optimal policy.*

Finally, let us briefly mention the case of bounded dividend rates, where the idea of obtaining the optimal policy is similar (in fact simpler in view of the fact that no Skorohod problem has to be involved in this case).
Suppose V is a concave solution to the HJB equation (5.4). Define

$$a^*(x) \equiv \arg\max_{\alpha\le a\le\beta}\left(\frac{1}{2}\sigma^2a^2V''(x) + (a\mu-\delta)V'(x) - \gamma V(x) + M\left(1-V'(x)\right)^+\right),$$

and

$$M^*(x) \equiv M1_{\{x\ge x_1\}},$$

where x_1 is defined by (5.5). The function $a^*(x)$ is the optimal feedback risk control function while the function $M^*(x)$ represents the optimal dividend rate payments, when the level of the reserve is x.

Theorem 6.3 *Let $R_t^*; t\ge 0$, be a solution to the following stochastic differential equation*

$$\begin{aligned} dR_t^* &= [a^*(R_t^*)\mu - \delta - M^*(R_t^*)]\,dt + a^*(R_t^*)\sigma dW_t, \\ R_0^* &= x. \end{aligned} \tag{6.9}$$

Then for $\pi^* \equiv (a_t^*, c_t^*; t\ge 0) = (a^*(R_t^*), M^*(R_t^*); t\ge 0)$, we have

$$J_x(\pi^*) = V(x), \quad \forall x \ge 0. \tag{6.10}$$

Corollary 6.4 *The function V presented in Section 5 is the optimal return function and π^* is the optimal policy.*

The proofs of Theorem 6.3 and Corollary 6.4 are similar to those of Theorem 6.1 and Corollary 6.2 which are omitted here.
Next we summarize all the results we obtained in the following tables for easy reference.

Range for $\frac{\delta}{\mu}$	x_α	x_β	$a^*(x)$	Risk α ever attained	Risk β ever attained	x_1	x_1 is the first point the possible maximal risk is attained at
$\frac{2\delta}{\mu} < \alpha$	positive and finite; see (4.9)	positive and finite; see (4.16)	(i) α, for $x \in [0, x_\alpha]$; (ii) increases from α to β on $[x_\alpha, x_\beta]$; see (4.15); (iii) β, for $x \geq x_\beta$	yes	yes	positive; see (4.25)	no
$\alpha < \frac{2\delta}{\mu} < \beta$ / $\frac{2\delta}{\mu} = \alpha$	0	positive and finite; see (4.30)	(i) increases from $2\delta/\mu$ to β on $[0, x_\beta]$; (ii) β, for $x \geq x_\beta$	no / yes	yes	positive; see (4.33)	no
$\frac{\delta}{\mu} < \beta \leq \frac{2\delta}{\mu}$	0	0	β	no	yes	positive; see (4.36)	no
$\frac{\delta}{\mu} \geq \beta$ (trivial case)	0	0	any	N/A	N/A	0	no (optimal policy is to declare bankruptcy immediately)

Table 1: The case of $M = \infty$

Range for M	x_α	x_β	$a^*(x)$	Risk α ever attained	Risk β ever attained	x_1	x_1 is the first point the possible maximal risk is attained at
$M > M_\beta$	positive and finite; see (4.9)	positive and finite; see (4.16)	(i) $\alpha,$ for $x \in [0, x_\alpha]$; (ii) increases from α to β on $[x_\alpha,\ x_\beta]$; see (4.15); (iii) $\beta,$ for $x \geq x_\beta$	yes	yes	positive; see (5.22)	no
$M_\alpha < M < M_\beta$ $M = M_\beta$	positive and finite; see (4.9)	∞	(i) $\alpha,$ for $x \in [0, x_\alpha]$; (ii) increases from α to $\frac{2\mu(\delta+M)}{\mu^2+2\gamma\sigma^2}$ on $[x_\alpha, x_1]$; see (4.15); (iii) $\frac{2\mu(\delta+M)}{\mu^2+2\gamma\sigma^2}$, for $x \geq x_1$	yes	no yes	positive; see (5.24)	yes
$M_0(\alpha) < M \leq M_\alpha$	∞	∞	α	yes	no	positive; see (5.34)	no
$M \leq M_0(\alpha)$	∞	∞	α	yes	no	0	yes

Table 2: The case of $M < \infty$ and $\dfrac{2\delta}{\mu} < \alpha$

Range for M	x_α	x_β	$a^*(x)$	Risk α ever attained	Risk β ever attained	x_1	x_1 is the first point the possible maximal risk is attained at
$M > M_\beta$	0	positive and finite; see (4.30)	(i) increases from $\frac{2\delta}{\mu}$ to β on $[0,\ x_\beta]$; see (4.29); (ii) β, for $x \geq x_\beta$	yes, if $\alpha = \frac{2\delta}{\mu}$; no, if $\alpha > \frac{2\delta}{\mu}$	yes	positive; see (5.22)	no
$M_0(\frac{2\delta}{\mu}) < M < M_\beta$ $M = M_\beta$	0	∞	(i) increases from $\frac{2\delta}{\mu}$ to $\frac{2\mu(\delta+M)}{\mu^2+2\gamma\sigma^2}$ on $[0,\ x_1]$; see (4.29); (ii) $\frac{2\mu(\delta+M)}{\mu^2+2\gamma\sigma^2}$, for $x \geq x_1$	yes, if $\alpha = \frac{2\delta}{\mu}$; no, if $\alpha > \frac{2\delta}{\mu}$	no yes	positive; see (5.43)	yes
$M_\alpha < M \leq M_0(\frac{2\delta}{\mu})$	0	∞	$\frac{2\mu(\delta+M)}{\mu^2+2\gamma\sigma^2}$	no	no	0	yes
$M \leq M_\alpha$	0	∞	α	yes	no	0	yes

Table 3: The case of $M < \infty$ and $\alpha \leq \dfrac{2\delta}{\mu} < \beta$.

Range for M	x_α	x_β	$a^*(x)$	Risk α ever attained	Risk β ever attained	x_1	x_1 is the first point the possible maximal risk is attained at
$M > M_0(\beta)$	0	0	β	no	yes	positive; see (5.51)	no
$M_\beta \leq M \leq M_0(\beta)$	0	0	β	no	yes	0	yes
$M_\alpha < M < M_\beta$ $M = M_\alpha$	0	∞	$\frac{2\mu(\delta+M)}{\mu^2+2\gamma\sigma^2}$	no yes	no	0	yes
$M < M_\alpha$	∞	∞	α	no	no	0	yes

Table 4: The case of $M < \infty$ and $\frac{\delta}{\mu} < \beta \leq \frac{2\delta}{\mu}$.

7 Economic Interpretation and Conclusions

The optimal policies obtained in the previous sections have clear economic meaning and are very easy to implement. Let us now elaborate.

First consider the case when the dividend distribution rate can be unbounded. The risk control policy is characterized by two critical reserve levels: x_α and x_β. The values of these two levels are further determined by the three parameters: the minimum risk allowed (α), the maximum risk allowed (β), and the ratio between the debt rate and profit rate ($\frac{\delta}{\mu}$). If the company has very little debt compared to the potential profit (so that $\frac{2\delta}{\mu} \le \alpha$), then both the critical reserve levels, x_α and x_β, are positive and finite. In this case, the company will minimize the business activity (i.e., take the minimum risk α) when the reserve is below the level x_α, then gradually increase the business activity when the reserve is between x_α and x_β, and then maximize the business activity (i.e., take the maximum risk β) when the reserve ever reaches or goes beyond the level x_β.

Next, if the company has a higher debt-profit ratio (so that $\alpha < \frac{2\delta}{\mu} < \beta$), then the company has to be a bit more aggressive in the sense that $x_\alpha = 0$ and x_β is positive and finite. In this case, no matter how small the reserve is the company will never take the minimum risk; rather it will start with the risk level $\frac{2\delta}{\mu}$ and gradually increase to the maximum risk level β when the reserve hits the level x_β and goes above this level. This can be explained by the fact that when the debt rate is high one needs to gamble on the higher potential profits in order to get out of the "bankruptcy zone" as fast as possible, even at the expense of assuming higher risk. The company becomes more aggressive when the debt-profit ratio is even higher (precisely when $\frac{\delta}{\mu} < \beta \le \frac{2\delta}{\mu}$), in which case the maximum allowable risk β is taken throughout while the two critical levels x_α and x_β are both zero. Finally, when the debt-profit ratio is so high that the debt-profit ratio is greater than the maximum risk possible, then the company should declare bankruptcy and go out of business immediately. This is due to the fact that the expected net cash flow is negative in this case, no matter what the company's policy might be.

On the other hand, the optimal dividend policy is always of a threshold type with the threshold being equal to x_1. Namely, the reserve should be kept below the critical level x_1 while distributing any excess as dividends. Moreover, in the case of unbounded dividend rate, the maximum business activity is always taken *before* dividend distributions ever take place.

As for the case of bounded dividend rate, the situation is much more complicated, depending on the value of the maximum dividend rate allowed. However, the fundamental structure of the optimal policies is the same as that of the case of the unbounded dividend rate. That is, the optimal risk control policy is characterized by two critical reserve levels, x_α and x_β, while the optimal dividend distribution policy is determined by another critical reserve level x_1. One striking difference, however, is that company may need to pay dividends *before* the maximum risk level β is ever taken; refer to Tables 2–4 for details. The economic reason for such a behavior is the following. When there is a significant constraint on the dividend rate there might be not necessary to pursue the business aggressively because the accumulated liquid assets could not be paid out as dividends fast enough anyway.

In conclusion, we would like to point out at an intricate interplay between the liability and restrictions on the risk control of a financial company. The sheer number of qualitatively different optimal policies, which appears due to different possible relationship between exogenous parameters, shows the multiplicity of different economic environments which a financial company faces depending on the size of the debt and on the size of available business activity.

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